臺灣大學數學系 103 學年度碩士班甄試試題
科目：高等微積分

2013.10.18

請為每一題預留充分書寫空間，依題號次序 1.(a)(b)(c)，2.(a)(b)，…作答。若未依題目次序，其跳題作答之部份不予批閱、計分。

Answer the following questions in order.
All functions and sequences are real-valued.

1. (25%=5+5+15)

   (a) State the definition that a sequence (a_n) is a *Cauchy sequence.*

   (b) State the definition that a function $F : D \subset \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on D.

   (c) Let $f(x)$ be a continuous function on $\mathbb{R}$ and (x_n) be a Cauchy sequence. Use an $\epsilon - \delta/\epsilon - N$ type argument to show that $(f(x_n))$ is also a Cauchy sequence.

2. (25%=5+20)

   (a) State the definition that a sequence of functions $(h_n), h_n : D \subset \mathbb{R} \to \mathbb{R}$, *converges to* H *uniformly* on D.

   (b) Let (g_n) be a sequence of differentiable functions on (a,b) such that $\lim_{n\to\infty} g_n(x) = G(x)$ exists for all $x \in (a,b)$. Suppose that there exists a constant $M > 0$ such that $\sup_{x\in(a,b)} |g_n'(x)| < M$ for all n. Show that (g_n) converges to G uniformly on (a,b).

3. (25%=7+8+10) Let $\lambda > 0, J(\lambda) = \displaystyle\int_0^\infty \frac{dx}{(1+x)x^{2\lambda}}$ and $\Gamma(\lambda) = \displaystyle\int_0^\infty x^{\lambda-1}e^{-x}\,dx.$

   (a) Show that for $\lambda > 0$ the improper integral $\Gamma(\lambda)$ converges.

   (b) Find the range of $\lambda > 0$ on which the improper integral $J(\lambda)$ converges.

   (c) Show that $J(\lambda) = \Gamma(2\lambda)\Gamma(1-2\lambda)$ when both sides are meaningful. Hint. Express $(1+x)^{-1}$ as an integral.

4. (25%=10+15) Let $u_k, p_k, k = 1, \ldots, n$, be positive numbers and $p_1 + \cdots + p_n = 1$.

   (a) Evaluate the limit
   $$\lim_{t\to\infty} \left(\sum_{k=1}^n p_k u_k^t\right)^{\frac{1}{t}}.$$

   (b) Evaluate the limit
   $$\lim_{t\to 0} \left(\sum_{k=1}^n p_k u_k^t\right)^{\frac{1}{t}}.$$

   Hint. Use $u^t = \exp(t\ln u)$ and Taylor expansions.