TEST CASES FOR LOW MACH NUMBER FLOWS

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Contents

1. Introduction 1
2. Advection of a vortex. 1
2.1. Governing equations. 1
2.2. Domain. 2
2.3. Exact solution, initial condition. 2
2.4. Boundary conditions 3
2.5. Test cases 4
2.6. Output 4
2.7. Example 4

1. Introduction

When attempting to compute unsteady, variable density flows at very small or zero Mach number using standard finite volume compressible flow solvers one faces at least the following difficulties: (i) Spatial pressure variations vanish as the Mach number $M \longrightarrow 0$, but they do affect the velocity field at leading order; (ii) the resulting spatial homogeneity of the leading order pressure implies an elliptic divergence constraint for the energy flux; (iii) violations of this constraint crucially affect the transport of mass, preventing a code to properly advect even a constant density distribution. We propose a suite of test problems for low Mach number variable density flows. These problems are designed to assess the accuracy and the efficiency of numerical methods and test their capability to cope with the above mentioned difficulties. For these test problems either the exact solution or at least some properties of the exact solution are known. This allows a meaningful validation of new computational approaches and the comparison of numerical results obtained with different methods. All test problems can be run with trivial geometries and straight-forward boundary conditions.

2. Advection of a vortex.

2.1. Governing equations.

The governing equations are the compressible Euler equations for a calorically perfect gas:

$$
\begin{aligned}
&\rho_t + \nabla \cdot (\rho \vec{v}) = 0 \\
&(\rho \vec{v})_t + \nabla \cdot (\rho \vec{v} \circ \vec{v}) + \nabla p = 0 \\
&(\rho e)_t + \nabla \cdot ((\rho e + p) \vec{v}) = 0 \\
&p = (\gamma - 1)(\rho e - \frac{1}{2} \rho \vec{v} \cdot \vec{v})
\end{aligned} \tag{1}
$$

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These equations support special two-dimensional solutions consisting of a single "vortex" advected at constant speed:

$$\mathcal{U}(\vec{x},t) = \mathcal{U}(\vec{x} - \vec{v}_a t, 0)$$

We use $\mathcal{U}$ to indicate a solution component or a set of solution components of the governing equations, e.g., $\mathcal{U} := p$ or $\mathcal{U} := \{\rho, \vec{v}\}$. $\vec{x} := (x, y)$ and t are the space and the time coordinate, respectively, and $\vec{v}_a$ is the advection velocity. Vortex solutions $\mathcal{U}$ of (1) can be constructed as follows. Let $\vec{x}_{c,0}$ be the position of the vortex center at time $\vec{s} = 0$. At any time

$$\vec{x}_c(t) = (x_c(t), y_c(t)) = \vec{x}_{c,0} + \vec{v}_a t \tag{2}$$

and $\mathcal{U}$ only depends on $\vec{x} - \vec{x}_c$. In particular, $\mathcal{U}$ depends on the distance r between $\vec{x}$ and the center of the vortex $\vec{x}_c$ and on the angle θ between some fixed direction and the direction of $\vec{x} - \vec{x}_c$. Taking the direction associated to the x-coordinate as fixed direction one has

$$r(\vec{x}, \vec{s}) := \sqrt{(x - x_c(t))^2 + (y - y_c(t))^2} \qquad \theta(\vec{x}, \vec{s}) := \arctan \frac{y - y_c(t)}{x - x_c(t)} \tag{3}$$

$$\begin{aligned} \rho(\vec{x},t) &= \rho_r(r) \\ \vec{v}(\vec{x},t) &= \vec{v}_a + u_r(r)(-\sin\theta, \cos\theta) \\ p(\vec{x},t) &= p_r(r) \end{aligned} \tag{4}$$

For given functions ρ_r and u_r, the pressure p_r is solution of the ordinary differential equation

$$\frac{d}{dr} p_r = \rho_r(r) \frac{u_r(r)^2}{r} \tag{5}$$

with boundary conditions, e.g., at the center of the vortex. In principle ρ_r and u_r are arbitrary the only restriction beeing that (5) is integrable.

In practice it is convenient to consider cases in which ρ_r is either constant or a monotonically increasing function of r and u_r has a compact support. The first condition is required to avoid the solution undergoing Rayleigh-Taylor instability. The second condition allows one to define meaningful boundary conditions on a finite domain.

2.2. **Domain.** We consider exact and approximate solutions of (1) on the rectangular domain

$$\Omega := \{\vec{x} \in \mathbb{R}^2 : 0 \leq x \leq 4 \quad 0 \leq y \leq 1\}$$

2.3. **Exact solution, initial condition.** The exact solution consists of a vortex of radius $R = 0.4$. At time $\vec{s} = 0$ the vortex is located at the left end of Ω at

$$\vec{x}_{c,0} = (0.5, 0.5) \tag{6}$$

The advection velocity $\vec{v}_a$ is in the horizontal direction

$$\vec{v}_a = (u_a, 0)$$

and the tangential velocity $u_r(r)$ is defined as follows

$$u_r(r) = u_a \begin{cases} 2r/R & \text{if } 0 \leq r < R/2, \\ 2(1 - r/R) & \text{if } R/2 \leq r < R, \\ 0 & \text{if } R \leq r. \end{cases}$$

With the above equations and (2), (3), (4) the exact velocity field only depends on the parameter u_a and on the time t. The velocity field at $t = 0$ is the initial condition.

To complete the construction of the exact solution (and of the initial condition) the density and the pressure fields must be specified. Two problems are considered: one with constant density and one with variable density.

2.3.1. *Constant density case.* In this case $\rho_r(r) := \rho_0 := 1$ and the pressure is obtained by integrating (5) with boundary condition $p_r(R) := p_R := 1/\gamma$. This yields

$$p_r(r) = \begin{cases} p_0 + 2\rho_0 u_a^2 \frac{r^2}{R^2} & \text{if } 0 \le r < R/2, \\ p_1 + \rho_0 u_a^2 \left(2\frac{r^2}{R^2} - 8\frac{r}{R} + 4\log r\right) & \text{if } R/2 \le r < R, \\ p_R & \text{if } R \le r. \end{cases}$$

where the integration constants p_0 and p_1 are

$$p_0 = p_R + 2\rho_0 u_a^2(1 - 2\log 2) \qquad p_1 = p_R + 6\rho_0 u_a^2 - 4\rho_0 u_a^2 \log R$$

2.3.2. *Variable density case.* In this case

$$\rho_r(r) = \begin{cases} \rho_0 + (\rho_R - \rho_0)\frac{r^2}{R^2} & \text{if } 0 \le r < R, \\ \rho_R & \text{if } R \le r. \end{cases}$$

and (5) with boundary condition $p_r(R) := p_R := 1/\gamma$ yields

$$p_r(r) = \begin{cases} p_0 + 2\rho_0 u_a^2 \frac{r^2}{R^2} + (\rho_R - \rho_0)u_a^2 \frac{r^4}{R^4} & \text{if } R/2 \le r < R, \\ p_1 + 4\rho_0 u_a^2 \log r + (\rho_R - \rho_0)u_a^2 \dfrac{r^4}{R^4} & \\ \quad - \dfrac{8}{3}(\rho_R - \rho_0)u_a^2 \dfrac{r^3}{R^3} + 2\rho_R u_a^2 \dfrac{r^2}{R^2} & \text{if } R/2 \le r < R, \\ \quad - 8\rho_0 u_a^2 \dfrac{r}{R} & \\ p_R & \text{if } R \le r. \end{cases}$$

where the integration constants p_0 and p_1 are

$$p_0 = p_R - \frac{1}{6}(\rho_R - 13\rho_0)u_a^2 - 4\rho_0 u_a^2 \log 2$$
$$p_1 = p_R - \frac{1}{3}(\rho_R - 19\rho_0)u_a^2 - 4\rho_0 u_a^2 \log R.$$

With the specification of ρ_r, p_r and (2), (3), (4) the construction of the exact solution (and of the initial condition) are completed. Notice that $p_r = p_R = 1/\gamma$ for $r \ge R$. Thus the Mach number based on the advection velocity u_a and on the speed of sound outside the vortex (which is the square root of $\gamma p/\rho$) is simply equal to u_a and one can study the behaviour of numerical solutions of (1) at low Mach numbers by considering different values of the advection velocity u_a.

Numerical methods for (1) usually compute, at some discrete times $t^0, t^1, \dots, t^N$ numerical approximations $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^N$ to some function $\mathcal{F}$ of an exact solution $\mathcal{U}$ of (1). Finite volume methods, for instance, compute numerical approximations to sets of exact cell averages. These methods require mapping the initial condition constructed above into a set of cell averages i.e. one has to integrate the initial condition on a subset of Ω. This is usually done numerically. The issue is important (depending on the roughness of the grid and on the smoothness of the initial condition, different quadrature rules may yield quite different initial cell averages) and should be considered in the comparison of numerical results.

2.4. **Boundary conditions.** Periodic boundary conditions are imposed on the left and on the right sides of Ω ($0 \times [0, 1]$ and $4 \times [0, 1]$, respectively). On the bottom and on the top sides ($[0, 4] \times 0$ and $[0, 4] \times 1$, respectively) the vertical component of the velocity is required to be zero.

2.5. **Test cases.** Numerical solutions of (1) with the initial conditions and the boundary conditions described above should be computed for three values of the advection velocity (the Mach number): $u_a = 0.1$, $u_a = 0.01$ and $u_a = 0.001$.

The numerical integration is to be carried out until the center of the vortex $\vec{x}_c$ has reached the point $(3.5, 0)$ at the right end of the domain, that is, using (2), (6), until $t^N := 3/u_a$.

Each computation should be run on 3 different grids h_1, h_2 and h_3. For Cartesian grid methods these are equally spaced grids of 80×20 cells (81×21 nodes), 160×40 cells (161×41 nodes) and 320×80 cells (321×81 nodes), respectively. For methods based on unstructured grids h_1, h_2 and h_3 should be uniform with 1600, 6400 and 25600 degrees of freedom per unknown, respectively.

What one expects to see is that both the efficiency (measured, for instance, as the number of time steps required for the numerical integration from $t = 0$ to $t = t^N$) and / or the accuracy (measured, e.g., through some suitable norm of $\mathcal{F}^N - \mathcal{F}(\mathcal{U}(\vec{x}, t^N))$) of standard numerical methods for compressible flows degenerate as u_a is decreased. It would be interesting to see how adaptive grid methods and methods of order higher than 2 behave with respect to the efficiency and accuracy issues.

2.6. **Output.** For a given test case, i.e. a pair (u_a, h) for either the constant density or for the variable density cases, the following output is required:

1. Number of iterations needed for the integration over $[0, t^N]$, CPU time per iteration, CPU type.
2. Time history of the error norms
$$E_2(f) := \|f(\mathcal{F}^k) - f(\mathcal{F}(\mathcal{U}(\vec{x}, t^k)))\|_2 \; / \; \|f(\mathcal{F}(\mathcal{U}(\vec{x}, 0)))\|_2$$
where $\|f\|$ is the L_2-norm of f, $f := \{\rho,\ \rho\vec{v},\ \rho e,\ p,\ \vec{v},\ \omega\}$ and ω is the vorticity:
$$\|f\| := \left(\int_\Omega f^2 \, dx \, dy \right)^{\frac{1}{2}} \qquad \omega := \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$
3. Contour lines of ω (values at $u_a\{-3,\ -1,\ 1,\ 3,\ 5,\ 7\}$), minimum values of the density (constant density cases) and density contour lines (variable density cases, values at $0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$) at time 0, $t^N/3$, $t^N 2/3$ and t^N.

2.7. **Example.** Consider, as an example, a variable density case with $u_a := 0.1$ and on a Cartesian equally spaced grid of 80×20 cells.

With a second order explicit method at $\Delta t := t^{k+1} - t^k := 0.02, \quad k = 0, 1, \ldots, N-1$ it takes $N = 1500$ time steps to integrate over $[0, 30]$ with a CFL number of ≈ 0.83. One single iteration costs about 0.14 CPU seconds on a 400 MHz Pentium Pro.

The contour lines of ω at $\{-.3,\ -.1,\ .1,\ .3,\ .5,\ .7\}$ should look something like Figure 1.

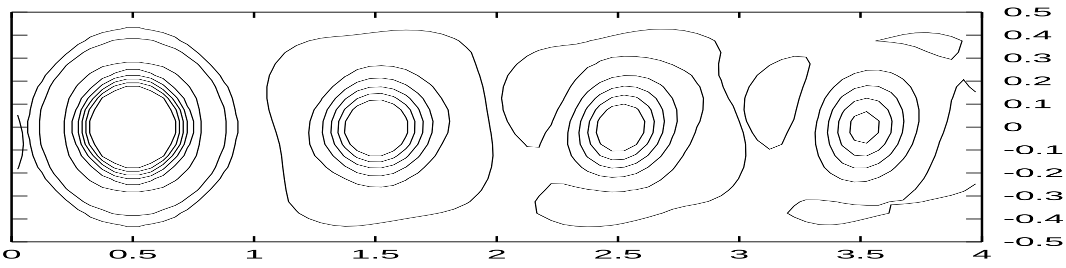

FIGURE 1. Advection of a vortex: vorticity contour lines.