Definite Quaternion Algebras over Function Fields and Brandt Matrices

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Notations

k : rational function field $\mathbb{F}_q(t)$, q is power of p, p an odd prime.

A : polynomial ring $\mathbb{F}_q[t]$.

∞ : infinite place, corresponding to the valuation of the degree.

k_∞ : $\mathbb{F}_q((\frac{1}{t}))$, i.e., the completion of k at ∞.

P : monic irreducible in A, i.e. finite prime.

$\bar{k}_\infty$: a fixed algebraic closure of k_∞.

$\bar{k}$: the algebraic closure of k inside k_∞.

$\overline{\mathbb{F}_q(t)}$: the algebraic closure of $\mathbb{F}_q$ inside $\bar{k}$.

v_∞ : the valuation on k_∞ s.t. $v_\infty(a) = -\deg(a)$ for all $a \in A$.

For us : k, A, k_∞ play the role of $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{R}$ respectively.

Definite quaternion algebras

Let P_0 be a fixed finite prime, $\mathcal{D}$ be the ("definite") quaternion algebra over k which ramifies only at ∞ and P_0.

Let $R \subset \mathcal{D}$ be a maximal order (A- rank 4).
Interested in left ideals I of R inside $\mathcal{D}$.

The left ideal classes can be put into 1-1 correspondence with isomorphism classes of rank 2 supersingular Drinfeld A-modules in A-characteristic P_0.

Let R^I be the right order of I, and set $w(I) = \#(R^I)^\times/(q-1)$.
If $[\phi]$ is class of Drinfeld A-modules corresponds to I, $w(\phi) = w(I)$ counts its automorphisms, then **Mass Formula** (Gekeler) says

$$\sum_{[\phi]} \frac{1}{w(\phi)} = \frac{q^{\deg P_0} - 1}{q^2 - 1} = \zeta_A(-1)(1 - q^{\deg P_0}).$$

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Drinfeld A-modules

Let (L, ι) (denoted by L simply) be an A-field, i.e. a field L together with $\mathbb{F}_q$-algebra homomorphism $\iota : A \to L$.
The kernel of ι is called the *A-characteristic of L.* This A-characteristic is a prime ideal (P), here P is a prime (monic irreducible) in A or zero.

Consider the twist polynomial ring : ($\tau(x) = x^q$)

$$L\{\tau\} = \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/L})$$

A rank 2 Drinfeld A-modules ϕ over L with A-characteristic P is an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to L\{\tau\}$, which satisfies

$$\phi_t = \iota(t) + g\tau + \Delta\tau^2, \Delta \neq 0.$$

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Supersingular Drinfeld A-modules

Let ϕ and ϕ' be two Drinfeld modules. A *morphism* $u : \phi \to \phi'$ over L is an element $u \in L\{\tau\}$ such that for all $a \in A$

$$u\phi_a = \phi'_a u.$$

We have accordingly endomorphisms, isomorphisms, and automorphisms of Drinfeld modules. A non-zero morphism is called an isogeny.

Given ϕ of rank 2 over L, and prime $P \in A$. The P-torsion of ϕ

$$\phi[P] = \{x \in \overline{L} : \phi_P(x) = 0\},$$

where $\overline{L}$ is fixed algebraic closure of L, is a finite A-module isomorphic to $(A/(P))^2$, if P is **not** the A-characteristic of L. In case the A-characteristic is $(P_0) \neq 0$, either $\phi[P_0] \cong A/(P)$ or ϕ is *supersingular*, i.e. $\phi[P_0] = 0$.

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Quaternion algebras as endomorphism algebras

Supersingular Drinfeld A-modules ϕ are always definable over finite A-field L, in fact, quadratic extension of $\mathbb{F}_{P_0} := A/(P_0)$.

If ϕ is of rank 2, $\mathrm{End}_L(\phi) \otimes_A k = \mathcal{D} = \mathcal{D}(P_0, \infty)$ is a quaternion division algebra over k. This quaternion algebra is "definite", in the sense it splits at primes differ from the characteristic P_0 and ∞.

Then $\mathrm{End}_L(\phi)$ is a maximal order in $\mathcal{D}$. Left ideal classes of $\mathrm{End}_L(\phi)$ correspond bijectively to the isomorphism classes of rank 2 supersingular Drinfeld A-modules over $L = \overline{\mathbb{F}_{P_0}}$.

The group $G = \mathrm{Gal}(\overline{\mathbb{F}_{P_0}}/\mathbb{F}_{P_0})$ acts on the left ideal classes by acting on the corresponding supersingular Drinfeld A-modules, the types (i.e. conjugacy classes) of maximal orders in $\mathcal{D}$ correspond bijectively to the orbits of isomorphism classes of supersingular Drinfeld A-modules under the action of G.

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Brandt matrices

Fix mixmal order R. For left ideal I, set $I^{-1} = \{b \in \mathcal{D} : IbI \subset I\}$, a right ideal for R whose left order is the right order of I.

Let $\{I_1, ..., I_n\}$ be left ideals of R representing the distinct ideal classes, with $I_1 = R$. Let R_i be the right order of I_i, and $w_i = \#(R_i^\times)/(q-1)$. Let $M_{ij} = I_j^{-1} I_i$, which is a left ideal of R_j with right order R_i. For any element $b \in M_{ij}$, $\mathrm{Nr}(b)$ denotes its reduced norm, and define $N_{ij} = f/g$ where f and g are the unique monic polynomials in A s.t. the quotients $\mathrm{Nr}(b)/N_{ij}$ are all in A with no common factor.
For each monic $m \in A$, let

$$B_{ij}(m) = \frac{\#\{b \in M_{ij} : (\mathrm{Nr}(b)/N_{ij}) = (m)\}}{(q-1)w_j}$$

and $B(m) = (B_{ij}(m)) \in \mathrm{Mat}_n(\mathbb{Z})$.
Also set $B(0) = (B_{ij}(0))$, with $B_{ij}(0) = \frac{1}{(q-1)w_j}$.

Supersingular Drinfeld Modules and Brandt Matrices

For each i, let ϕ_i be a supersingular Drinfeld module rank 2 corresponding to I_i. Then $\mathrm{End}(\phi_i) \cong R_i$. Moreover, one has

$$M_{ij} \cong \mathrm{Hom}(\phi_i, \phi_j), b \mapsto u_j b u_i^{-1},$$

where $u_i : \phi_1 \to \phi_i$ is the isogeny corresponding to I_i.

Note that given two isogenies u and u' from ϕ_i to ϕ_j, the finite A-submodule scheme $\ker(u)$ and $\ker(u')$ are equal if and only if $u' = \alpha u$, where $\alpha \in \mathrm{Aut}(\phi_j)$. Any finite A-submodule scheme C of ϕ_i is the kernel of some isogeny with **height** $h, 0 \leq h \leq 2$. The *Euler-Poincaré characteristic* of C is the ideal $(P_0^h d_1 d_2)$, if $C(\overline{L}) \cong A/(d_1) \times A/(d_2)$.

The entry $B_{ij}(m)$ is exactly the number of finite A-submodule schemes C of ϕ_i whose Euler-Poincaré characteristic is (m) and $\phi_i / C \cong \phi_j$.

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About Brandt matrices

(1) The row sums $\sum_j B_{ij}(m)$ are independent of i and equal to

$$\sigma(m)_{P_0} := \sum_{m'} q^{\deg(m')}$$

sum is over all monic polynomial $m'|m$ which is prime to P_0.

(2) If $(m, m') = 1$, then $B(m)B(m') = B(mm')$.

(3) If $B(P_0) \neq 1$, it is a permutation matrix of order 2 and

$$B(P_0^\ell) = B(P_0)^\ell.$$

(4) If $P \neq P_0$ is another monic prime, then for $\ell \geq 2$,

$$B(P^\ell) = B(P^{\ell-1})B(P) - q^{\deg(P)}B(P^{\ell-2}).$$

(5) The $B(m)$ generate a commutative subring $\mathbb{B}$ of $\mathrm{Mat}_n(\mathbb{Z})$.

(6) For all i, j the symmetry relation

$$w_j B_{ij}(m) = w_i B_{ji}(m).$$

(7) The algebra $\mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple, and isomorphic to a product of totally real number fields.

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Class numbers of imaginary quadratic fields

Let a be an element in $k \subset k_\infty$. If $a \neq 0$, then we define

$$\begin{cases} a > 0 & \text{if } a \in (k_\infty^\times)^2, \\ a < 0 & \text{if } a \in k_\infty^\times - (k_\infty^\times)^2. \end{cases}$$

If $d \in A$ with $d < 0$ let $h(d)$ be class number of $O_d = A[\sqrt{d}]$ and let $u(d) = \#(O_d^\times/\mathbb{F}_q^\times)$ ($u(d) = q+1$ or 1).

For $a \in A$ with $a < 0$ the *Hurwitz class number* is given by

$$H(a) = \sum_{df^2 = a, f \text{ monic}} \frac{h(d)}{u(d)}.$$

$$H_{P_0}(a) = \begin{cases} 0 & \text{if } P_0 \text{ splits in } O_a, \\ \frac{2}{q-1} H(a) & \text{if } P_0 \text{ is inert in } O_a, \\ \frac{1}{q-1} H(a) & \text{if } P_0 \text{ ramified but prime to conductor of } O_a, \\ H_{P_0}(a/P_0^2) & \text{if } P_0 \text{ divides the conductor of } O_a. \end{cases}$$

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Trace formula

We have analogue of Eichler's **trace formula**,

$$\operatorname{tr} B(m) = \sum_{m' \in A, (m')=(m)} \left\{ \sum_{s \in A, s^2 \leq 4m'} H_{P_0}(s^2 - 4m') \right\},$$

for all monic polynomial $m \in A$.

Set also $H_{P_0}(0) = \frac{q^d - 1}{(q-1)(q^2-1)}$, then Mass formula amounts to

$$\operatorname{tr} B(0) = H_{P_0}(0).$$

Theta series

Fix addtive characters as $\sigma : \mathbb{F}_q \to \mathbb{C}^\times$, and $\psi_\infty : k_\infty \to \mathbb{C}^\times$,
$\sigma(\xi) = \exp(\frac{2\pi i}{p} \operatorname{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\xi))$,

$\psi_\infty(y) = \sigma(\operatorname{Res}_\infty(ydt))$.

Let n be the class number of the maximal order R, choose representatives $I_i, i = 1, \cdots, n$, of the left ideal classes, and set $M_{ij} = I_j^{-1} I_i$. For $x \in k_\infty^\times$, $y \in k_\infty$, define **Theta Series** for $\mathcal{D}$,

$$\theta_{ij}(x, y) = \sum_{b \in M_{ij}} \phi_\infty(\frac{\operatorname{Nr}(b)}{N_{ij}} x t^2) \cdot \psi_\infty(\frac{\operatorname{Nr}(b)}{N_{ij}} y),$$

where ϕ_∞ is the characteristic function of $\mathcal{O}_\infty$, and $N_{ij} = f/g$ where f and g are the unique monic polynomials in A s.t. the quotients $\operatorname{Nr}(b)/N_{ij}$ are all in A with no common factor.

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Automorphy of theta series

For each $a \in A$, let $B'_{ij}(a) = \#\{b \in M_{ij} : \mathrm{Nr}(b)/N_{ij} = a\}$. Then

$$(q-1)w_j \cdot B_{ij}(m) = \sum_{(a)=(m)} B'_{ij}(a).$$

We may rewrite the theta series as

$$\theta_{ij}(x,y) = \sum_{a \in A, \deg(a) \le v_\infty(x)-2} B'_{ij}(a)\psi_\infty(ay).$$

One has $\theta_{ij}(x, y+a) = \theta_{ij}(x,y)$ for $a \in A$.
Also $\theta_{ij}(\alpha x, \beta x + y) = \theta_{ij}(x,y)$ for $\alpha \in \mathcal{O}_\infty^\times$, $\beta \in \mathcal{O}_\infty$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$. Assume $v_\infty(x) > v_\infty(y)$, $v_\infty(cx) > v_\infty(cy+d)$, and $c \equiv 0 \pmod{P_0}$. Then

$$\theta_{ij}(g \circ (x,y)) = q^{-2v_\infty(cy+d)} \cdot \theta_{ij}(x,y).$$

Automorphy of theta series

For each $a \in A$, let $B'_{ij}(a) = \#\{b \in M_{ij} : \mathrm{Nr}(b)/N_{ij} = a\}$. Then

$$(q-1)w_j \cdot B_{ij}(m) = \sum_{(a)=(m)} B'_{ij}(a).$$

We may rewrite the theta series as

$$\theta_{ij}(x,y) = \sum_{a \in A, \deg(a) \leq v_\infty(x)-2} B'_{ij}(a)\psi_\infty(ay).$$

One has $\theta_{ij}(x, y+a) = \theta_{ij}(x,y)$ for $a \in A$.
Also $\theta_{ij}(\alpha x, \beta x + y) = \theta_{ij}(x,y)$ for $\alpha \in \mathcal{O}_\infty^\times$, $\beta \in \mathcal{O}_\infty$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$. Assume $v_\infty(x) > v_\infty(y)$, $v_\infty(cx) > v_\infty(cy+d)$, and $c \equiv 0 \pmod{P_0}$. Then

$$\theta_{ij}(g \circ (x,y)) = q^{-2v_\infty(cy+d)} \cdot \theta_{ij}(x,y).$$

Functions on ∞-adic space

Introducing complex-valued functions on $\mathrm{GL}_2(k_\infty)$:

$$\theta'_{ij}(g) = q^{-v_\infty(x)}\theta_{ij}(x, y)$$

where $g = \gamma \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \gamma_\infty \alpha$ for some $\gamma \in \Gamma_0(P_0) \cap \mathrm{SL}_2(A)$, $\gamma_\infty \in \Gamma_\infty$, $\alpha \in k_\infty^\times$. Moreover, let

$$\Theta_{ij}(g) = \sum_{\epsilon \in \mathbb{F}_q^\times} \theta'_{ij}\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right).$$

Then Θ_{ij} are complex-valued functions on the double coset space

$$\Gamma_0(P_0)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times.$$

Definite Shimura curves

Let Y be the genus 0 curve over k associated with the quaternion algebra $\mathcal{D}$, which is defined by:

$$Y(M) = \{x \in \mathcal{D} \otimes_k M : \mathrm{tr}(x) = \mathrm{Nr}(x) = 0\}/M^\times.$$

Here M is any k-algebra. The group $\mathcal{D}^\times$ acts on Y by conjugation. If K is a quadratic extension of k, then one can identify $Y(K) = \mathrm{Hom}(K, \mathcal{D})$.

To each embedding $f : K \to \mathcal{D}$ we let $y = y_f$ be the image of the unique K-line on the quadric $\{x \in \mathcal{D} \otimes_k K : \mathrm{tr}(x) = N(x) = 0\}$ on which conjugation by $f(K^\times)$ acts by the character $a \mapsto a/\sigma(a)$, σ is the non-trivial automorphism of K/k. Note that y_f is one of the 2 fixed points of $f(K^\times)$ acting on $Y(K)$; the other is the image of the line where conjugation acts by the character $a \mapsto \sigma(a)/a$.

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Notations continued

k_P : completion of k at a finite prime P.

A_P : closure of A in k_P.

R_P : $= R \otimes_A A_P$, $K_P := K \otimes_k k_P$, and $\mathcal{D}_P := \mathcal{D} \otimes_k k_P$.

$\hat{k}$: $\prod'_P k_P$, the finite adele ring of k.

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For quadratic order $O_d \subset K$ one has

$$\hat{O}_d^\times \backslash \hat{K}^\times / K^\times \cong \operatorname{Pic} O_d.$$

For left ideal classes of the maximal order R, one has bijection with double cosets in

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Special points

Our definite Shimura curve X_{P_0} is defined as

$$\left(\hat{R}^\times \backslash \hat{\mathcal{D}}^\times \times Y\right) / \mathcal{D}^\times.$$

This is union of curves of genus 0, with components in bijection with the left ideal classes of R. Thus if there are n left ideal classes, $\mathrm{Pic}(X_{P_0}) \cong \mathbb{Z}^n$, generated by $e_i, i = 1, \ldots, n$, which are classes of degree 1 on each component of X_{P_0}.

The *special points* (Gross points) on X_{P_0} over K are points in the image of $\hat{R}^\times \backslash \hat{\mathcal{D}}^\times \times Y(K)$ in $X_{P_0}(K)$. We say the point $x = (g, y)$ has discriminant d if $f(K) \cap g^{-1}\hat{R}g = f(O_d)$, where $f : K \to \mathcal{D}$ is the embedding corresponding to y. Note that here the discriminant of a special point is well defined up to multiplication by elements in $(\mathbb{F}_q^\times)^2$.

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Hecke correspondences

Given P. Let $\mathcal{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(k_P)$. The vertices are the classes of A_P-lattices in k_P^2, and two such vertices are adjacent if the "distance" between the lattice classes is 1.

The Hecke correspondence t_P sends vertex v to the formal sum of its $q^{\deg(P)} + 1$ neighbors on the tree.

Identifying $\mathrm{PGL}_2(A_P)\backslash \mathrm{PGL}_2(k_P)$ with vertices of the Bruhat-Tits tree, for $g_P \in \mathrm{PGL}_2(A_P)\backslash \mathrm{PGL}_2(k_P)$ one has:

$$t_P(g_P) = \sum_{\deg(u)\leq\deg(P)} \begin{pmatrix} 1 & u \\ 0 & P \end{pmatrix} g_P + \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} g_P.$$

When $P \neq P_0$, one has $R_P^\times\backslash\mathcal{D}_P^\times/k_P^\times \cong \mathrm{PGL}_2(A_P)\backslash \mathrm{PGL}_2(k_P)$.

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Correspondence on Shimura curve

View X_{P_0} as $(\hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \hat{k}^\times) \times Y / (\mathcal{D}^\times / k^\times)$. This leads to global Hecke correspondence t_P on X_{P_0} for all P.

As t_P and $t_{P'}$ are commute for any prime P and P', one defines t_m for every ideal (m) of A:

$$t_{mm'} = t_m t_{m'}, \quad \text{if } m \text{ and } m' \text{ are relatively prime,}$$

$$t_{P^\ell} = t_{P^{\ell-1}} t_P - q^{\deg P} t_{P^{\ell-2}}, \quad \text{for } P \neq P_0,$$

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Let $\mathbb{T}$ be the $\mathbb{Z}$ algebra generated by all $t_m, m \in A$ monic. Then $\mathbb{T} \cong \mathbb{B}$ as $\mathbb{Z}$-algebras. Passing to $\mathrm{Pic}(X_{P_0})$, one shows that, for the basis $e_i, 1 \leq i \leq n$:

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Gross pairing

Following B. Gross, we define a height pairing $<,>$ on $\mathrm{Pic}(X_{P_0})$ with values in $\mathbb{Z}$ by setting

$$< e_i, e_j >= 0, \text{ if } i \neq j;$$

$$< e_i, e_i >= w_i.$$

This pairing gives an isomorphism of $\mathrm{Pic}(X_{P_0}))^\vee = \mathrm{Hom}(\mathrm{Pic}(X_{P_0}), \mathbb{Z})$ with the subgroup of $\mathrm{Pic}(X_{P_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with basis $\{\check{e}_i = e_i/w_i : i = 1, ..., n\}$. Since $w_j B_{ij}(m) = w_i B_{ji}(m)$ always hold, one has the following identity, for all classes e and e' in $\mathrm{Pic}(X_{P_0})$,

$$< t_m e, e' >=< e, t_m e' > .$$

Automorphic forms

Let $\mathcal{O}_\infty$ be the valuation ring of k_∞, with uniformizer π_∞. We are interested in automorphic forms of level $P_0\infty$, i.e. complex-valued functions on the double coset space

$$\Gamma_0(P_0)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times,$$

where $\Gamma_0(P_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) : c \equiv 0 \mod P_0 \right\}$,

and $\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) : c \in \pi_\infty \mathcal{O}_\infty \right\}$.

From Brandt matrices we have constructed theta series θ_{ij}. These theta series then give rise automorphic forms of **Drinfeld type**.

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Automorphic forms of Drinfeld type

An automorphic form f is *of Drinfeld type* if it satisfies the following *harmonic* properties: for any $g \in \mathrm{GL}_2(k_\infty)$

$$(1) \quad f(g \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = -f(g),$$

$$(2) \quad \sum_{\kappa \in \mathrm{GL}_2(\mathcal{O}_\infty)/\Gamma_\infty} f(g\kappa) = 0.$$

All the functions Θ_{ij} constructed from the quaternion algebra $\mathcal{D}$ are of Drinfeld type.

Let $M(\Gamma(P_0))$ be the space of all automorphic forms of Drinfeld type of level $P_0\infty$. For each monic $m \in A$ one also has Hecke operators T_m on the space $M(\Gamma(P_0))$. This gives a commutative algebra of Hecke operators on automorphic forms of Drinfeld type. This algebra is again isomorphic to the algebra of $\mathbb{B}$.

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A canonical pairing

Moreover we have for all $1 \leq i, j \leq n$ and any monic m the identity,

$$T_m \Theta_{ij} = \sum_{\ell} B_{i\ell}(m) \Theta_{\ell j}.$$

The multiplicity one theorem for automorphic forms then implies that the theta series $\Theta_{\ell j}$ generate a subspace inside $M(\Gamma(P_0))$ which is a free $\mathbb{B} \otimes \mathbb{C}$-module of rank one.

We have a pairing :

$$\phi : \mathrm{Pic}(X_{P_0}) \times \mathrm{Pic}(X_{P_0}) \longrightarrow M(\Gamma(P_0)),$$

$$\phi(e, e') \begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = q^{-r+2} \Bigg(\deg e \cdot \deg e' + \sum_{m \text{ monic}, \deg m \leq r-2} < e, t_m e' > \sum_{(\lambda)=(m)} \psi_\infty(\lambda u) \Bigg)$$

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Isomorphism of Hecke modules

This pairing is equivariant w.r.t. the Hecke action: for all $m \in A$.

$$T_m \phi(e, e') = \phi(t_m e, e') = \phi(e, t_m e').$$

We **claim** that the theta series $\Theta_{\ell j}$ actually generate $M(\Gamma(P_0))$. It follows that our pairing induces an isomorphism of Hecke modules:

$$(\mathrm{Pic}(X_{P_0}) \otimes_{\mathbb{Z}} \mathbb{C}) \otimes_{\mathbb{T}_{\mathbb{C}}} (\mathrm{Pic}(X_{P_0}) \otimes_{\mathbb{Z}} \mathbb{C}) \cong M(\Gamma_0(P_0)).$$

The dimension of $M(\Gamma(P_0))$ therefore equals to the number of left ideal classes of R. It also equals to $g(\Gamma(P_0)) + 1$ (Gekeler), where $g(\Gamma(P_0))$ is the genus of the Drinfeld modular curve $X_0(P_0)$.

The **claim** is essentially Jacquet-Langlands correspondence over the function field k.

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Jacquet-Langlands revisited

Given a Hecke character $\eta : \mathbb{A}_k^\times / k^\times$. Let $\mathcal{A}_0(\eta)$ be the space of automorphic cusp forms for $GL_2(k)$ with central character η and $\mathcal{A}'(\eta)$ be the space of automorphic forms for $\mathcal{D}^\times$ with central character η. Jacquet-Langlands correspondence describes the connection between $\mathcal{A}'(\eta)$ and $\mathcal{A}_0(\eta)$, namely:

If an irreducible admissible representation $\rho' = \otimes \rho'_v$ is a constituent of $\mathcal{A}'(\eta)$ and ρ'_v is infinite dimensional at ∞ and P_0, then there exist an irreducible admissible representation $\rho (= \rho'^{\mathsf{JL}})$ which is a constituent of $\mathcal{A}_0(\eta)$ so that

$$L(s, \omega \otimes \rho) = L(s, \omega \otimes \rho')$$

for any Hecke character ω.

Note that $\rho = \otimes \rho_v$ where $\rho_v = \rho'_v$ for finite place $v \neq P_0$. On the other hand ρ_{P_0} is from theta correspondence of ρ'_v and ρ_v, and is special or supercuspidal.

Conversely, suppose $\rho = \otimes \rho_v$ is a constituent of $\mathcal{A}_0(\eta)$. If the representation ρ_{P_0} is special or supercuspidal, then there is a constituent $\rho' = \otimes \rho'_v$ of $\mathcal{A}'(\eta)$ s.t. $\rho_v = \rho_v'^{\mathsf{JL}}$.

Jacquet-Langlands correspondence gives an isomorphism (as Hecke modules) between

$$\{ \text{ non-constant functions on } \hat{R}^\times \backslash \hat{\mathcal{D}}^\times / \mathcal{D}^\times \}$$

and

$$\left\{ \begin{array}{c} \text{automorphic cusp forms for } \Gamma_0(P_0) \text{ giving special} \\ \text{representation at } \infty \text{ with trivial central character} \end{array} \right\}.$$

The End. Thank You.