臺灣大學應用數學科學研究所 103 學年度碩士班甄試試題

科目：微積分

2013. 10. 18

(1) (15 pts) Find the length of the spiral given in polar coordinates by $r = \exp(\theta)$, $-\infty < \theta \leq 0$.

(2) (15 pts) Determine whether or not the following limit exists, and find its value if it exists:

$$\lim_{n\to\infty}\left[n - \frac{n}{e}\left(1+\frac{1}{n}\right)^n\right].$$

(3) (15 pts) Determine the maximum and minimum values of the function $f(x, y, z) = \cos(\pi(x + y + z))$ subject to the constraints $x^2 + y^2 + z^2 = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

(4) (15 pts) (a) (5 pts) Derive $1/n + \int_1^n (1/[x] - 1/x)dx$ where $[\cdot]$ is gauss symbol.

(b) (10 pts) Present a proper argument to prove the existence of the limit

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right).$$

You might consider using **(a)** to answer **(b)**.

(5) (15 pts) A real-valued function f defined on (a, b) is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, and $0 < \lambda < 1$. Prove that every convex function is continuous, and prove that every increasing convex function of a convex function is convex.

(6) (25 pts) (Newton Iteration) Suppose that f is a smooth real function defined for all real x, such that $|f'(x)| \geq \epsilon > 0$ and $|f^{(2)}(x)| \leq M$ for all x where $M > 0$.

(a) Show that $f(x) = 0$ has a unique root r_0.

(b) Given x_0, define a sequence by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Show that

$$|x_{n+1} - r_0| \leq |x_n - r_0|^2 M/\epsilon.$$

(c) Show that the sequence $\{x_n\}$ converges to r_0 provided that $|f(x_0)| < \epsilon^2/M$.