臺灣大學數學系111學年度第1學期博士班一般資格考試

科目：偏微分方程

2022.09.12

1. (25%) Assume that $u \geq 0$ is a C^2 solution of $\Delta u = 0$ on Ω, where $\Omega \subset \mathbb{R}^2$ is an open set.
   (a) Show that u is a constant if $\Omega = \mathbb{R}^2$.
   (b) Let $\Omega = \{(x,y) \,|\, x > 0, y > 0\}$. We further assume that $u(x,0) = 0$ for $x \geq 0$, $u(0,y) = 0$ for $y \geq 0$, $u(x,y) \leq x^a y^a$, and u is continuous on the closure of Ω. Show that
   (b1) $u = 0$ if $0 \leq a < 1$,
   (b2) u is non-unique if $a \geq 1$,
   (b3) $u = 0$ if $a \geq 1$ and the $L^2(\Omega)$ norm of u is bounded

2. (25%) Solve the following equations:
   (a) $(u + u_x)^2 + u_y^2 = 1, u(x,0) = 0$,
   (b) $u_x + u_y + u_z = y^2 u, u(0,0,z) = z$.

3. (25%) Consider the heat equation:

$$\begin{cases} u_t = u_{xx}, & (x,t) \in (-\infty,\infty) \times (0,\infty) \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

   where $f(x)$ is continuous and $0 \leq f(x) \leq 1$.
   (a) Show that the equation has a solution u which satisfies $\lim_{t \to 0^+} u(x,t) = f(x)$ for $x \in \mathbb{R}$.
   (b) Show that there is an $f(x)$ with $0 \leq f(x) \leq 1$ such that the equation has a solution u satisfying

$$\limsup_{t \to \infty} u(0,t) = 1 \text{ and } \liminf_{t \to \infty} u(0,t) = 0.$$

4. (25%) Consider the equation

$$\begin{cases} u_{xy} - \alpha u_{xx} = 0, & (x,y) \in \mathbb{R}^2 \\ u(0,y) = y, u_y(0,y) = 1 - y, & y \in \mathbb{R}. \end{cases}$$

   Solve the equation with $\alpha = 0$ and $\alpha = 1$ respectively.