On values of Modular Forms at Algebraic Points

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In value distribution theory the exponential function e^z is a key. This fuction also enjoys the following extraordinary properties: (Hermite-Lindemann-Weierstrass 1880) For $\alpha \neq 0 \in \overline{\mathbb{Q}}$, e^α is **transcendental**. Moreover, if algebraic numbers $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are **algebraically independent**, i.e. for any polynomial $P \neq 0 \in \overline{\mathbb{Q}}(x_1, \ldots, x_n)$,

$$P(e^{\alpha_1}, \ldots, e^{\alpha_n}) \neq 0$$

Tools for proving this come from **Complex Analysis**.

Let $\mathcal{H}$ be the complex upper half plane.
We are also interested in values of "natural" holomorphic functions taking at "**algebraic points**" of $\mathcal{H}$.

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Siegel-Schneider

The modular function $j : SL_2(\mathbb{Z})\backslash\mathcal{H} \cong \mathbb{C}$ which parametrizes isomorphism classes of complex elliptic curves.
This function j can be employed for proving the Picard theorem.

This j also has beautiful transcendence property:
(Siegel-Schneider 1930) If $\alpha \in \overline{\mathbb{Q}} \cap \mathcal{H}$ and α is not quadratic, then $j(\alpha)$ is transcendental.
If α is (imaginary) quadratic, then $j(\alpha)$ is actually an algebraic integer, as known to Kronecker.

Call $\alpha \in \mathcal{H}$ **algebraic point** if $j(\alpha) \in \overline{\mathbb{Q}}$. Thus unless an algebraic point $\alpha \in \mathcal{H}$ is imaginary quadratic number, it must be a transcendental number.
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Arithmetic modular forms

(Meromorphic) Modular form $f : \mathcal{H} \longrightarrow \mathbb{C} \cup \{\infty\}$, of weight k, here k is a fixed integer, satisfying for all $z \in \mathcal{H}$

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z),$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Modular forms are required to be meromorphic at ∞, i.e. with Fourier expansion:

$$f(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z}.$$

Call f **arithmetic** modular form if all coefficients $a_n \in \overline{\mathbb{Q}}$.

Note one can replace $SL_2(\mathbb{Z})$ by its congruence subgroups Γ , and requiring f to be meromorphic at all "cusps".

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Reformulating works of Siegel-Schneider, one has

Theorem. Let f be arithmetic modular form of **nonzero** weight k. Let $\alpha \in \mathcal{H}$ is an algebraic point which is neither zero nor pole of f, then $f(\alpha)$ is transcendental.

Open Problem. Let f as above, $\alpha_1, \ldots, \alpha_n \in \mathcal{H}$ be algebraic points which are neither zeros nor poles of f. Suppose that the α_i are pairwise non-isogenous, are the values $f(\alpha_1), \ldots, f(\alpha_n)$ algebraically independent?

Here α and $\beta \in \mathcal{H}$ are said to be non-isogenous, if the elliptic curves they correspond are not isogenous.

Note that the value $f(\alpha)$ is always an algebraic multiple of the k-th power of a period (of the elliptic curve corresponding to α) dividing by π.

Can prove the linearly independence over $\overline{\mathbb{Q}}$ of these values.

Values at algebraic points

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World of Positive characteristic

$\mathbb{F}_q :=$ the finite field of q elements.
$k := \mathbb{F}_q(\theta) :=$ the rational function field in the variable θ over $\mathbb{F}_q$.
$\bar{k} :=$ fixed algebraic closure of k.

$k_\infty := \mathbb{F}_q((\frac{1}{\theta}))$, completion of k with respect to the infinite place.
$\overline{k_\infty} :=$ a fixed algebraic closure of k_∞ containing $\bar{k}$.
$\mathbb{C}_\infty :=$ completion of $\overline{k_\infty}$ with respect to the canonical extension of the infinite place.

Non-archimedean analytic function theory on $\mathbb{C}_\infty$, and on Drinfeld upper-half space $\mathcal{H}_\infty$ which is $\mathbb{C}_\infty - k_\infty$.

Natural non-archimedean analytic functions come from Drinfeld modules theory.

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Drinfeld modules

Let $\tau : x \mapsto x^q$ be the Frobenius endomorphism of $\mathbb{G}_a/\mathbb{F}_q$.

Let $\mathbb{C}_\infty[\tau]$ be the twisted polynomial ring :

$$\tau c = c^q \tau, \quad \text{for all } \; c \in \mathbb{C}_\infty.$$

A Drinfeld $\mathbb{F}_q[t]$-module ρ of rank r (over $\mathbb{C}_\infty$) is a $\mathbb{F}_q$-linear ring homomorphism (Drinfeld 1974) $\rho : \mathbb{F}_q[t] \to \mathbb{C}_\infty[\tau]$ given by $(\Delta \neq 0)$

$$\rho_t = \theta + g_1\tau + \cdots + g_{r-1}\tau^{r-1} + \Delta\tau^r,$$

Drinfeld exponential $\exp_\rho(z) = \sum_{h=0}^{\infty} c_h z^{q^h}$, $c_h \in \bar{k}$, on $\mathbb{C}_\infty$ linearizes this t-action :

$$\begin{array}{ccc}
\mathbb{C}_\infty & \xrightarrow{\exp_\rho} & \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \\
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Analogue of Hermite-Lindemann-Weierstrass, and Siegel-Schneider:

Theorem 1.(Yu 1986) Let ρ be a Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$, with associated exponential map $\exp_\rho(z)$ on $\mathbb{C}_\infty$.
If $\alpha \neq 0 \in \bar{k}$, then $\exp_\rho(\alpha)$ is transcendental over k.

Theorem 2.(A. Thiery 1995) Suppose the Drinfeld module ρ is of rank 1. If $\alpha_1, \ldots, \alpha_n \in \bar{k}$ are linearly independent over k , then $\exp_\rho(\alpha_1), \ldots, \exp_\rho(\alpha_n)$ are algebraically independent over k.

Drinfeld upper-half space $\mathcal{H}_\infty$ parametrizes isomorphism classes of rank 2 Drinfeld modules, let $j = g_1^{q+1}/\Delta$:

$$j : GL_2(\mathbb{F}_q[\theta])\backslash\mathcal{H}_\infty \cong \mathbb{C}_\infty.$$

Call $\alpha \in \mathcal{H}_\infty$ **algebraic point** if $j(\alpha) \in \bar{k}$. Drinfeld modules corresponding to algebraic points can be defined over $\bar{k}$.

Transcendence theory

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We have the following

Theorem 3.(Yu 1986) If $\alpha \in \bar{k} \cap \mathcal{H}_\infty$ and α is not quadratic over k, then $j(\alpha)$ is transcendental over k. Moreover, for those α quadratic over k, $j(\alpha)$ are integral over $\mathbb{F}_q[\theta]$.

If Drinfeld module ρ is of rank r, kernel of $\exp_\rho$ is a discrete free $\mathbb{F}_q[\theta]$-module $\Lambda_\rho \subset \mathbb{C}_\infty$ of rank r. Moreover

$$\exp_\rho(z) = z \prod_{\lambda \neq 0 \in \Lambda_\rho} (1 - \frac{z}{\lambda}).$$

The nonzero elements in Λ_ρ are the **periods** of the Drinfeld module ρ. They are all transcendental over $\bar{k}$ by Theorem 1.

Morphisms of Drinfeld modules $f : \rho_1 \to \rho_2$ are the twisting polynomials $f \in \bar{k}[\tau]$ satisfying $(\rho_2)_t \circ f = f \circ (\rho_1)_t$.

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Algebraic relations among periods

Isomorphisms from ρ_1 to ρ_2 are given by constant polynomials $f \in \bar{k} \subset \bar{k}[\tau]$ such that $f\,\Lambda_{\rho_1} = \Lambda_{\rho_2}$.

The endomorphism ring of Drinfeld module ρ can be identified with

$$R_\rho = \{\alpha \in \bar{k} |\ \alpha\Lambda_\rho \subset \Lambda_\rho\}.$$

The field of fractions of R_ρ, denoted by K_ρ, is called the field of multiplications of ρ. One has that $[K_\rho : k]$ always divides the rank of the Drinfeld module ρ.

Drinfeld module ρ of rank 2 is said to be without Complex Multiplications if $K_\rho = k$, and with CM if $[K_\rho : k] = 2$.
If ρ has CM, there are non-trivial algebraic relations among its periods.

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Drinfeld modular forms

Modular form $f : \mathcal{H}_\infty \longrightarrow \mathbb{C}_\infty \cup \{\infty\}$, of weight k and type m, here k is a fixed integer, $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, satisfying for all $z \in \mathcal{H}_\infty$

$$f(\frac{az+b}{cz+d}) = (\det \gamma)^m (cz+d)^k f(z),$$

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q[\theta]).$$

Modular forms are required to be "rigid" meromorphic functions, and at ∞ with "Fourier" expansion:

$$f(z) = \sum_{n=n_0}^{\infty} a_n q_\infty(z)^n, \quad q_\infty(z) = \sum_{a \in \mathbb{F}_q[\theta]} \frac{1}{z-a}.$$

Call f **arithmetic** modular form if all coefficients $a_n \in \bar{k}$.

Note one can replace $GL_2(\mathbb{F}_q[\theta])$ by its congruence subgroups Γ , and requiring f to be meromorphic at all "cusps".

Here one also proves

Theorem.(Yu) Let f be arithmetic modular form of **nonzero** weight k. Let $\alpha \in \mathcal{H}_\infty$ is an algebraic point which is neither zero nor pole of f, then $f(\alpha)$ is transcendental over k.

Open Problem. Let f as above, $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_\infty$ be algebraic points which are neither zeros nor poles of f. Suppose that the α_i are pairwise non-isogenous, are the values $f(\alpha_1), \ldots, f(\alpha_n)$ algebraically independent over k?

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Again the value $f(\alpha)$ is equal to an element of $\bar{k}$ times k-th power of a period (of the rank 2 Drinfeld modules defined over $\bar{k}$ corresponding to α) dividing by the Carlitz period (of rank 1 Drinfeld $\mathbb{F}_q[t]$-module).

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The CM points are those $\alpha \in \mathcal{H}_\infty$ which are quadratic over k, hence correspond to Drinfeld modules with CM.

Theorem.(C.-Y. Chang 2010) Let f be arithmetic modular form of **nonzero** weight. Let $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_\infty$ be CM points which are neither zeros nor poles of f. Suppose that the α_i are pairwise non-isogenous, then the values $f(\alpha_1), \ldots, f(\alpha_n)$ are algebraically independent over k.

Here CM points α and $\beta \in \mathcal{H}_\infty$ are non-isogenous, precisely when they belong to different quadratic extension of k.

Method for proving algebraic independence in positive characteristic, developed in the last 10 years, by Anderson, Brownawell, Chang, Papanikolas, and Yu. Crucial step by Papanikolas 2008.

Realizing a program of **Grothendieck** in positive characteristic.

We are interested in finitely generated extension of $\bar{k}$ generated by a set S of special values, denoted by K_S. In particular we want to determine all algebraic relations among elements of S.
From known algebraic relations, one can guess the transcendence degree of K_S over $\bar{k}$, and the goal is to prove that is indeed the specific degree in question.

To proceed, we construct a t-motive M_S for this purpose, so that it has the **GP** property and its "periods" $\Psi_S(\theta)$ from "rigid analytic trivialization "generate also the field K_S, then computing the dimension of the motivic Galois (algebraic) group Γ_{M_S}.

GP property of the motive M_S requires:

$$\dim \Gamma_{M_S} = \text{tr.deg}_{\bar{k}}\ \bar{k}(\Psi_S(\theta)).$$

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