台灣大學數學系

九十三學年度博士班入學考試題

代數

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(1)

Let $a, b \in \mathbb{N}$ and a, b are relatively prime. Consider the set $S = \{xa + yb|$ x and y are nonnegative integers $\}$. Show that $ab - a - b \notin S$. For any integer $c \geq ab - a - b + 1$, show that $c \in S$.

(2)

If $f(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree n, show that $f(x) = 0$ cannot have n real roots, multiplicity counted, between two consecutive integers.

(3)

Let F be a field of characteristic $\neq 2$ and V a vector space over F. If $B(u, v)$ is a nonzero skew-symmetric bilinear function defined in V, show that it cannot be expressed as a product of two linear functions: $B(u, v) \neq f_1(u) f_2(v)$, where $f_1, f_2 : V \to F$ are linear functions.

(4)

Show that the ring $\mathbb{Z}[\frac{-1+\sqrt{-7}}{2}]$ is a Euclidean domain.

(5)

(a) If p is prime and G is a subgroup of the symmetric group S_p that contains a transposition and a p-cycle, prove that $G = S_p$.

(b) Determine the Galois group of the polynomial $2x^5 - 5x^4 + 5 \in \mathbb{Q}[x]$.