國立臺灣大學數學系

九十六學年度上學期博士班資格考試題

科目：代數

2007.09

Notations:
$\mathbb{R}$: the field of real numbers,
$\mathbb{Q}$: the field of rational numbers,
$M_n(F)$: the set of $n \times n$ matrices over a field F.

(1) If $A \in M_n(F)$ is a matrix of rank r, what is the nullity of the linear transformation $L_A : X \mapsto AX$ on $M_n(F)$?

(2) Let K be an algebraically closed field, and T be a linear transformation on an n-dimensional vector space whose matrix relative to a basis is

$$\begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & \lambda_1 \\ 0 & 0 & \cdot & \cdot & \lambda_2 & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & \lambda_{n-1} & \cdot & \cdot & 0 & 0 \\ \lambda_n & 0 & \cdot & \cdot & 0 & 0 \end{bmatrix}.$$

Show that T is diagonalizable if and only if for each k, $1 \le k \le n$, if $\lambda_k = 0$, then $\lambda_{n+1-k} = 0$. [Consider T^2.]

(3) Let G be a group and $\rho_a : x \mapsto a^{-1}xa$ be the inner automorphism determined by $a \in G$. Let ϕ be an endomorphism of G such that $\phi \circ \rho_a = \rho_a \circ \phi, \forall a \in G$. Show that:
(a) $H = \{x \in G : h(h(x)) = h(x)\}$ is a normal subgroup of G.
(b) G/H is an abelian group.

(4) Let G be a group of order p^5, p is a prime. Show that G must have a maximal abelian normal subgroup with order $\ge p^3$.

(5) Let R be the ring of all continuous real valued functions defined on the interval $[0,1]$.
(a) Give an example of a nonzero proper ideal of R.
(b) Show that R is not a ring direct sum of its proper ideals.

(6) Find all prime ideals of the ring $\mathbb{R}[x, y]$.

(7) Find the Galois group of the polynomial $x^{15} - 1$ over $\mathbb{Q}$.

(8) Let $F \subset E \subset \mathbb{R}$ be fields. If E/F is a radical extension $[E = F(a_1, \cdots, a_n), a_i^{n_i} \in F(a_1, \cdots, a_{i-1})$ for some $n_i]$ of odd degree $m > 1$, show that E is not a normal extension of F.