A generalized Lagrangian scheme for hyperbolic balance laws

Application to compressible multifluid flow

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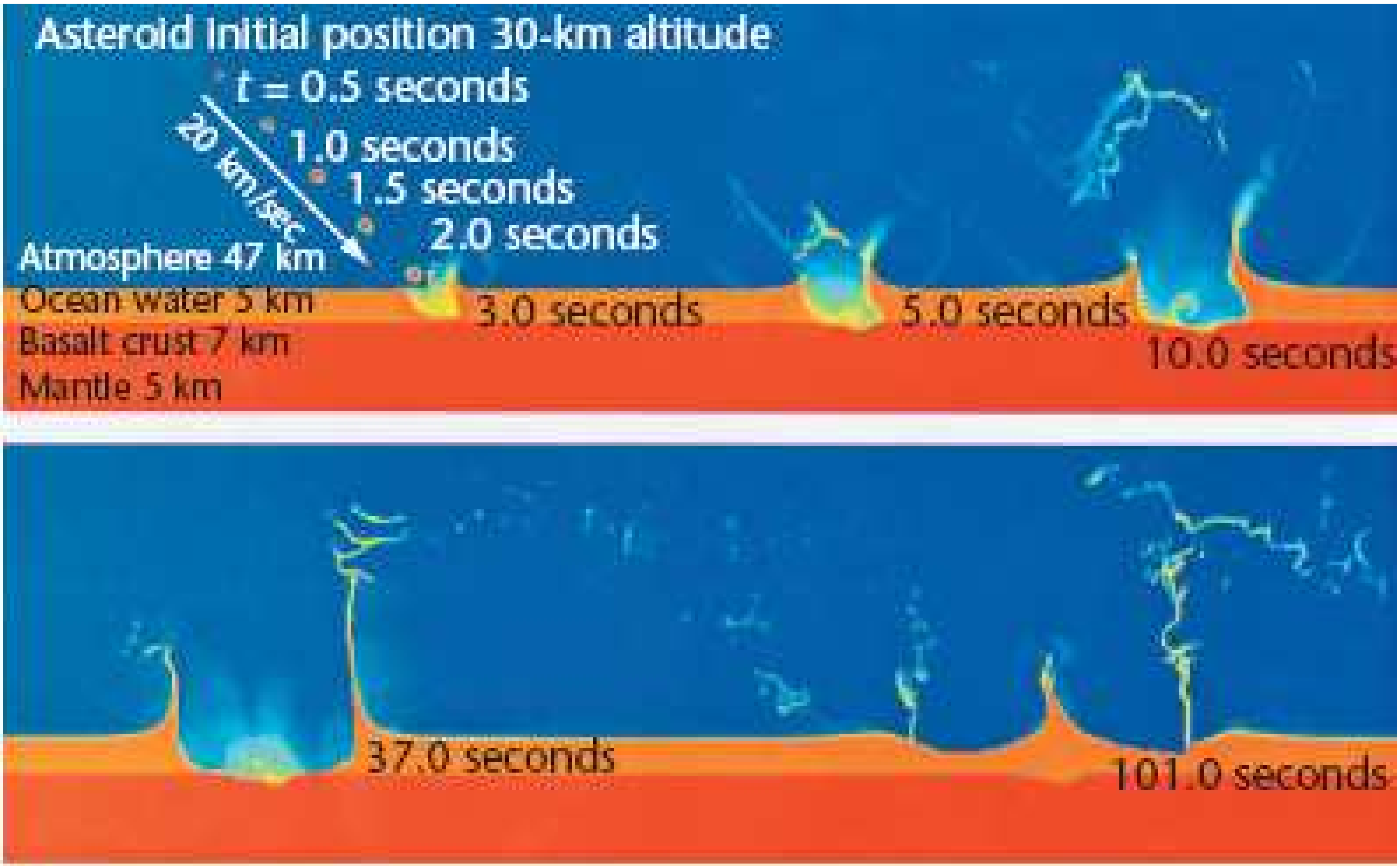
Outline

- Model scientific problem
- Previous work & current work motivation
- Mathematical formulation
  - Hyperbolic balance law in generalized coordinate
  - Grid movement conditions
  - Examples to shallow water/compressible flow model
- Numerical discretization
  - Generalized Riemann problem
  - Godunov-type flux-based wave propagation method
- Sample examples
- Future work

Model Scientific Problem

Asteroid impact problem (a geophysical example)



Fundamental Challenges

- Mathematical model aspect
  - Incompressible or compressible flow modelling
    - Equations of motion & constitutive laws
      - Gas phase: air
      - Liquid phase: ocean water
      - Solid phase: asteroid, basalt crust, mantle
  - Interface conditions
    - Mass transfer, cavitation, fracture, $\cdots$

- Numerical method aspect
  - Multiphase, multiscale, Eulerian or Lagrangian solver
  - Discretization based on structured, unstructured, or overlapping grid

Previous Work

- Fluid-mixture interface-capturing method (JCP 1998, 1999, 2001, 2004, Shock Waves 2006)
  - Shock over MORB (Mid-Ocean Ridge Basalt) liquid
  - Falling liquid drop problem
- Volume-of-fluid interface tracking method (JCP 2006)
  - Shock-bubble interaction
- Surface tracking for moving boundaries (Hyp 2006)
  - Falling rigid object in water tank
  - Moving cylindrical vessel
- Unified coordinate method (2006, with Hui & Hu)
  - Supersonic NACA0012 over heavier gas

Current Work

- Motivated by well-known facts that
  - Lagrangian method can resolve material or slip lines sharply if there is no grid tangling
  - Generalized curvilinear grid is often superior to Cartesian when employed in numerical methods for complex fixed or moving geometries

Current Work

- Motivated by well-known facts that
  - Lagrangian method can resolve material or slip lines sharply if there is no grid tangling
  - Generalized curvilinear grid is often superior to Cartesian when employed in numerical methods for complex fixed or moving geometries
- Aim is to devise Lagrange-like moving grid approach for nonlinear hyperbolic system of balance law

$$\frac{\partial}{\partial t} q\left(\vec{x},\ t\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial x_j} f_j\left(q,\ \vec{x}\right) = \psi\left(q,\ \vec{x}\right)$$

in general $N_d \geq 1$ space dimension that is more robust than aforementioned Eulerian-based method

Mathematical Formulation

Begin by considering canonical hyperbolic balance law

$$\frac{\partial}{\partial t} q\left(\vec{x},\ t\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial x_j} f_j\left(q,\ \vec{x}\right) = \psi\left(q,\ \vec{x}\right)$$

in Cartesian coordinate system

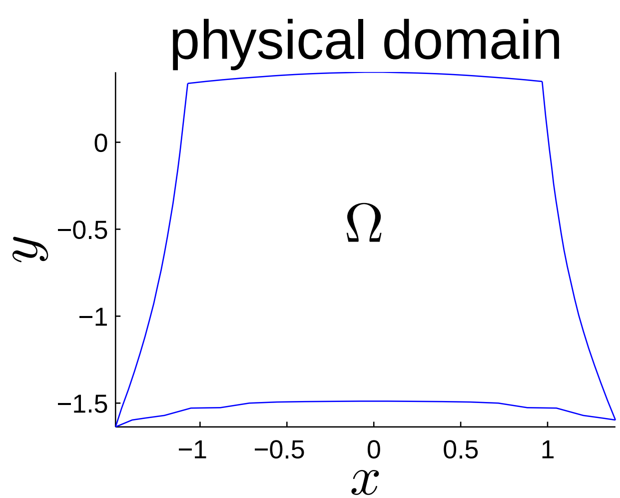
- Hyperbolic if $\sum_{j=1}^{N_d} \alpha_j \left(\partial f_j / \partial q\right)$ is diagonalizable with real eigenvalues, $\alpha_j \in \mathbb{R}$
- $q \in \mathbb{R}^m$: vector of m state quantities
- $f_j \in \mathbb{R}^m$: flux vector, $j = 1, 2, \cdots, N_d$, $\psi \in \mathbb{R}^m$: sources
- $\vec{x} = (x_1, x_2, \cdots, x_{N_d})$: spatial vector, $t \geq 0$: time

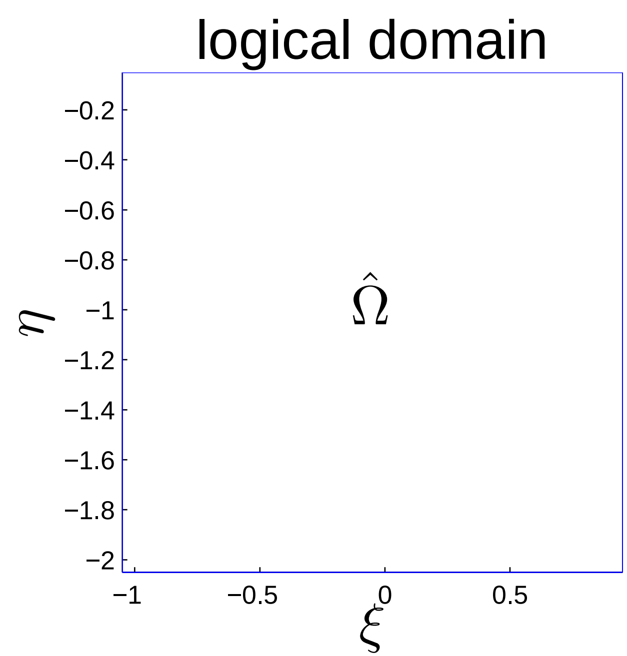
Mathematical Formulation

Now consider a general non-rectangular domain Ω in $N_d = 2$ & introduce coordinate change $(\vec{x},\ t) \mapsto (\vec{\xi},\ \tau)$ via

$$\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_{N_d}), \qquad \xi_j = \xi_j(\vec{x}, t), \qquad \tau = t,$$

that maps Ω to a logical domain $\hat{\Omega}$ & also transforms balance law to a new form





Mathematical Formulation

That is, using chain rule of partial differentiation, derivatives in physical space become

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \sum_{i=1}^{N_d} \frac{\partial \xi_i}{\partial t} \frac{\partial}{\partial \xi_i}, \qquad \frac{\partial}{\partial x_j} = \sum_{i=1}^{N_d} \frac{\partial \xi_i}{\partial x_j} \frac{\partial}{\partial \xi_i} \quad \text{for } j = 1, 2, \cdots, N_d,$$

yielding strong form of balance law in transformed space

$$\frac{\partial}{\partial \tau} \tilde{q}\left(\vec{\xi}, \tau\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial \xi_j} \tilde{f}_j \left(\tilde{q}, \nabla \vec{\xi}\right) = \tilde{\psi}\left(\tilde{q}, \nabla \vec{\xi}\right)$$

with

$$\tilde{q} = Jq, \quad \tilde{f}_j = J\left(q \frac{\partial \xi_j}{\partial t} + \sum_{k=1}^{N_d} f_k \frac{\partial \xi_j}{\partial x_k}\right) \quad \tilde{\psi} = J\psi, \quad J = \det\left(\partial \vec{\xi} / \partial \vec{x}\right)^{-1}$$

Mathematical Formulation

Assume existence of inverse transformation

$$t = \tau, \qquad x_j = x_j(\vec{\xi}, t) \qquad \text{for } j = 1, 2, \cdots, N_d,$$

To find basic geometric-metric relations between different coordinates, employ elementary differential rule

$$\frac{\partial(\tau, \vec{\xi})}{\partial(t, \vec{x})} = \frac{\partial(t, \vec{x})}{\partial(\tau, \vec{\xi})}^{-1},$$

yielding in $N_d = 3$ case, for example, as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial_t \xi_1 & \partial_{x_1} \xi_1 & \partial_{x_2} \xi_1 & \partial_{x_3} \xi_1 \\ \partial_t \xi_2 & \partial_{x_1} \xi_2 & \partial_{x_2} \xi_2 & \partial_{x_3} \xi_2 \\ \partial_t \xi_3 & \partial_{x_1} \xi_3 & \partial_{x_2} \xi_3 & \partial_{x_3} \xi_3 \end{pmatrix} = \frac{1}{J} \begin{pmatrix} J & 0 & 0 & 0 \\ J_{01} & J_{11} & J_{21} & J_{31} \\ J_{02} & J_{12} & J_{22} & J_{32} \\ J_{03} & J_{13} & J_{23} & J_{33} \end{pmatrix}$$

Mathematical Formulation

Here

$$J = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right| = \text{det}\left(\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right),$$

$$J_{11} = \left| \frac{\partial(x_2, x_3)}{\partial(\xi_2, \xi_3)} \right|, \quad J_{21} = \left| \frac{\partial(x_1, x_3)}{\partial(\xi_3, \xi_2)} \right|, \quad J_{31} = \left| \frac{\partial(x_1, x_2)}{\partial(\xi_2, \xi_3)} \right|,$$

$$J_{12} = \left| \frac{\partial(x_2, x_3)}{\partial(\xi_3, \xi_1)} \right|, \quad J_{22} = \left| \frac{\partial(x_1, x_3)}{\partial(\xi_1, \xi_3)} \right|, \quad J_{32} = \left| \frac{\partial(x_1, x_2)}{\partial(\xi_3, \xi_1)} \right|,$$

$$J_{13} = \left| \frac{\partial(x_2, x_3)}{\partial(\xi_1, \xi_2)} \right|, \quad J_{23} = \left| \frac{\partial(x_1, x_3)}{\partial(\xi_2, \xi_1)} \right|, \quad J_{33} = \left| \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} \right|,$$

$$J_{0j} = -\sum_{i=1}^{N_d} J_{ij} \partial_\tau x_i, \qquad j = 1, 2, 3,$$

and so numerically computatable $\nabla \xi_j, j = 1, 2, 3$, as

$$\nabla \xi_j = (\partial_t \xi_j,\ \nabla_{\vec{x}} \xi_j) = (\partial_t \xi_j,\ \partial_{x_1} \xi_j,\ \partial_{x_2} \xi_j,\ \partial_{x_3} \xi_j) = \frac{1}{J} (J_{0j},\ J_{1j},\ J_{2j},\ J_{3j})$$

Mathematical Formulation

Note that to complete the model

- When $\partial_\tau \vec{x} = 0$ (stationary case)
  - $\partial_t \vec{\xi} = 0$ & $\nabla_{\vec{x}} \vec{\xi}$ time-independent; no more condition

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- When $\partial_\tau \vec{x} = \vec{u}_0$, $\vec{u}_0$ is constant (quasi-stationary case)
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- When $\partial_\tau \vec{x} = \vec{u}_0$, $\vec{u}_0$ is constant (quasi-stationary case)
  - Both $\partial_t \vec{\xi}$ & $\nabla_{\vec{x}} \vec{\xi}$ time-independent; no more condition
- Otherwise $\partial_\tau \vec{x} \neq 0$ (genuine moving case)
  - Both $\partial_t \vec{\xi}$ & $\nabla_{\vec{x}} \vec{\xi}$ time-dependent; require N_d degree of freedom for $\partial_\tau \vec{x}$ & numerical approach to compute $\nabla_{\vec{\xi}} \vec{x}$ over time

Grid Movement Conditions

Here we are interested in

- Lagrange-like condition $\partial_\tau \vec{x} = h_0 \vec{u}$, $\vec{u}$ velocity, $h_0 \in [0, 1]$
- Compatibility conditions for $\partial_\tau \partial_{\xi_j} x_i$ & $\partial_{\xi_j} \partial_\tau x_i$, *i.e.*,

$$\frac{\partial}{\partial \tau}\left(\frac{\partial x_i}{\partial \xi_j}\right) + \frac{\partial}{\partial \xi_j}\left(-\frac{\partial x_i}{\partial \tau}\right) = 0 \qquad \text{for} \quad i, j = 1, 2, \cdots, N_d$$

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- General 1-parameter $\partial_\tau \vec{x} = h \vec{u}$ with $h \in [0,1]$ chosen by
  - J or grid-angle preserving, yielding

$$\mathcal{A}_0 h + \sum_{j=1}^{N_d} \mathcal{A}_j \partial_{\xi_j} h = 0 \quad (\text{1st order PDE constraint})$$

- General k-parameter, $k > 1$, $\cdots$

Shallow Water Equations

With bottom topography included, for example

- Cartesian coordinate case

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu_i \end{pmatrix} + \sum_{j=1}^{N_d} \frac{\partial}{\partial x_j}\begin{pmatrix} hu_j \\ hu_i u_j + \frac{1}{2} g h^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\partial B}{\partial x_i} \end{pmatrix}, \quad i = 1, \cdots, N_d$$

- Generalized coordinate case

$$\frac{\partial}{\partial \tau}\begin{pmatrix} hJ \\ hJu_i \end{pmatrix} + \sum_{j=1}^{N_d} \frac{\partial}{\partial \xi_j} J \begin{pmatrix} hU_j \\ hu_i U_j + \frac{1}{2} g h^2 \frac{\partial \xi_j}{\partial x_i} \end{pmatrix} = \begin{pmatrix} 0 \\ -ghJ\frac{\partial B}{\partial x_i} \end{pmatrix}$$

with $U_j = \partial_t \xi_j + \sum_{i=1}^{N_d} u_i \partial_{x_i} \xi_j, \quad j = 1, 2, \cdots, N_d$

h: water height, u_i: velocity in x_i-direction
B: bottom topography, g: gravitational constant

Shallow Water Equations

With $\partial_\tau \vec{x} = h_0 \vec{u}$ & grid-metric conditions, complete model system in $N_d = 2$ transformed space, for example, takes

$$\frac{\partial}{\partial \tau}\begin{pmatrix} hJ \\ hJu_1 \\ hJu_2 \\ A \\ B \\ C \\ D \end{pmatrix} + \frac{\partial}{\partial \xi_1}\begin{pmatrix} hJU_1 \\ hJu_1U_1 + \frac{1}{2}gJh^2\frac{\partial \xi_1}{\partial x_1} \\ hJu_2U_1 + \frac{1}{2}gJh^2\frac{\partial \xi_1}{\partial x_2} \\ -h_0u_1 \\ -h_0u_2 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \xi_2}\begin{pmatrix} hJU_2 \\ hJu_1U_2 + \frac{1}{2}gJh^2\frac{\partial \xi_2}{\partial x_1} \\ hJu_2U_2 + \frac{1}{2}gJh^2\frac{\partial \xi_2}{\partial x_2} \\ 0 \\ 0 \\ -h_0u_1 \\ -h_0u_2 \end{pmatrix} = \psi$$

Here $A = \partial_{\xi_1} x_1, \quad B = \partial_{\xi_1} x_2, \quad C = \partial_{\xi_2} x_1, \quad D = \partial_{\xi_2} x_2$

Remarks

- Hyperbolicity
  - In Cartesian coordinates, model is hyperbolic
  - In generalized coord., model is hyperbolic when $h_0 \neq 1$, & is weakly hyperbolic when $h_0 = 1$ & $N_d > 1$
- Canonical form
  - In Cartesian coordinates

$$\frac{\partial}{\partial t} q\left(\vec{x},\ t\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial x_j} f_j\left(q\right) = \psi\left(q\right)$$

  - In generalized coordinates: spatially varying fluxes

$$\frac{\partial}{\partial t} q\left(\vec{x},\ t\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial x_j} f_j\left(q,\ \nabla\vec{\xi}\right) = \psi\left(q,\ \nabla\vec{\xi}\right)$$

Compressible Euler Equations

With gravity effect included, for example

- Cartesian coordinate case

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u_i\\ E\end{pmatrix}+\sum_{j=1}^{N_d}\frac{\partial}{\partial x_j}\begin{pmatrix}\rho u_j\\ \rho u_i u_j + p\delta_{ij}\\ E u_j + p u_j\end{pmatrix}=\begin{pmatrix}0\\ -\rho\frac{\partial\phi}{\partial x_i}\\ -\rho\vec{u}\cdot\nabla\phi\end{pmatrix}, \quad i=1,\cdots,N_d$$

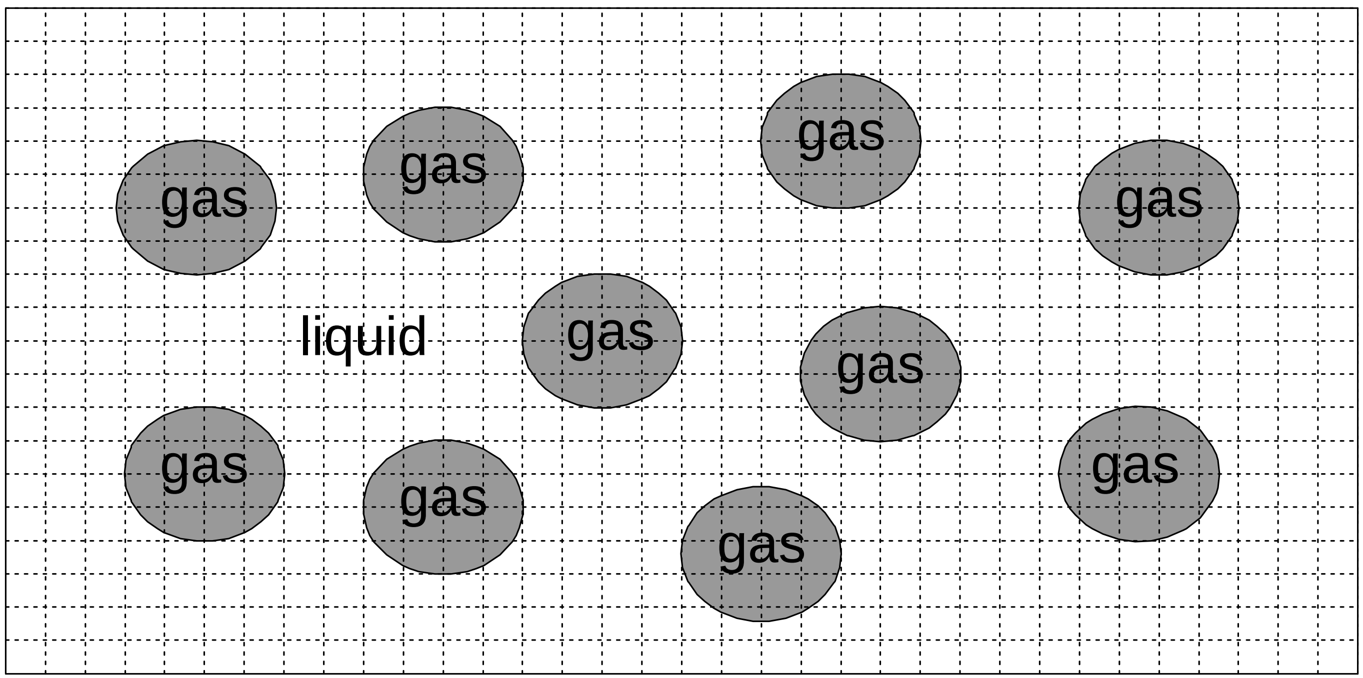
- Generalized coordinate case

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\rho J\\ \rho J u_i\\ JE\end{pmatrix}+\sum_{j=1}^{N_d}\frac{\partial}{\partial\xi_j}J\begin{pmatrix}\rho U_j\\ \rho u_i U_j + p\frac{\partial\xi_j}{\partial x_i}\\ EU_j + pU_j - p\frac{\partial\xi_j}{\partial t}\end{pmatrix}=\begin{pmatrix}0\\ -\rho J\frac{\partial\phi}{\partial x_i}\\ -\rho J\vec{u}\cdot\nabla\phi\end{pmatrix}$$

ρ: density, $p = p(\rho, e)$: pressure , e: internal energy
$E = \rho e + \rho\sum_{j=1}^{N_d} u_j^2/2$: total energy, ϕ: gravitational potential

Extension to Multifluid

- Assume homogeneous (1-pressure & 1-velocity) flow; *i.e.*, across interfaces $p_\iota = p$ & $\vec{u}_\iota = \vec{u}$, $\forall$ fluid phase ι



Extension to Multifluid

- Mathematical model: Fluid-mixture type
  - Use basic conservation (or balance) laws for single & multicomponent fluid mixtures
  - Introduce additional transport equations for problem-dependent material quantities near numerically diffused interfaces, yielding direct computation of pressure from EOS
- Model barotropic 2-phase flow problem with
  - fluid component 1 & 2 characterized by Tait EOS

$$p(\rho) = (p_{0\iota} + \mathcal{B}_\iota)\left(\frac{\rho}{\rho_{0\iota}}\right)^{\gamma_\iota} - \mathcal{B}_\iota, \quad \iota = 1, 2$$

Barotropic 2-Phase Flow

Define mixture pressure law (Shyue, JCP 2004)

$$p(\rho, \rho e) = \begin{cases} (p_{0\iota} + \mathcal{B}_\iota)\left(\dfrac{\rho}{\rho_{0\iota}}\right)^{\gamma_\iota} - \mathcal{B}_\iota & \text{if} \quad \alpha = 0 \text{ or } 1 \\ (\gamma - 1)\left(\rho e + \dfrac{\rho \mathcal{B}}{\rho_0}\right) - \gamma \mathcal{B} & \text{if} \quad \alpha \in (0,1) \end{cases}$$

Derived from $de = -p d(1/\rho)$ using

$$p(\rho, S) = \mathcal{A}(S)\,(p_0 + \mathcal{B})\left(\frac{\rho}{\rho_0}\right)^{\gamma} - \mathcal{B}$$

Here $\mathcal{A}(S) = e^{[(S-S_0)/C_V]}$, S, C_V: specific entropy & heat at constant vol. α: volume fraction of one fluid component

Barotropic 2-Phase Flow

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+\sum_{j=1}^{N_d} U_j \frac{\partial}{\partial \xi_j}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+\sum_{j=1}^{N_d} U_j \frac{\partial}{\partial \xi_j}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\sum_{j=1}^{N_d} \frac{\partial}{\partial \xi_j}\left(J\frac{\mathcal{B}}{\rho_0}\rho U_j\right)=0$$

  - Above equations are derived from energy equation & make use of homogeneous equilibrium flow assumption together with mass conservation law

Barotropic 2-Phase Flow

- α-based equations

$$\frac{\partial \alpha}{\partial \tau} + \sum_{j=1}^{N_d} U_j \frac{\partial \alpha}{\partial \xi} = 0, \qquad \text{with} \quad z = \sum_{\iota=1}^{2} \alpha_\iota z_\iota, \quad z = \frac{1}{\gamma - 1} \& \frac{\gamma \mathcal{B}}{\gamma - 1}$$

$$\frac{\partial}{\partial \tau}\left(J \frac{\mathcal{B}}{\rho_0} \rho\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial \xi_j}\left(J \frac{\mathcal{B}}{\rho_0} \rho U_j\right) = 0$$

- α-based equations (Allaire *et al.* , JCP 2002)

$$\frac{\partial \alpha}{\partial \tau} + \sum_{j=1}^{N_d} U_j \frac{\partial \alpha}{\partial \xi_j} = 0 \qquad \text{with} \quad z = \sum_{\iota=1}^{2} \alpha_\iota z_\iota \, , \quad z = \frac{1}{\gamma - 1} \& \frac{\gamma \mathcal{B}}{\gamma - 1}$$

$$\frac{\partial}{\partial \tau}\left(J \rho_1 \alpha\right) + \sum_{j=1}^{N_d} \frac{\partial}{\partial \xi_j}\left(J \rho_1 \alpha U_j\right) = 0$$

Multifluid Model

With $(x_\tau, y_\tau) = h_0(u, v)$ & sample EOS described above, our α-based model for multifluid flow is

$$\frac{\partial}{\partial \tau}\begin{pmatrix} J\rho \\ J\rho u \\ J\rho v \\ JE \\ x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \\ J\rho_1\alpha \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} J\rho U \\ J\rho u U + y_\eta p \\ J\rho v U - x_\eta p \\ JEU + (y_\eta u - x_\eta v)p \\ -h_0 u \\ -h_0 v \\ 0 \\ 0 \\ J\rho_1 \alpha U \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} J\rho V \\ J\rho u V - y_\xi p \\ J\rho v V + x_\xi p \\ JEV + (x_\xi v - y_\xi u)p \\ 0 \\ 0 \\ -h_0 u \\ -h_0 v \\ J\rho_1 \alpha V \end{pmatrix} = \tilde{\psi}$$

$$\frac{\partial \alpha}{\partial \tau} + U\frac{\partial \alpha}{\partial \xi} + V\frac{\partial \alpha}{\partial \eta} = 0,$$

plus α-averaged material quantities

Multifluid Model

For convenience, our multifluid model is written into

$$\frac{\partial q}{\partial \tau} + f_1\left(\frac{\partial}{\partial \xi}, q, \nabla\vec{\xi}\right) + f_2\left(\frac{\partial}{\partial \eta}, q, \nabla\vec{\xi}\right) = \tilde{\psi}$$

with

$$q = [J\rho, J\rho u, J\rho v, JE, x_\xi, y_\xi, x_\eta, y_\eta, J\rho_1\alpha, \alpha]^T$$

$$f_1 = \left[\frac{\partial}{\partial \xi}(J\rho U), \frac{\partial}{\partial \xi}(J\rho u U + y_\eta p), \frac{\partial}{\partial \xi}(J\rho v U - x_\eta p), \frac{\partial}{\partial \xi}(JEU + (y_\eta u - x_\eta v)p),\right.$$
$$\left.\frac{\partial}{\partial \xi}(-h_0 u), \frac{\partial}{\partial \xi}(-h_0 v), 0, 0, \frac{\partial}{\partial \xi}(J\rho_1\alpha U), U\frac{\partial \alpha}{\partial \xi}\right]^T$$

$$f_2 = \left[\frac{\partial}{\partial \eta}(J\rho V), \frac{\partial}{\partial \eta}(J\rho u V - y_\xi p), \frac{\partial}{\partial \eta}(J\rho v V + x_\xi p), \frac{\partial}{\partial \eta}(JEV + (x_\xi v - y_\xi u)p),\right.$$
$$\left.0, 0, \frac{\partial}{\partial \eta}(-h_0 u), \frac{\partial}{\partial \eta}(-h_0 v), \frac{\partial}{\partial \eta}(J\rho_1\alpha V), V\frac{\partial \alpha}{\partial \eta}\right]^T$$

Multifluid model: Remarks

- As before, under thermodyn. stability condition, our multifluid model in generalized coordinates is hyperbolic when $h_0 \neq 1$, & is weakly hyperbolic when $h_0 = 1$
- Our model system is written in quasi-conservative form with spatially varying fluxes in generalized coordinates
- Our grid system is a time-varying grid
- Extension of the model to general non-barotropic multifluid flow can be made in an analogous manner

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Numerical discretization?

Numerical Discretization

- In 2D, equations to be solved takes the form

$$\frac{\partial q}{\partial \tau} + f_1\left(\frac{\partial}{\partial \xi}, q, \nabla\vec{\xi}\right) + f_2\left(\frac{\partial}{\partial \eta}, q, \nabla\vec{\xi}\right) = \tilde{\psi}$$

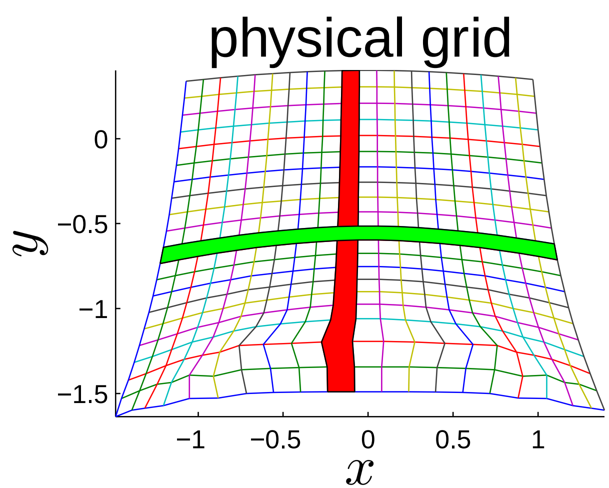
- A simple dimensional-splitting approach based on f-wave formulation of LeVeque *et al.* is used
  - Solve one-dimensional generalized Riemann problem (defined below) at each cell interfaces
  - Use resulting jumps of fluxes (decomposed into each wave family) of Riemann solution to update cell averages
  - Introduce limited jumps of fluxes to achieve high resolution

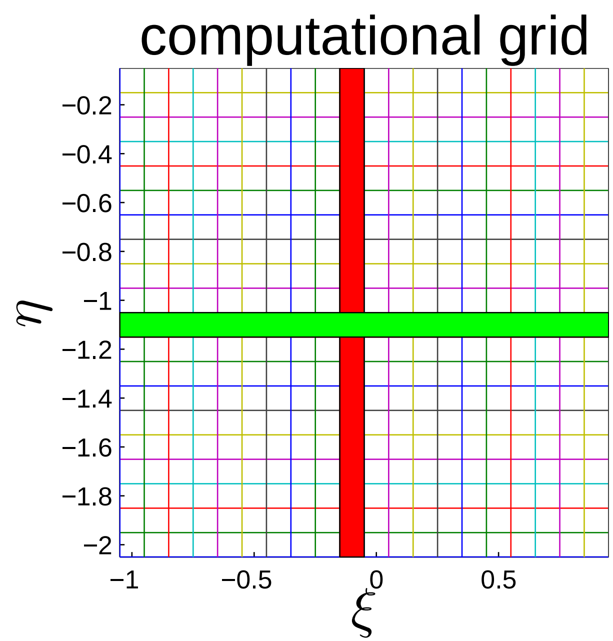
Numerical Discretization

Employ finite volume formulation of numerical solution

$$Q^n_{ij} \approx \frac{1}{\Delta\xi\Delta\eta}\int_{C_{ij}} q(\xi,\eta,\tau_n)\,dA$$

that gives approximate value of cell average of solution q over cell $C_{ij} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$ at time τ_n





Generalized Riemann Problem

Generalized Riemann problem of our multifluid model at cell interface $\xi_{i-1/2}$ consists of the equation

$$\frac{\partial q}{\partial \tau} + F_{i-\frac{1}{2},j}\left(\partial_\xi, q, \nabla\vec{\xi}\right) = 0$$

together with flux function

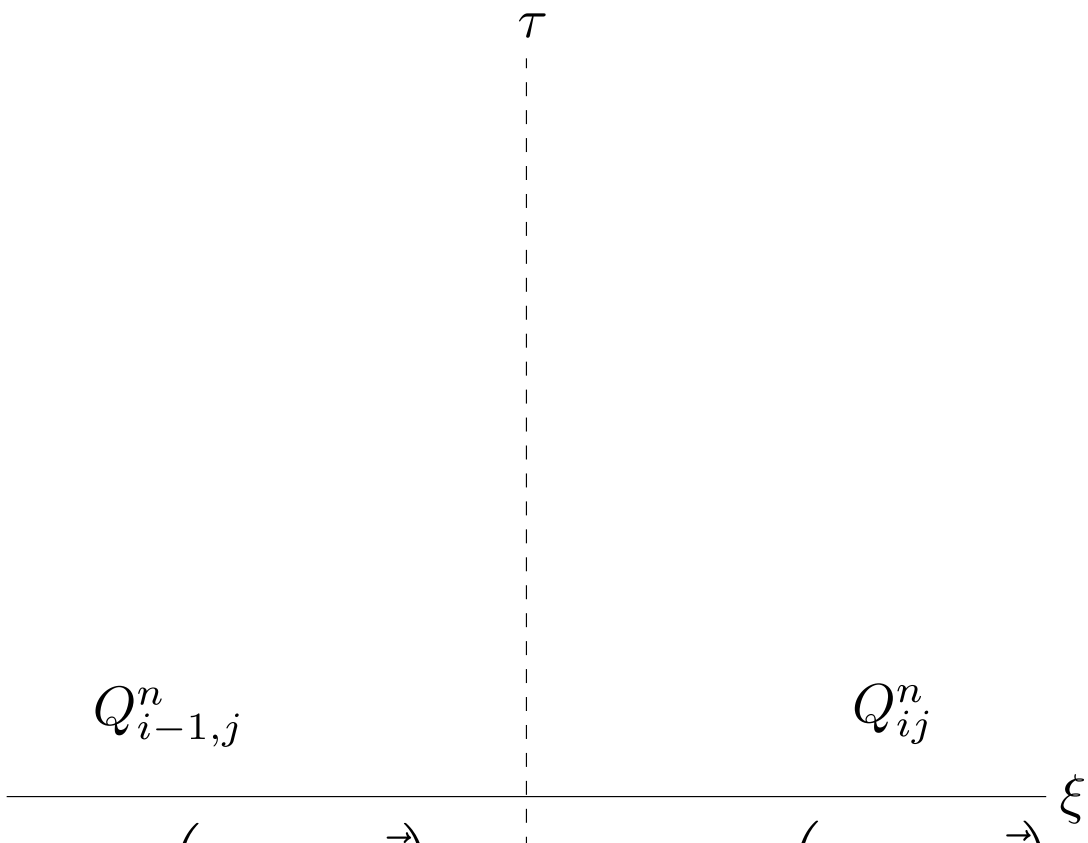
$$F_{i-\frac{1}{2},j} = \begin{cases} f_{i-1,j}\left(\partial_\xi, q, \nabla\vec{\xi}\right) & \text{for} \quad \xi < \xi_{i-1/2} \\ f_{ij}\left(\partial_\xi, q, \nabla\vec{\xi}\right) & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

and piecewise constant initial data

$$q(\xi, 0) = \begin{cases} Q^n_{i-1,j} & \text{for} \quad \xi < \xi_{i-1/2} \\ Q^n_{ij} & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

Generalized Riemann Problem

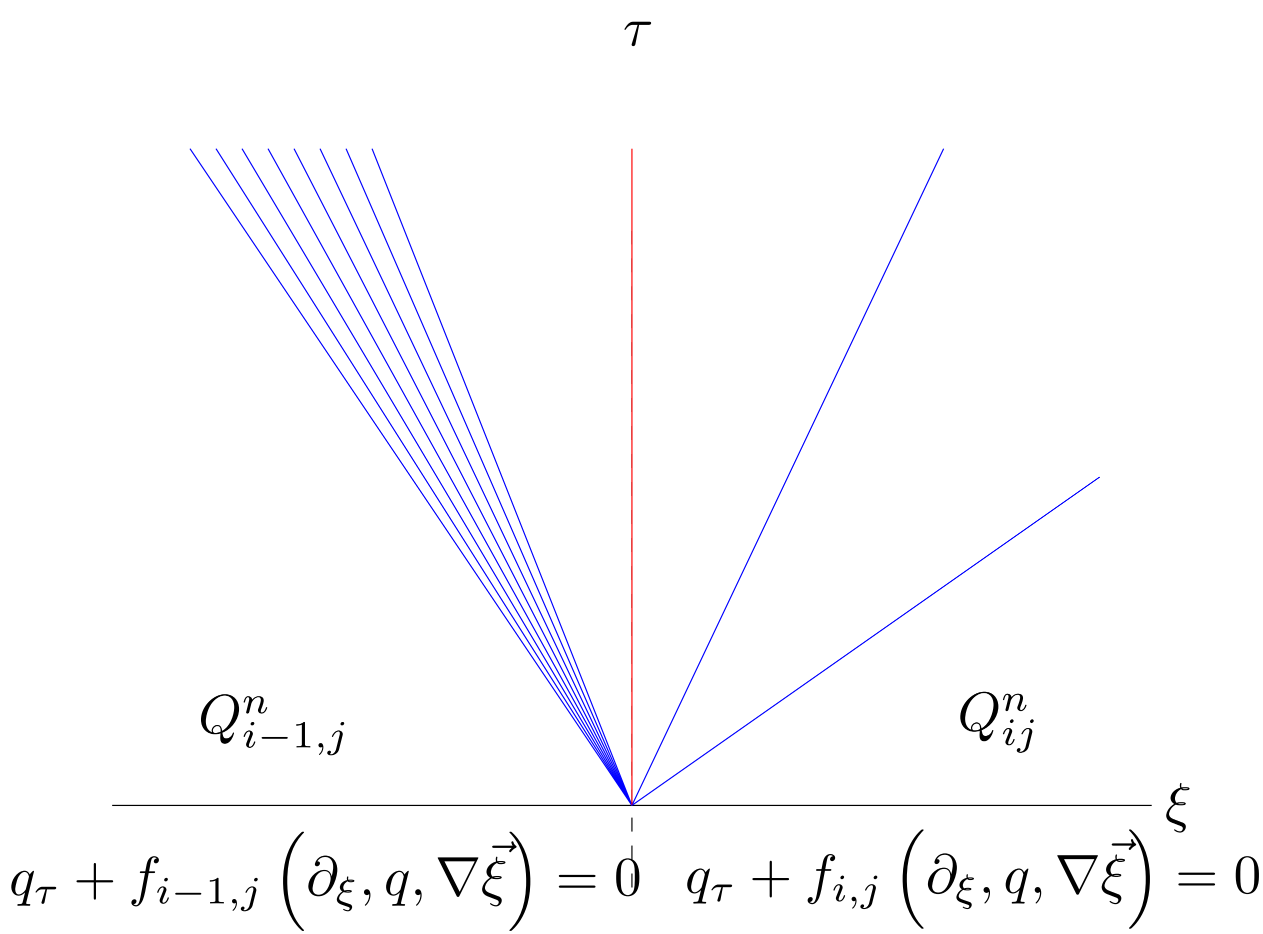
Generalized Riemann problem at time $\tau = 0$



$$q_\tau + f_{i-1,j}\left(\partial_\xi, q, \nabla\vec{\xi}\right) = 0 \qquad q_\tau + f_{i,j}\left(\partial_\xi, q, \nabla\vec{\xi}\right) = 0$$

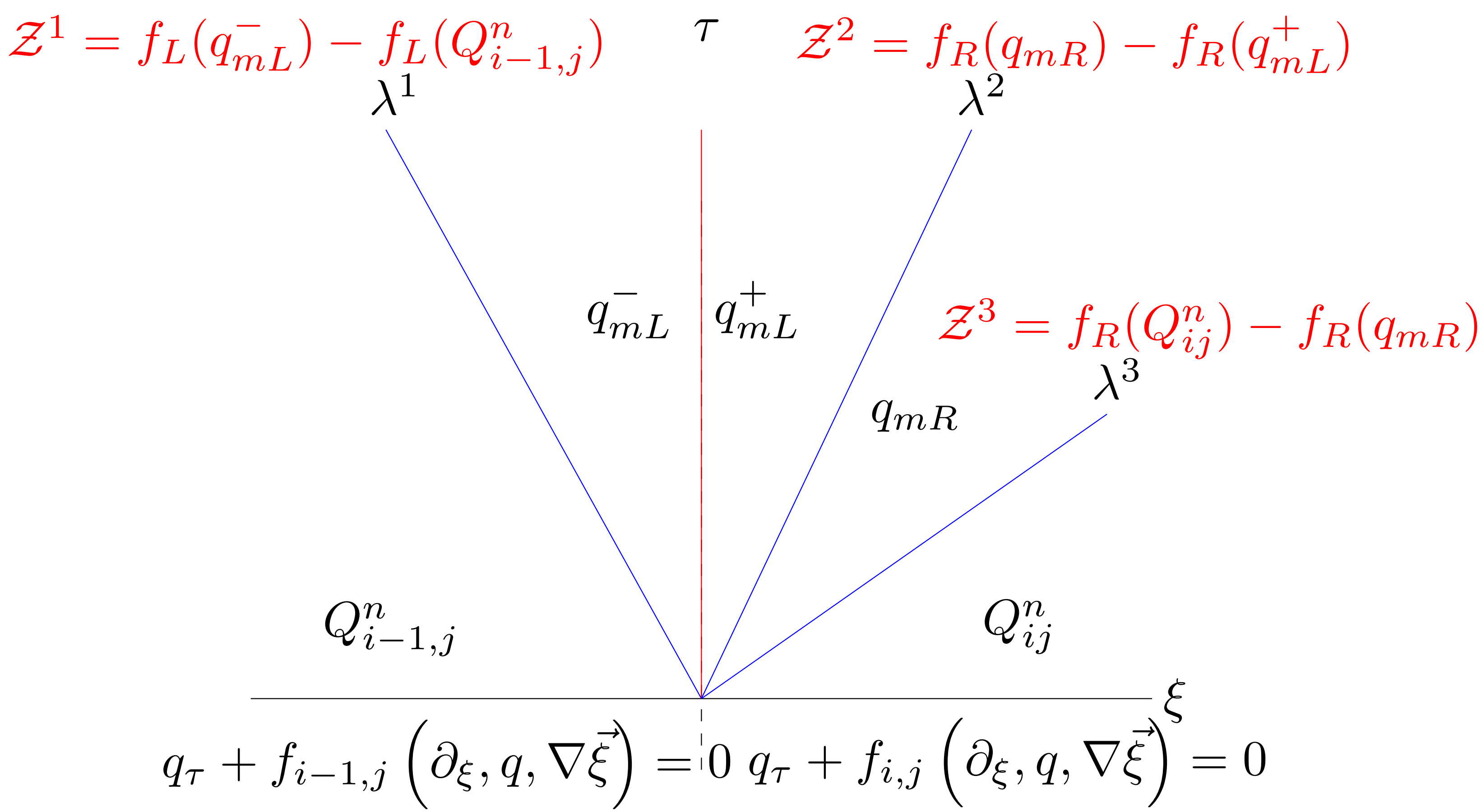
Generalized Riemann Problem

Exact generalized Riemann solution: basic structure



Generalized Riemann Problem

Shock-only approximate Riemann solution: basic structure



Numerical Discretization

Basic steps of a dimensional-splitting scheme

- ξ-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f_1\left(\frac{\partial}{\partial \xi}, q, \nabla\vec{\xi}\right) = 0$$

updating Q_{ij}^n to $Q_{i,j}^*$

- η-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f_2\left(\frac{\partial}{\partial \eta}, q, \nabla\vec{\xi}\right) = 0$$

updating Q_{ij}^* to $Q_{i,j}^{n+1}$

Numerical Discretization

That is to say,

- ξ-sweeps: we use

$$Q_{ij}^{*} = Q_{ij}^{n} - \frac{\Delta\tau}{\Delta\xi}\left(\mathcal{F}_{i+\frac{1}{2},j}^{-} - \mathcal{F}_{i-\frac{1}{2},j}^{+}\right) - \frac{\Delta\tau}{\Delta\xi}\left(\tilde{\mathcal{Z}}_{i+\frac{1}{2},j} - \tilde{\mathcal{Z}}_{i-\frac{1}{2},j}\right)$$

with $\displaystyle \tilde{\mathcal{Z}}_{i-\frac{1}{2},j} = \frac{1}{2}\sum_{p=1}^{m_w} \text{sign}\left(\lambda_{i-\frac{1}{2},j}^{p}\right)\left(1 - \frac{\Delta\tau}{\Delta\xi}\left|\lambda_{i-\frac{1}{2},j}^{p}\right|\right)\tilde{\mathcal{Z}}_{i-\frac{1}{2},j}^{p}$

- η-sweeps: we use

$$Q_{ij}^{n+1} = Q_{ij}^{*} - \frac{\Delta\tau}{\Delta\eta}\left(\mathcal{G}_{i,j+\frac{1}{2}}^{-} - \mathcal{G}_{i,j-\frac{1}{2}}^{+}\right) - \frac{\Delta\tau}{\Delta\eta}\left(\tilde{\mathcal{Z}}_{i,j+\frac{1}{2}} - \tilde{\mathcal{Z}}_{i,j-\frac{1}{2}}\right)$$

with $\displaystyle \tilde{\mathcal{Z}}_{i,j-\frac{1}{2}} = \frac{1}{2}\sum_{p=1}^{m_w} \text{sign}\left(\lambda_{i,j-\frac{1}{2}}^{p}\right)\left(1 - \frac{\Delta\tau}{\Delta\eta}\left|\lambda_{i,j-\frac{1}{2}}^{p}\right|\right)\tilde{\mathcal{Z}}_{i,j-\frac{1}{2}}^{p}$

Numerical Discretization

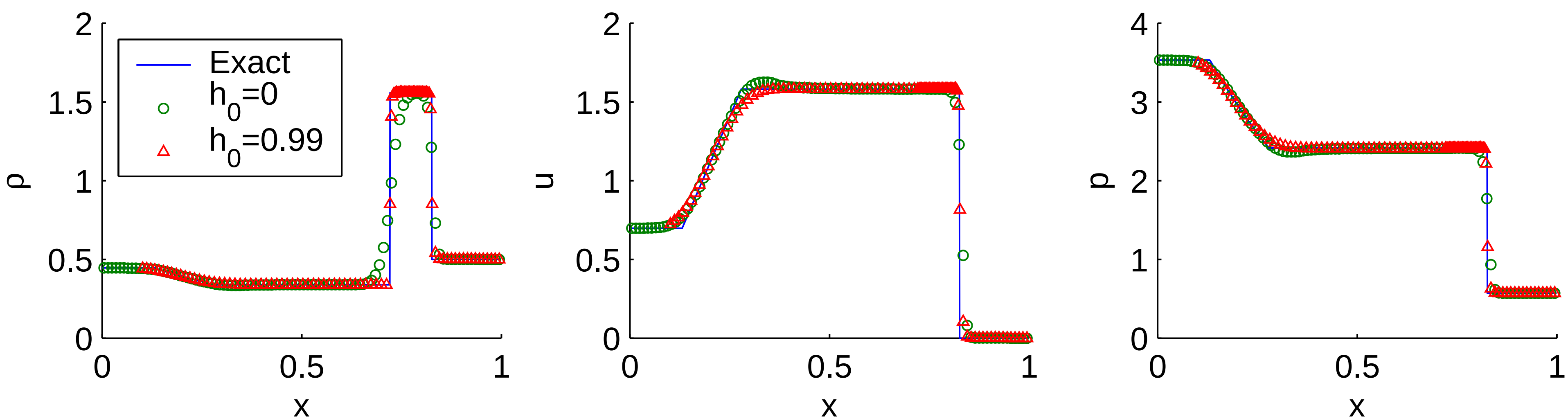
- Flux-based wave decomposition

$$f_{i,j} - f_{i-1,j} = \sum_{p=1}^{m_w} \mathcal{Z}^p_{i-1/2} = \sum_{p=1}^{m_w} \lambda^p_{i-1/2} \mathcal{W}^p_{i-1/2}$$

- Some care should be taken on the limited jump of fluxes $\tilde{\mathcal{W}}^p$, for $p = 2$ (contact wave), in particular to ensure correct pressure equilibrium across material interfaces
- MUSCL-type (slope limited) high resolution extension is not simple as one might think of for multifluid problems
- Splitting of discontinuous fluxes at cell interfaces: significance ?
- First order or high resolution method for geometric conservation laws: significance to grid uniformity ?

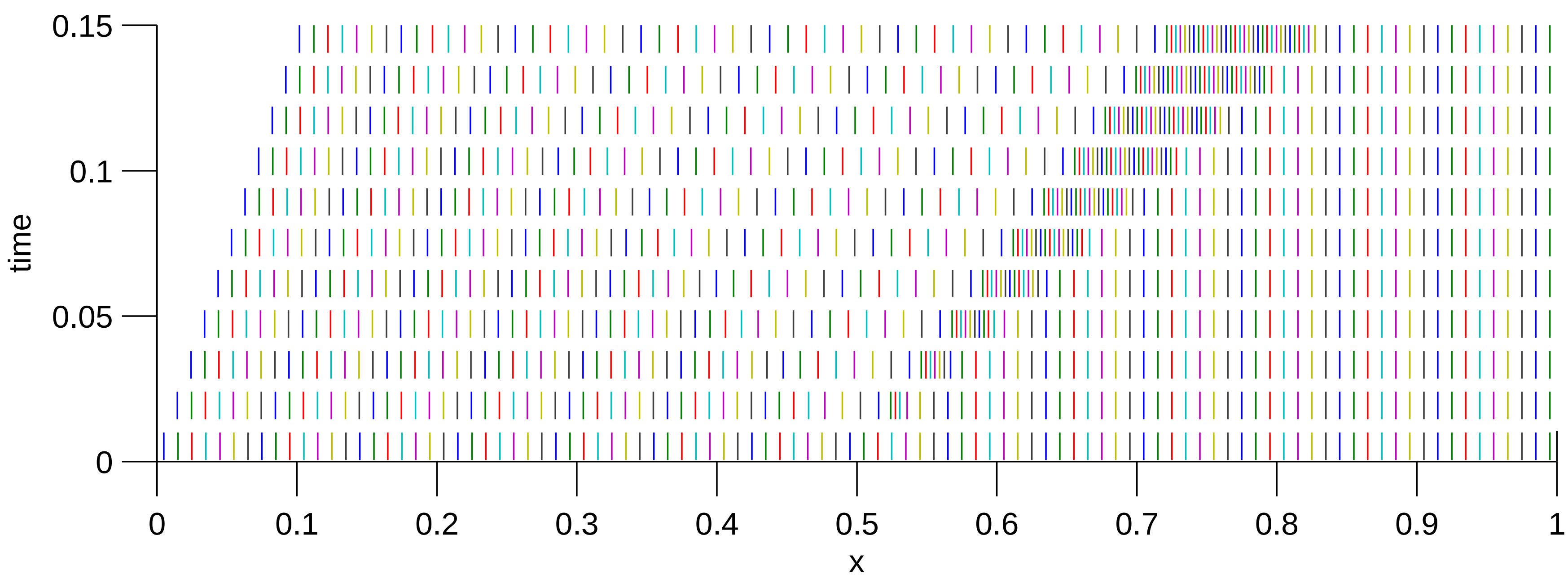
Lax's Riemann problem

- $h_0 = 0$ Eulerian result
- $h_0 = 0.99$ Lagrangian-like result
  - sharper resolution for contact discontinuity



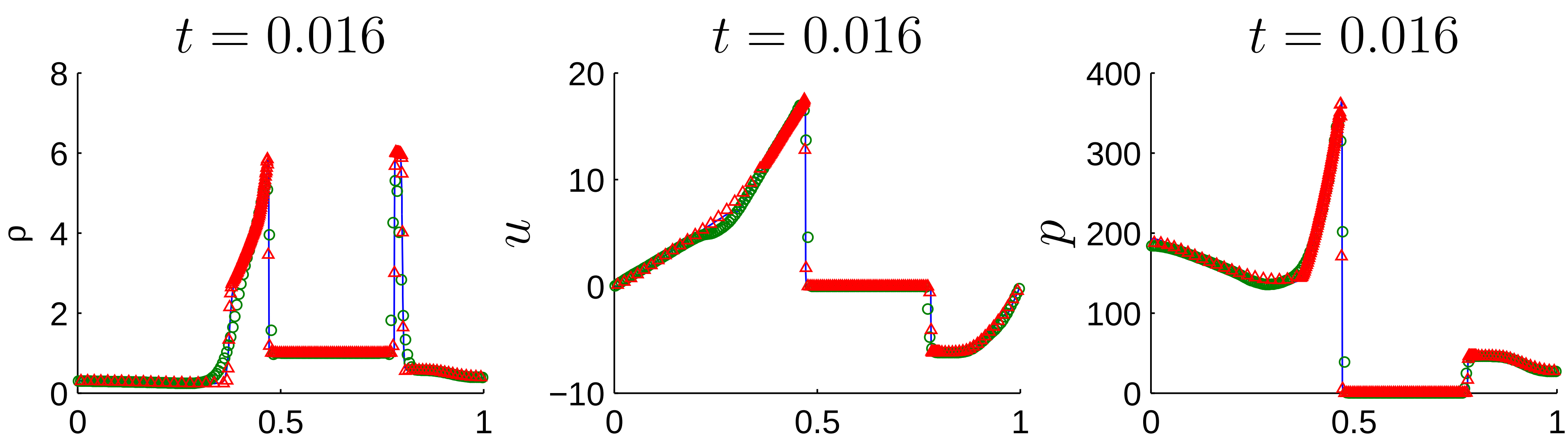
Lax's Riemann problem

- Physical grid coordinates at selected times
  - Each little dashed line gives a cell-center location of the proposed Lagrange-like grid system



Woodward-Colella's problem

- $h_0 = 0$ Eulerian result
- $h_0 = 0.99$ Lagrangian-like result
  - sharper resolution for contact discontinuity



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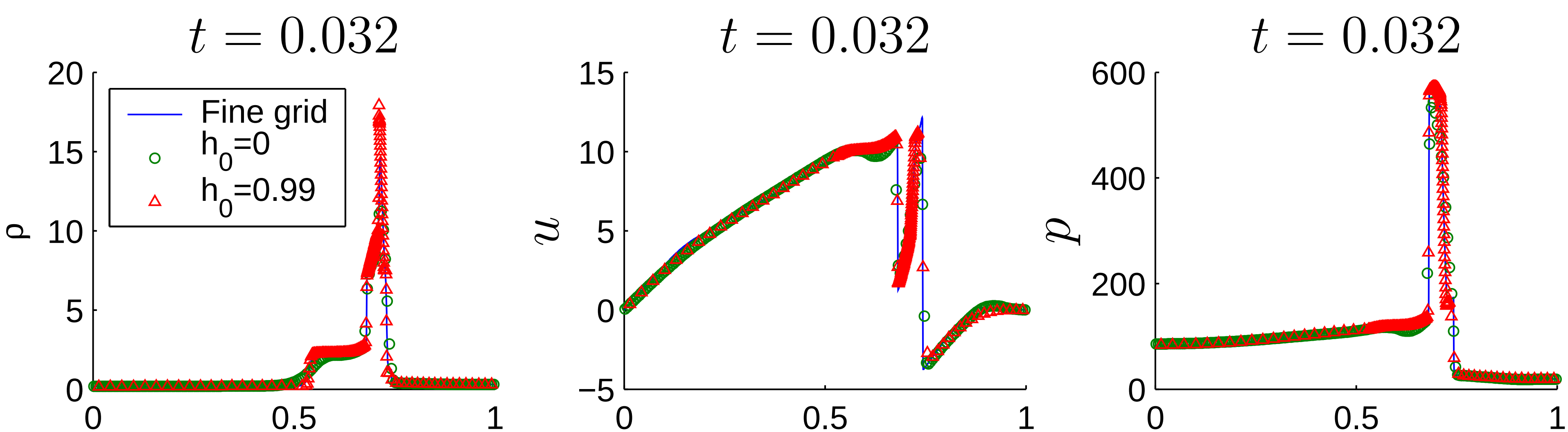

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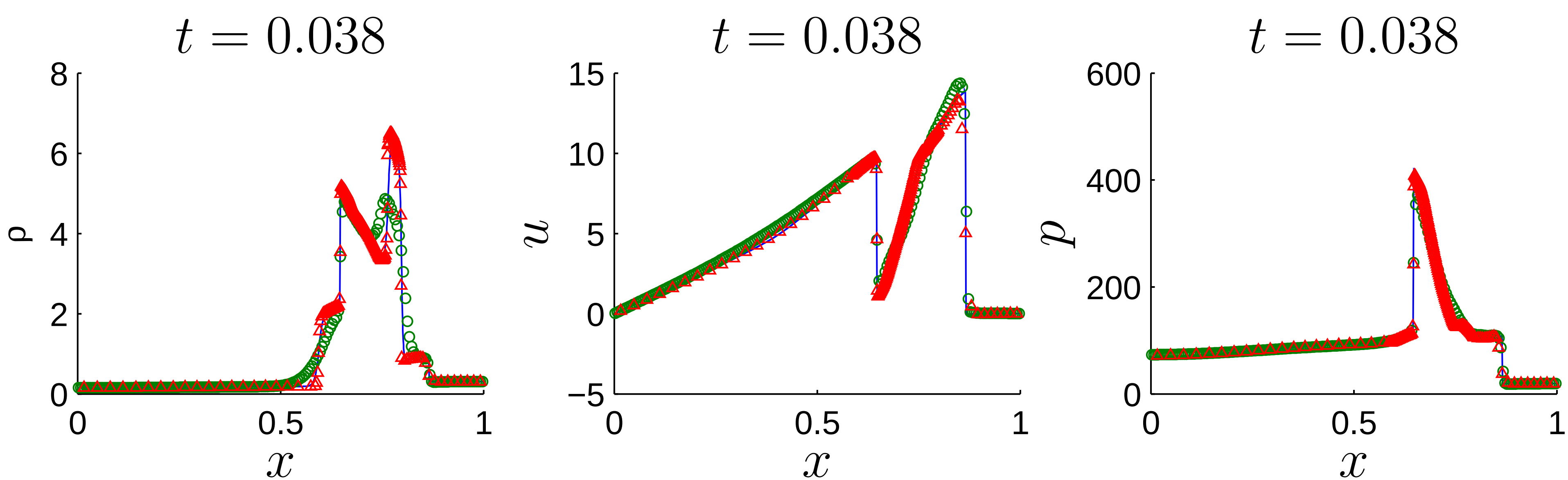

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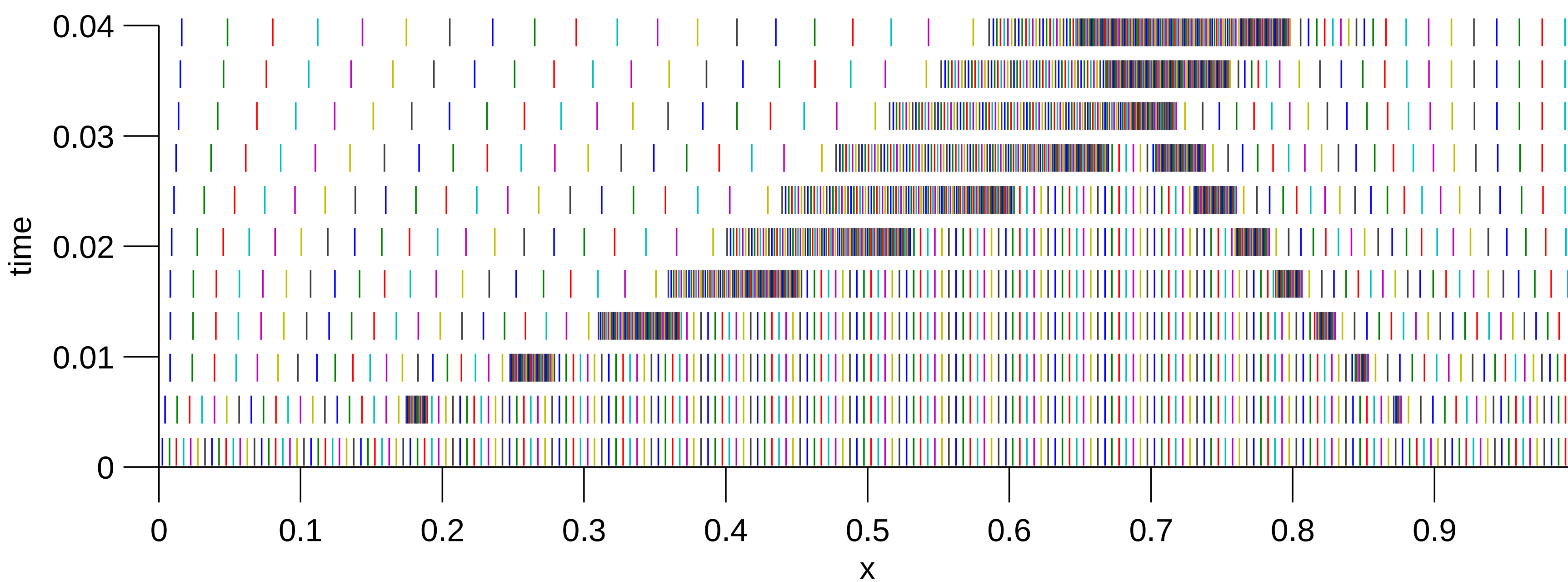
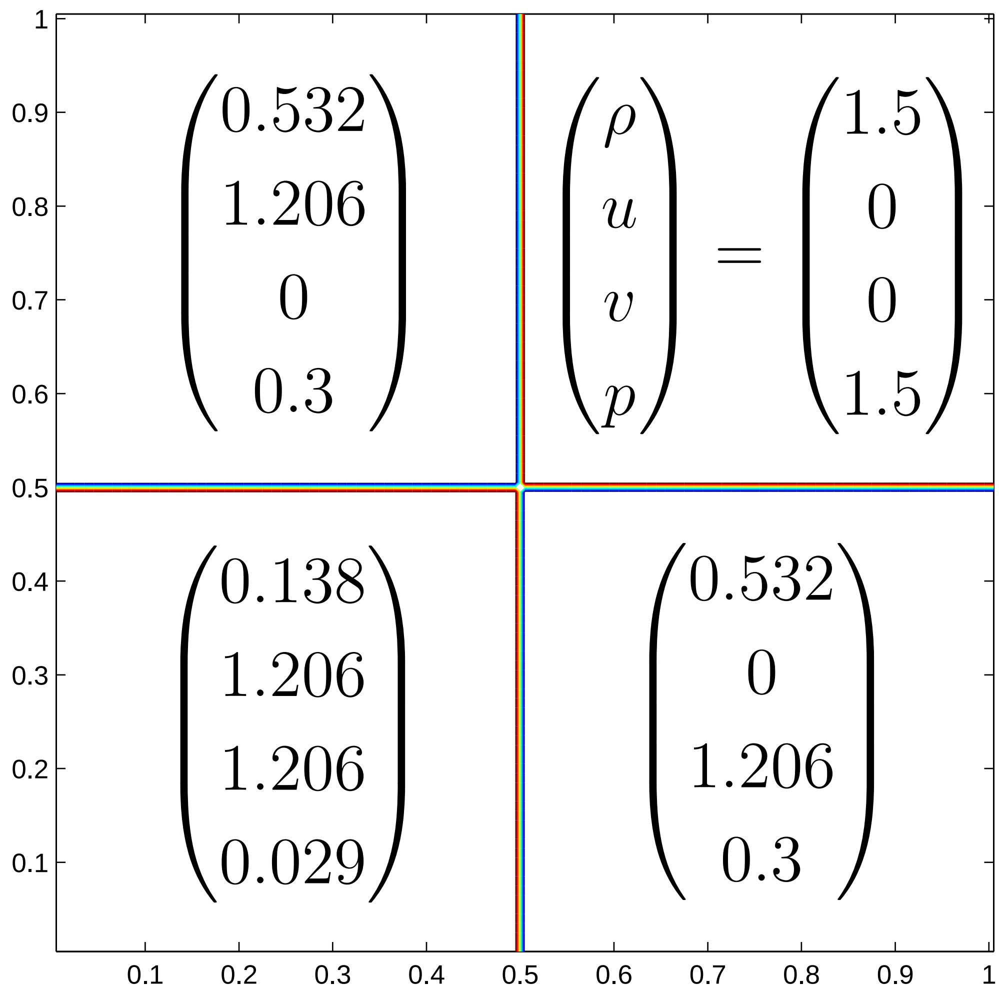


2D Riemann problem

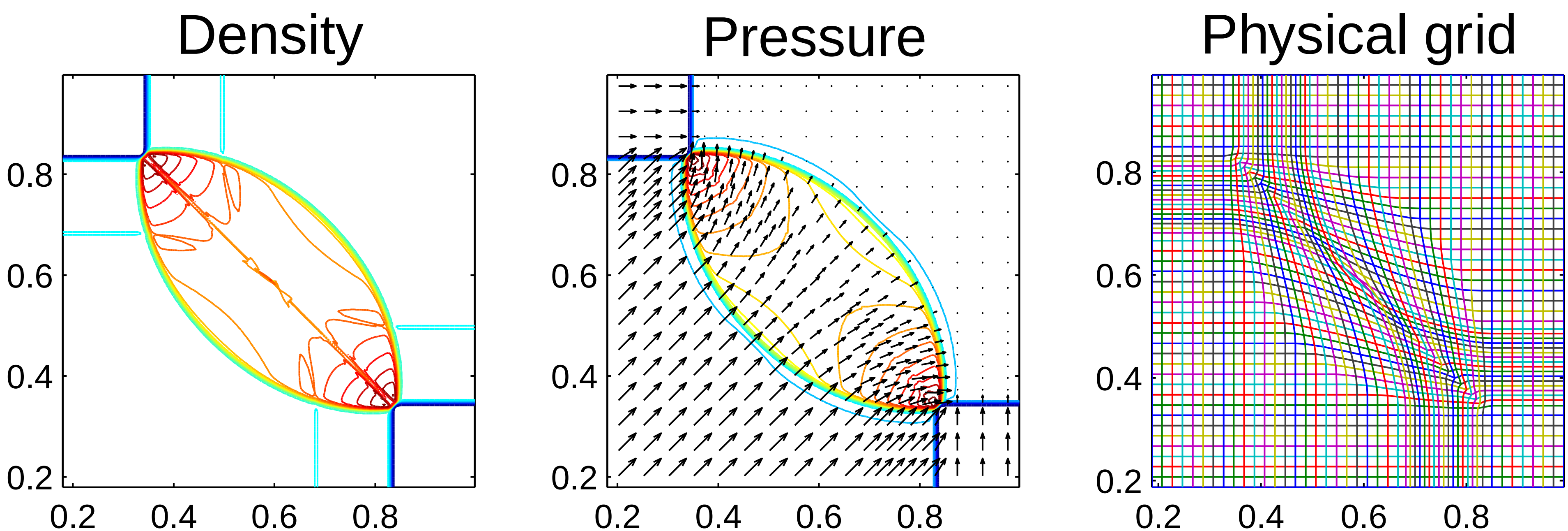
With initial 4-shock wave pattern



2D Riemann problem

With initial 4-shock wave pattern

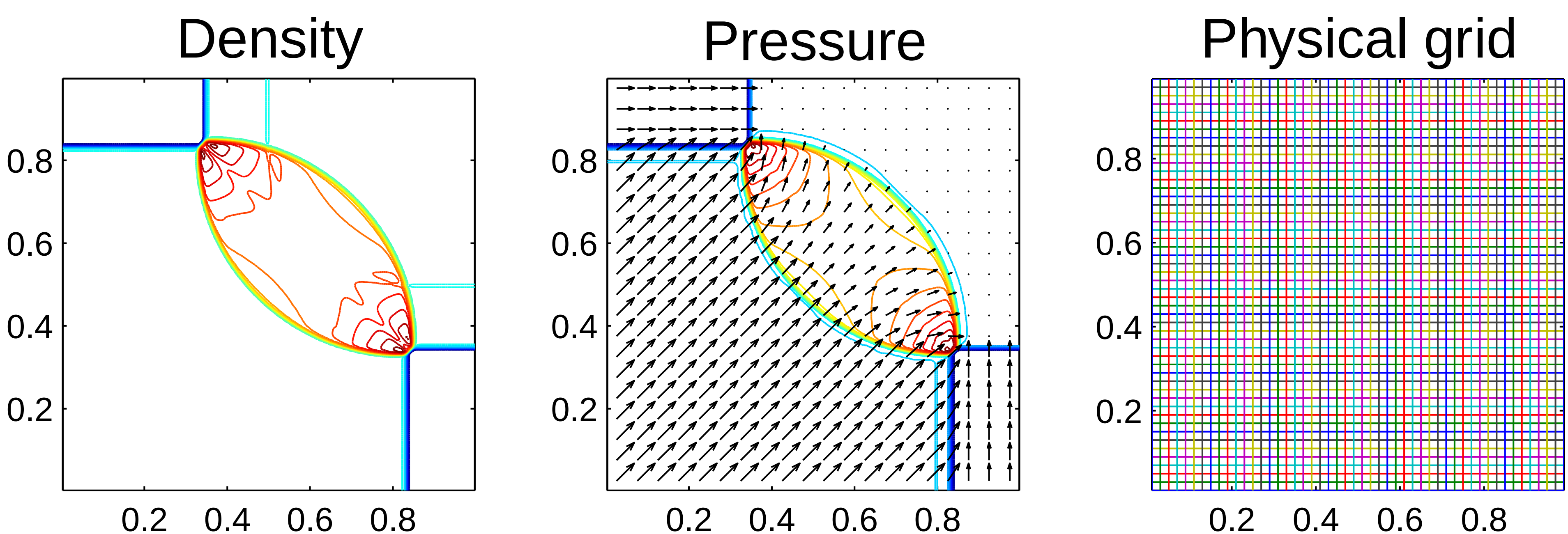
- Lagrange-like result
  - Occurrence of simple Mach reflection



2D Riemann problem

With initial 4-shock wave pattern

- Eulerian result
  - Poor resolution around simple Mach reflection



More Examples

- Two-dimensional case
  - Radially symmetric problem
  - Underwater explosion
  - Shock-bubble interaction
    - Helium bubble case
    - Refrigerant bubble case
- Three-dimensional case
  - Underwater explosion
  - Shock-bubble interaction
    - Helium bubble case
    - Refrigerant bubble case

Conclusion

- Have described fluid-mixture type algorithm in generalized moving-curvilinear grid
- Have shown results in 1, 2 & 3D to demonstrate feasibility of method for practical problems

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  - Efficient & accurate grid movement strategy
  - Static & Moving 3D geometry problems
  - Weakly compressible flow
  - Viscous flow extension
  - $\cdots$

Conclusion

- Have described fluid-mixture type algorithm in generalized moving-curvilinear grid
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- Future direction
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  - $\cdots$

Thank You

Euler in General. Coord.

Model system in quasi-linear form

$$\frac{\partial \tilde{q}}{\partial \tau} + A\frac{\partial \tilde{q}}{\partial \xi} + B\frac{\partial \tilde{q}}{\partial \eta} = \tilde{\psi}$$

$$A = \frac{\partial \tilde{f}}{\partial \tilde{q}} = \begin{bmatrix} \xi_t & \xi_x & \xi_y & 0 \\ \xi_x p_\rho - u\mathcal{U} & \xi_x u(1-p_E) + U & \xi_y u - \xi_x v p_E & \xi_x p_E \\ \xi_y p_\rho - v\mathcal{U} & \xi_x v - \xi_y u p_E & \xi_y v(1-p_E) + U & \xi_y p_E \\ (p_\rho - H)\mathcal{U} & \xi_x H - u\mathcal{U} p_E & \xi_y H - v\mathcal{U} p_E & U + p_E\mathcal{U} \end{bmatrix}$$

$$B = \frac{\partial \tilde{g}}{\partial \tilde{q}} = \begin{bmatrix} \eta_t & \eta_x & \eta_y & 0 \\ \eta_x p_\rho - u\mathcal{V} & \eta_x u(1-p_E) + V & \eta_y u - \eta_x v p_E & \eta_x p_E \\ \eta_y p_\rho - v\mathcal{V} & \eta_x v - \eta_y u p_E & \eta_y v(1-p_E) + V & \eta_y p_E \\ (p_\rho - H)\mathcal{V} & \eta_x H - u\mathcal{V} p_E & \eta_y H - v\mathcal{V} p_E & V + p_E\mathcal{V} \end{bmatrix}$$

with $H = (E+p)/\rho$, $\quad \mathcal{U} = U - \xi_t = \xi_x u + \xi_y v$, $\quad \mathcal{V} = V - \eta_t = \eta_x u + \eta_y v$

Euler in General. Coord. (Cont.)

Eigen-structure of matrix A is

$$\Lambda_A = \text{diag}\left(U - c\sqrt{\xi_x^2 + \xi_y^2},\ U,\ U,\ U + c\sqrt{\xi_x^2 + \xi_y^2}\right)$$

$$R_A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u - \alpha_1 c & u & \alpha_2 & u + \alpha_1 c \\ v - \alpha_2 c & v & -\alpha_1 & v + \alpha_2 c \\ H - \mathcal{U}_1 c & H - c^2/p_E & -\mathcal{U}_2 & H + \mathcal{U}_1 c \end{bmatrix}$$

$$L_A = \begin{bmatrix} (p_\rho + c\mathcal{U}_1)/2c^2 & -(\alpha_1 c + up_E)/2c^2 & -(\alpha_2 c + vp_E)/2c^2 & p_E/2c^2 \\ 1 - p_\rho/c^2 & up_E/c^2 & vp_E/c^2 & -p_E/c^2 \\ \mathcal{U}_2 & \alpha_2 & -\alpha_1 & 0 \\ (p_\rho - c\mathcal{U}_1)/2c^2 & (\alpha_1 c - up_E)/2c^2 & (\alpha_2 c - vp_E)/2c^2 & p_E/2c^2 \end{bmatrix}$$

with $(\alpha_1, \alpha_2) = (\xi_x, \xi_y)/\sqrt{\xi_x^2 + \xi_y^2}$, $\quad \mathcal{U}_1 = \alpha_1 u + \alpha_2 v$, $\quad \mathcal{U}_2 = -\alpha_2 u + \alpha_1 v$

Euler in General. Coord. (Cont.)

Eigen-structure of matrix B is

$$\Lambda_B = \text{diag}\left(V - c\sqrt{\eta_x^2+\eta_y^2},\ V,\ V,\ V + c\sqrt{\eta_x^2+\eta_y^2}\right)$$

$$R_B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u - \beta_1 c & u & \beta_2 & u + \beta_1 c \\ v - \beta_2 c & v & -\beta_1 & v + \beta_2 c \\ H - \mathcal{V}_1 c & H - c^2/p_E & -\mathcal{V}_2 & H + \mathcal{V}_1 c \end{bmatrix}$$

$$L_B = \begin{bmatrix} (p_\rho + c\mathcal{V}_1)/2c^2 & -(\beta_1 c + u p_E)/2c^2 & -(\beta_2 c + v p_E)/2c^2 & p_E/2c^2 \\ 1 - p_\rho/c^2 & u p_E/c^2 & v p_E/c^2 & -p_E/c^2 \\ \mathcal{V}_2 & \beta_2 & -\beta_1 & 0 \\ (p_\rho - c\mathcal{V}_1)/2c^2 & (\beta_1 c - u p_E)/2c^2 & (\beta_2 c - v p_E)/2c^2 & p_E/2c^2 \end{bmatrix}$$

with $(\beta_1, \beta_2) = (\eta_x, \eta_y)/\sqrt{\eta_x^2+\eta_y^2}$, $\mathcal{V}_1 = \beta_1 u + \beta_2 v$, $\mathcal{V}_2 = -\beta_2 u + \beta_1 v$

Grid Movement (Cont.)

- General 1-parameter case: $(x_\tau, y_\tau) = h(u, v)$, $\quad h \in [0, 1]$

  At given time instance, h can be chosen based on
  - Grid-angle preserving condition (Hui *et al.* JCP 1999)

$$\frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{\nabla \xi}{|\nabla \xi|} \cdot \frac{\nabla \eta}{|\nabla \eta|}\right) = \frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{-y_\eta x_\eta - y_\xi x_\xi}{\sqrt{y_\xi^2 + y_\eta^2}\sqrt{x_\xi^2 + x_\eta^2}}\right)$$

$$= \cdots$$

$$= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \quad (\text{1st order PDE})$$

with

$$\mathcal{A} = \sqrt{x_\eta^2 + y_\eta^2}\,(v x_\xi - u y_\xi)\,, \quad \mathcal{B} = \sqrt{x_\xi^2 + y_\xi^2}\,(u y_\eta - v x_\eta)$$

$$\mathcal{C} = \sqrt{x_\xi^2 + y_\xi^2}\,(u_\eta y_\eta - v_\eta x_\eta) - \sqrt{x_\eta^2 + y_\eta^2}\,(u_\xi y_\xi - v_\xi x_\xi)$$

Grid Movement (Cont.)

- General 1-parameter case: $(x_\tau, y_\tau) = h(u,v)$, $h \in [0,1]$

  Or alternatively, based on

  - Mesh-area preserving condition

$$
\begin{aligned}
\frac{\partial J}{\partial \tau} &= \frac{\partial}{\partial \tau}(x_\xi y_\eta - x_\eta y_\xi) \\
&= x_{\xi\tau}\, y_\eta + x_\xi\, y_{\eta\tau} - x_{\eta\tau}\, y_\xi - x_\eta\, y_{\xi\tau} \\
&= \cdots \\
&= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \quad \text{(1st order PDE)}
\end{aligned}
$$

with

$$
\mathcal{A} = u y_\eta - v x_\eta, \quad \mathcal{B} = v x_\xi - u y_\xi, \quad \mathcal{C} = u_\xi y_\eta + v_\eta x_\xi - u_\eta y_\xi - v_\xi x_\eta
$$

Grid Movement (Cont.)

To ensure $h \in [0,1]$, transformed variable $\tilde{h} = \kappa(h)$ is used, *e.g.*, Hui *et al.* employed $\kappa = \ln\left(\varepsilon h|\vec{u}|\right)$, ε normalized constant, yielding

$$\tilde{\mathcal{A}}\tilde{h}_\xi + \tilde{\mathcal{B}}\tilde{h}_\eta + \tilde{\mathcal{C}} = 0$$

- Grid-angle preserving case

$$\tilde{\mathcal{A}} = \sqrt{x_\eta^2 + y_\eta^2}\left(x_\xi \sin\theta - y_\xi \cos\theta\right), \quad \tilde{\mathcal{B}} = \sqrt{x_\xi^2 + y_\xi^2}\left(y_\eta \cos\theta - x_\eta \sin\theta\right)$$
$$\tilde{\mathcal{C}} = \sqrt{x_\xi^2 + y_\xi^2}\left[y_\eta(\cos\theta)_\eta - x_\eta(\sin\theta)_\eta\right] - \sqrt{x_\eta^2 + y_\eta^2}\left[y_\xi(\cos\theta)_\xi - x_\xi(\sin\theta)_\xi\right]$$

- Mesh-area preserving case

$$\tilde{\mathcal{A}} = y_\eta \cos\theta - x_\eta \sin\theta, \quad \tilde{\mathcal{B}} = x_\xi \sin\theta - y_\xi \cos\theta$$
$$\tilde{\mathcal{C}} = y_\eta(\cos\theta)_\xi - x_\eta(\sin\theta)_\xi + x_\xi(\sin\theta)_\eta - y_\xi(\cos\theta)_\eta$$

where $\vec{u} = (u,v) = |\vec{u}|(\cos\theta, \sin\theta)$

Grid Movement: Remarks

- Numerics: h- or $\tilde{h}$-equation constraint geometrical laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \end{pmatrix} - \frac{\partial}{\partial \xi}\begin{pmatrix} hu \\ hv \\ 0 \\ 0 \end{pmatrix} - \frac{\partial}{\partial \eta}\begin{pmatrix} 0 \\ 0 \\ hu \\ hv \end{pmatrix} = 0$$

- Usability: Mesh-area evolution equation

$$\frac{\partial J}{\partial \tau} - \frac{\partial}{\partial \xi}\left[h\left(u y_\eta - v x_\eta\right)\right] - \frac{\partial}{\partial \eta}\left[h\left(v x_\xi - u y_\xi\right)\right] = 0$$

- Initial & boundary conditions for h- or $\tilde{h}$-equation ?

Grid Movement: 2 Free Degrees

- 2-parameter case of Hui *et al.* (2005): $(x_\tau, y_\tau) = (U_g, V_g)$
  - Imposed conditions
    1. Grid-angle preserving
    2. Specialized grid-material line matching (see next)

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- Other 2-parameter case: $(x_\tau, y_\tau) = (hu, kv)$
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Roadmap of current work:

$$\boxed{(x_\tau, y_\tau) = h_0(u, v)} \longrightarrow \boxed{(x_\tau, y_\tau) = h(u, v)} \longrightarrow \cdots$$

Novel Conditions for h & k

- Mesh-area preserving case

$$\begin{aligned}\frac{\partial J}{\partial \tau} &= \frac{\partial}{\partial \tau}\left(x_\xi y_\eta - x_\eta y_\xi\right)\\ &= x_{\xi\tau}\, y_\eta + x_\xi\, y_{\eta\tau} - x_{\eta\tau}\, y_\xi - x_\eta\, y_{\xi\tau}\\ &= \cdots\\ &= (\mathcal{A}_1 h_\xi + \mathcal{B}_1 h_\eta + \mathcal{C}_1 h) + (\mathcal{A}_2 k_\xi + \mathcal{B}_2 k_\eta + \mathcal{C}_2 k) = 0,\end{aligned}$$

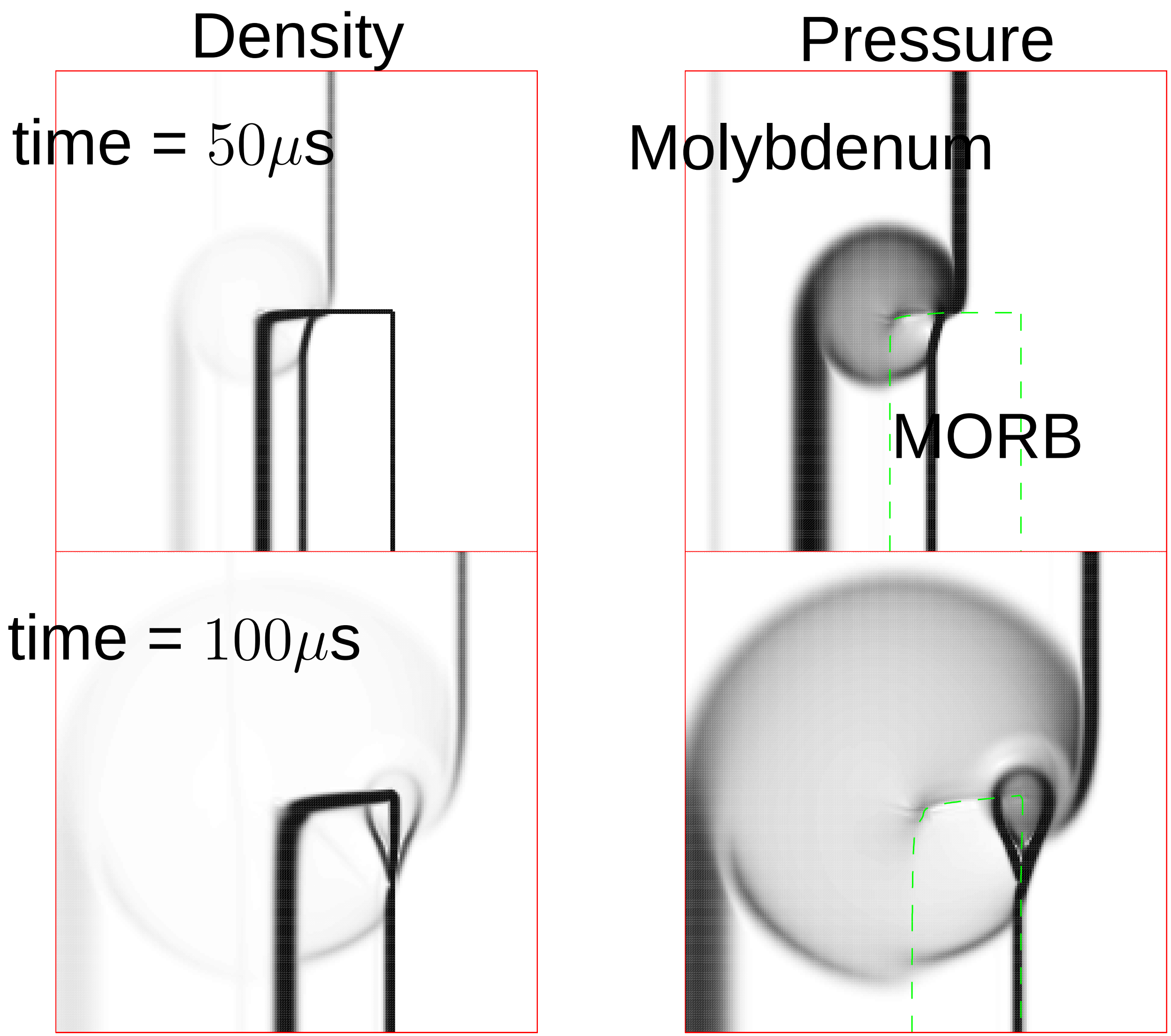
yielding, for example,

$$\mathcal{A}_1 h_\xi + \mathcal{B}_1 h_\eta + \mathcal{C}_1 h = 0$$
$$\mathcal{A}_2 k_\xi + \mathcal{B}_2 k_\eta + \mathcal{C}_2 k = 0$$

with

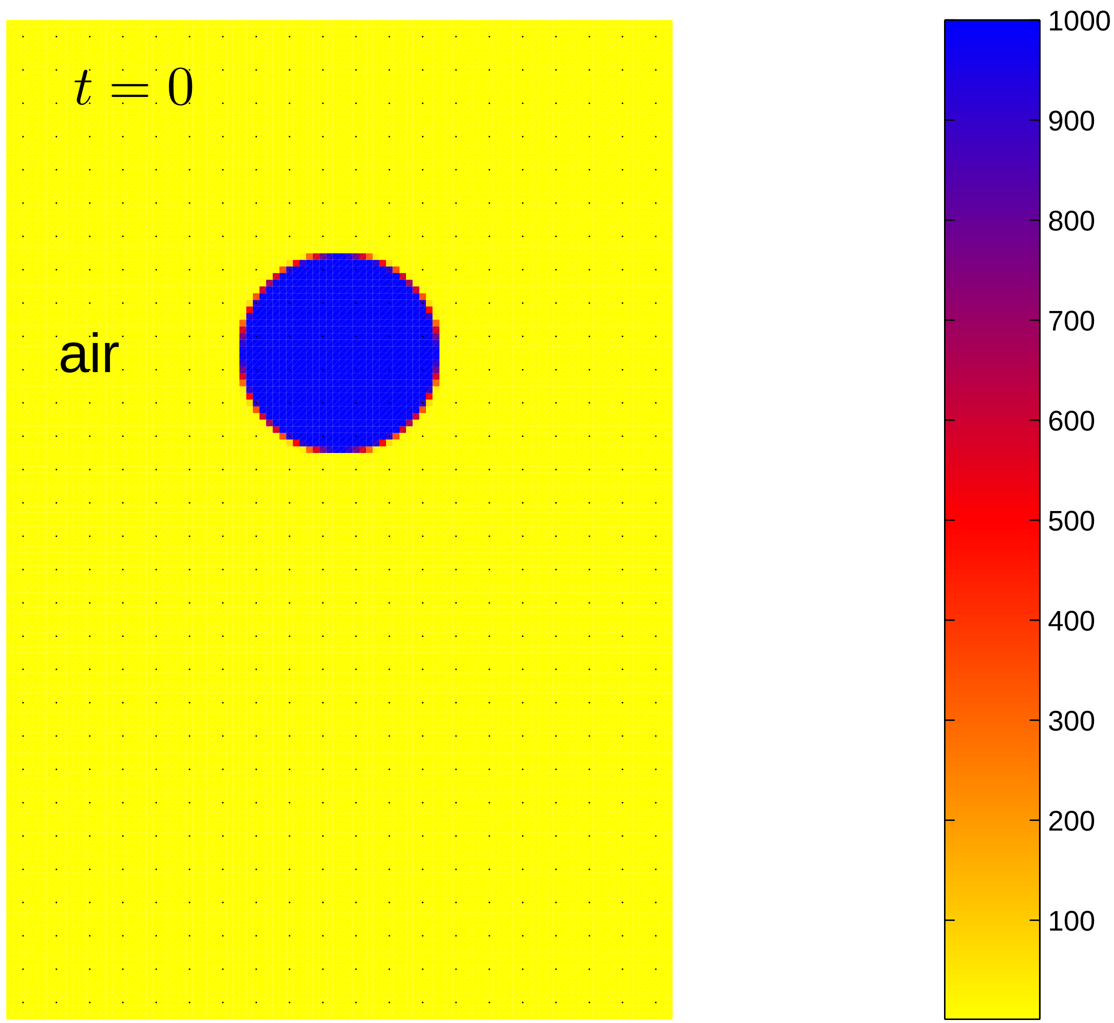
$$\begin{aligned}\mathcal{A}_1 &= u y_\eta, & \mathcal{B}_1 &= u y_\xi, & \mathcal{C}_1 &= u_\xi y_\eta - u_\eta y_\xi\\ \mathcal{A}_2 &= -v x_\eta, & \mathcal{B}_2 &= v x_\xi, & \mathcal{C}_2 &= v_\eta x_\xi - v_\xi x_\eta\end{aligned}$$

Shock in molybdenum over MORB

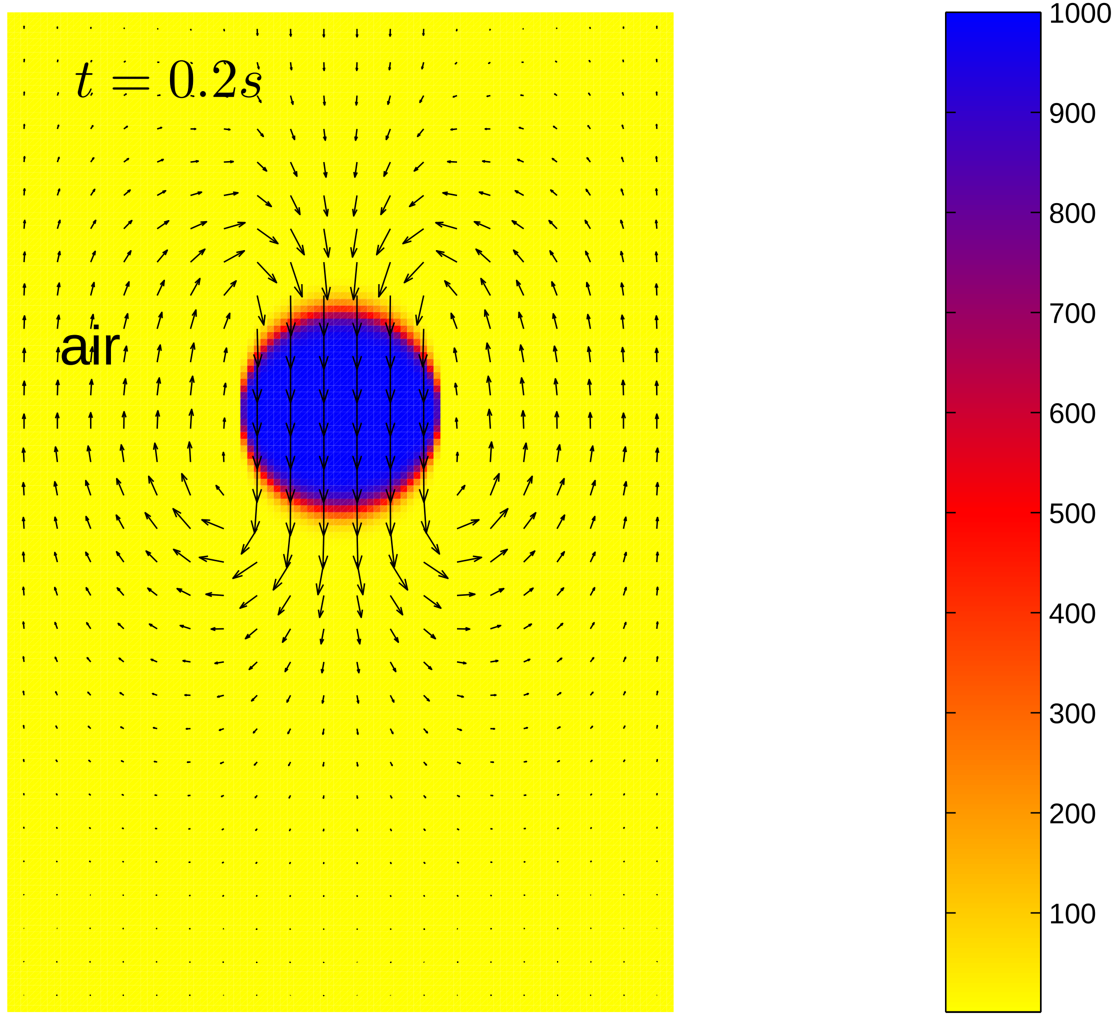


Falling Liquid Drop Problem

- Interface capturing with gravity

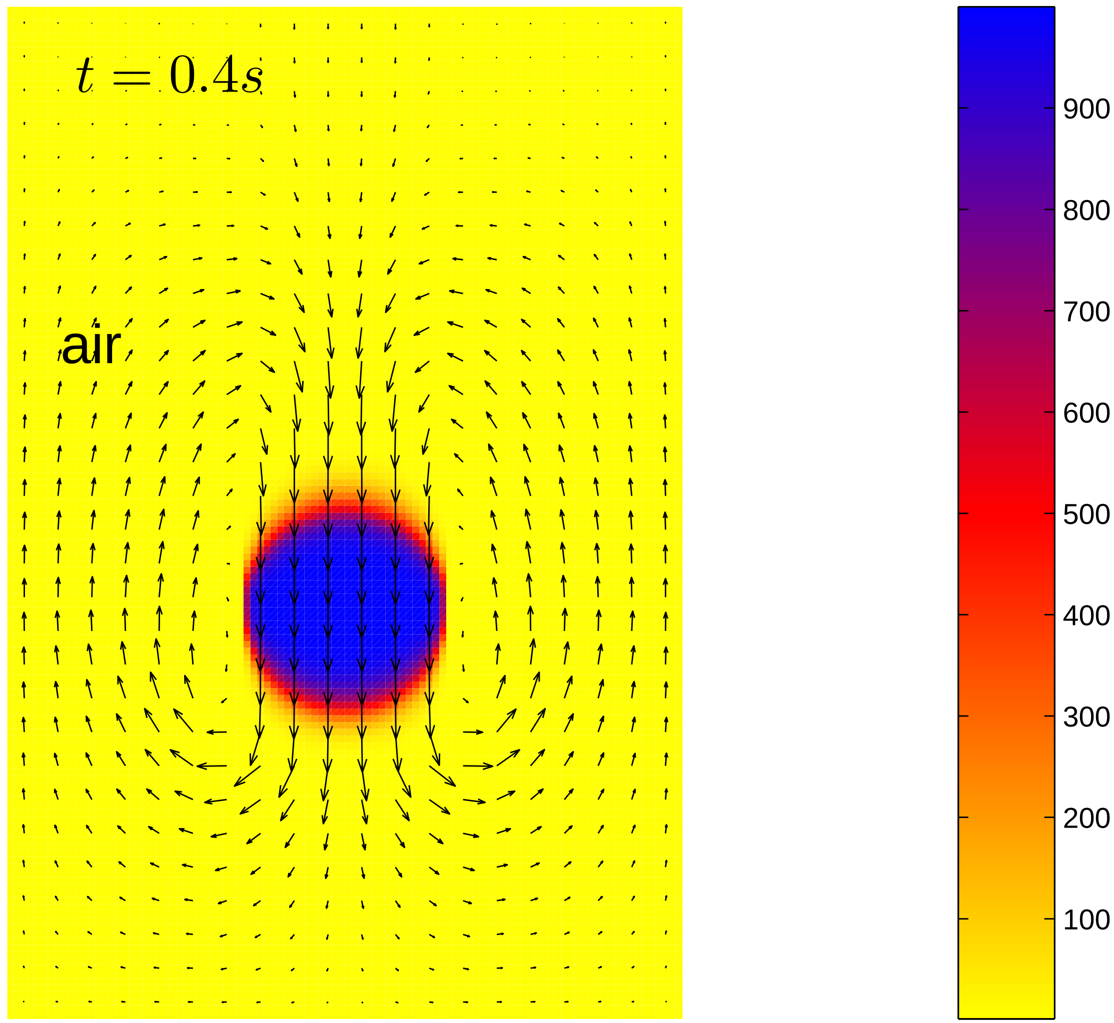


Falling Liquid Drop Problem



Falling Liquid Drop Problem

- Interface diffused badily



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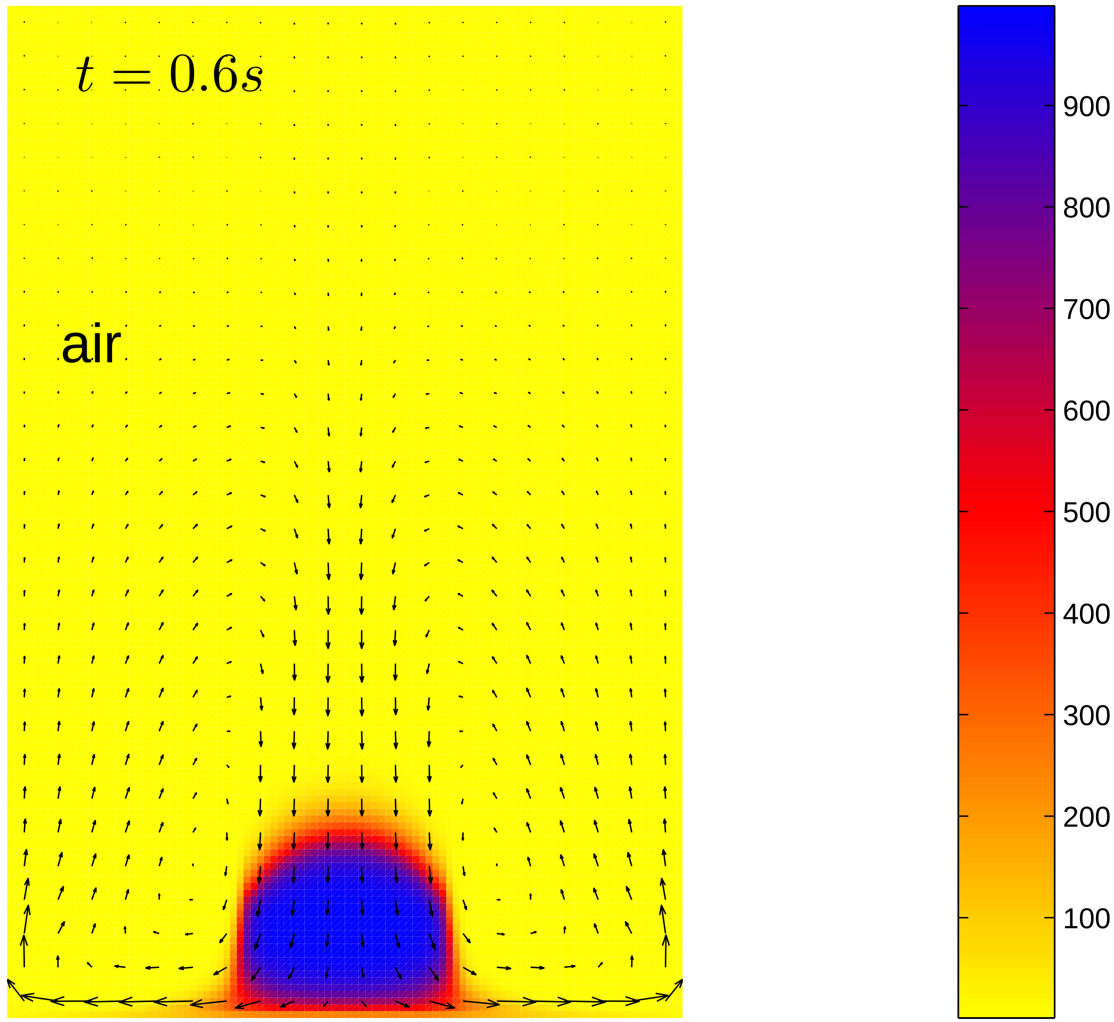

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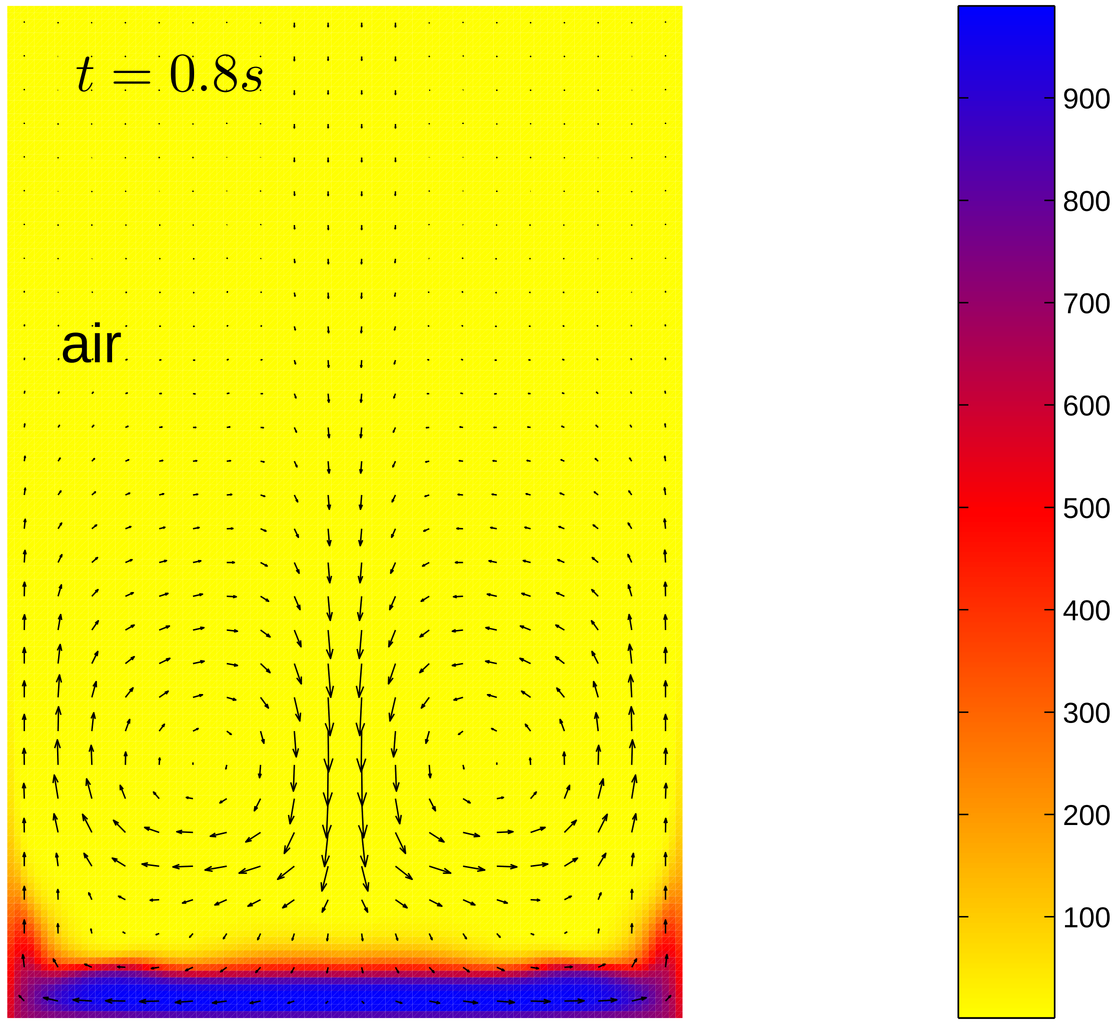

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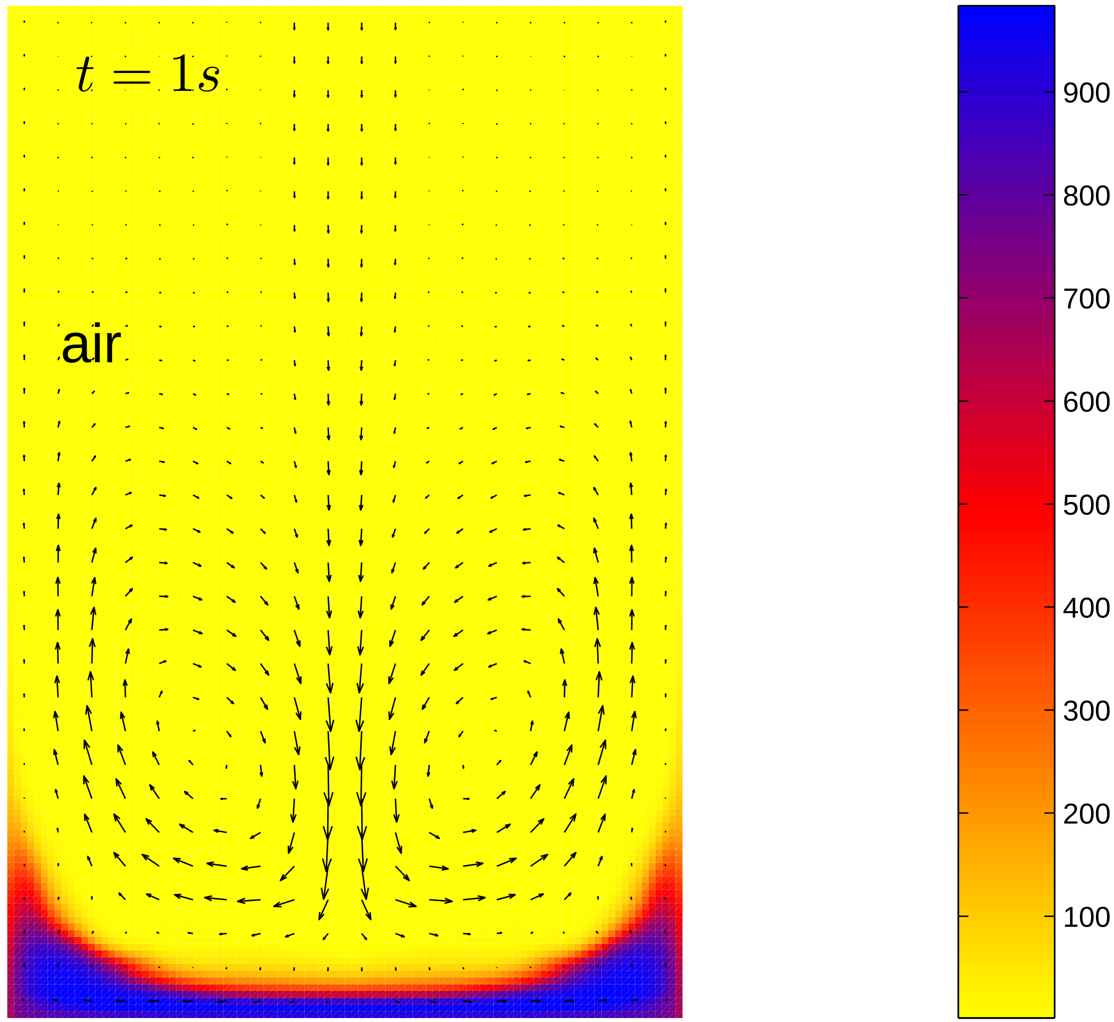
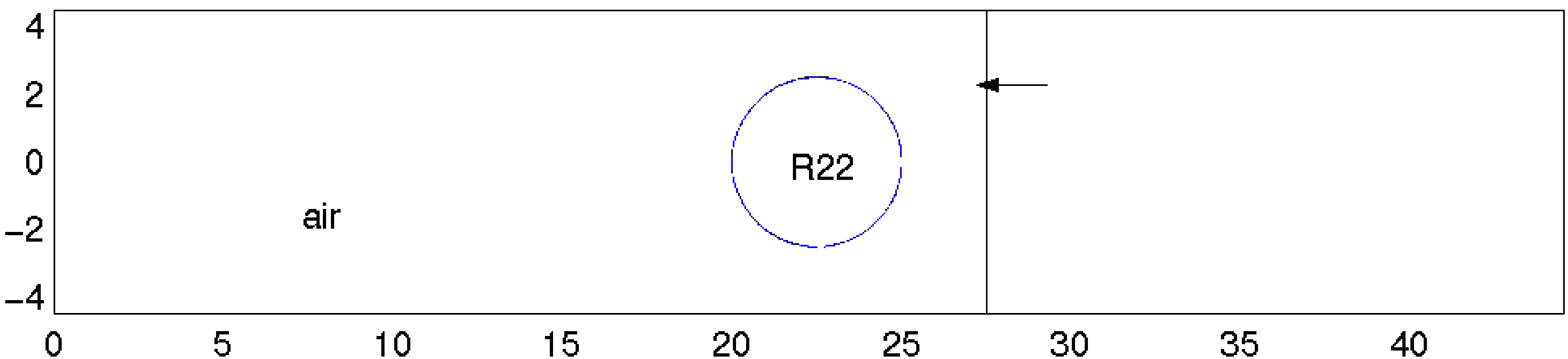


Shock-Bubble Interaction

- Volume tracking for material interface



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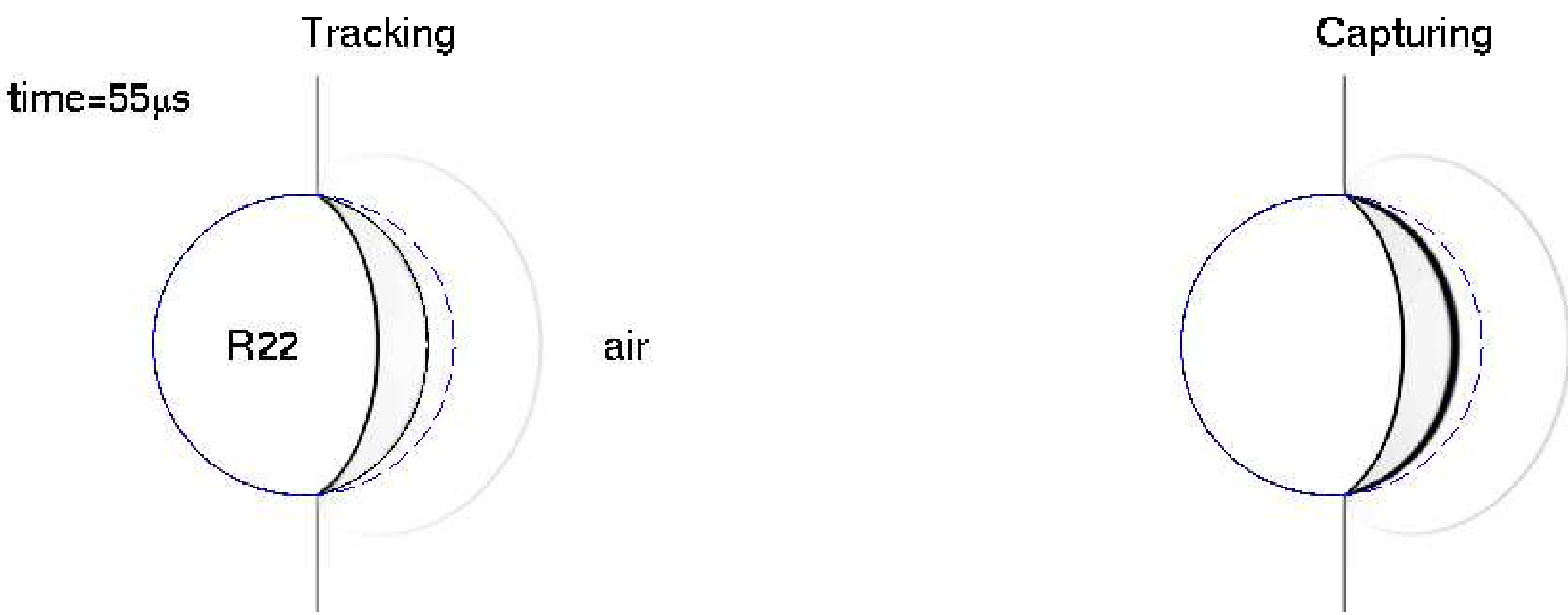

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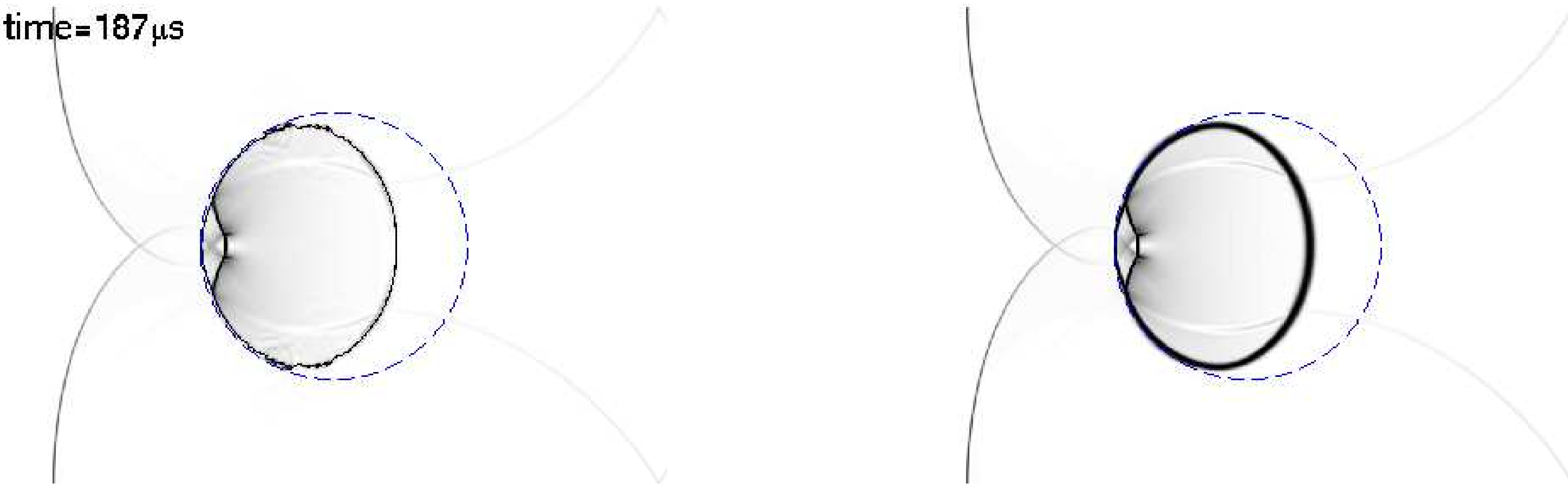

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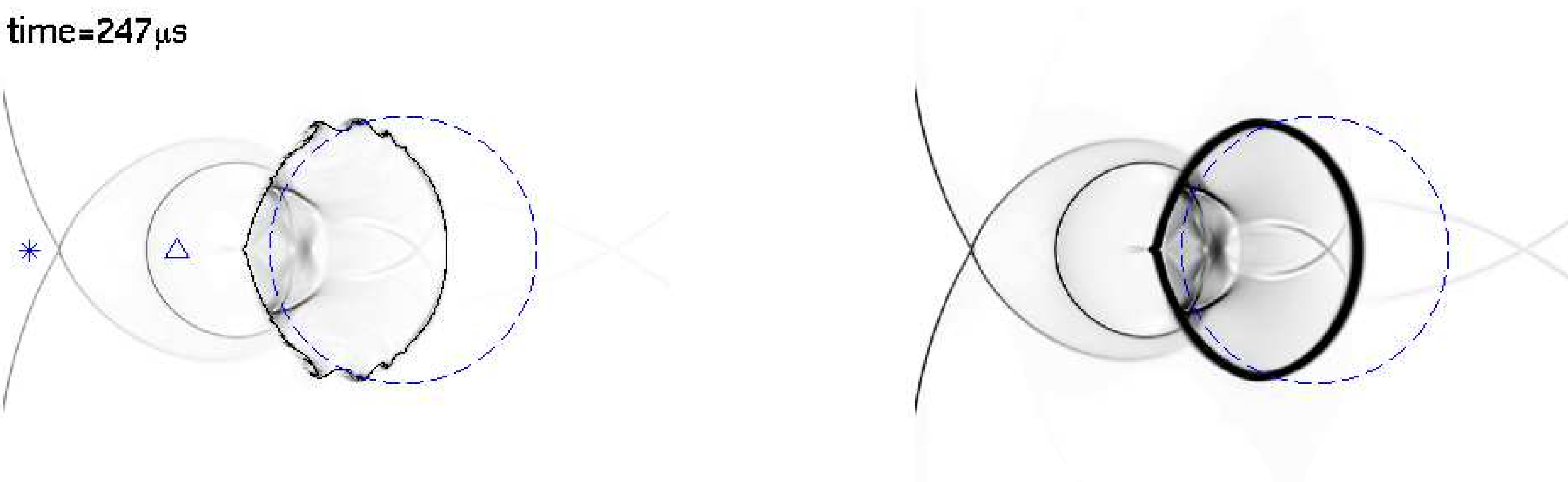

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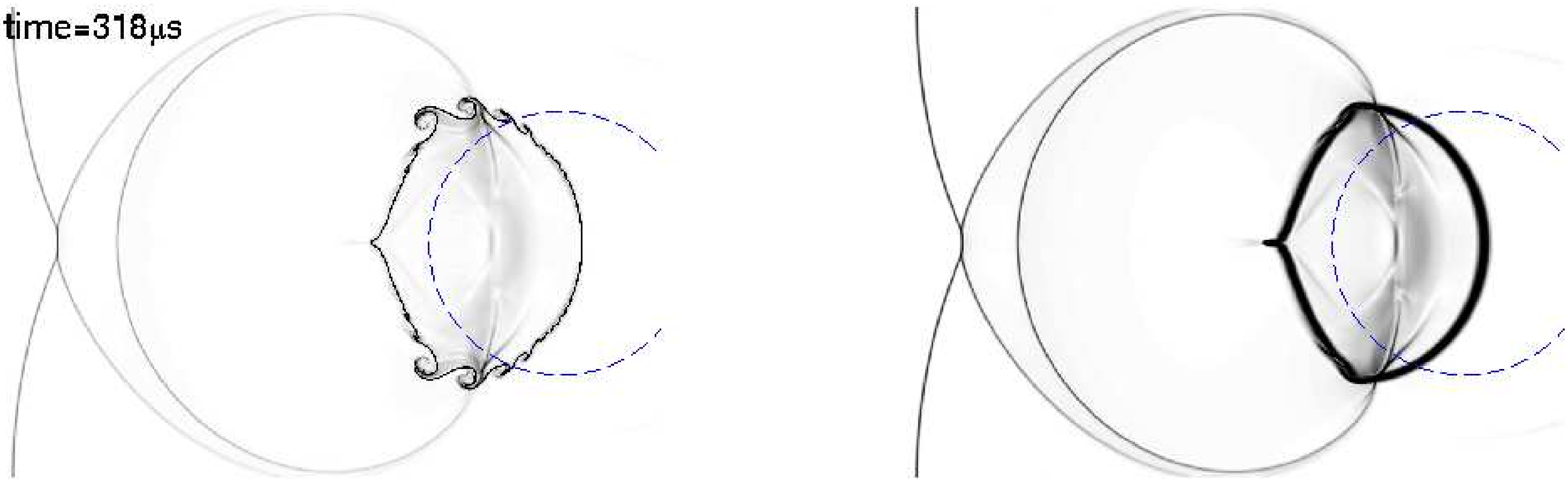

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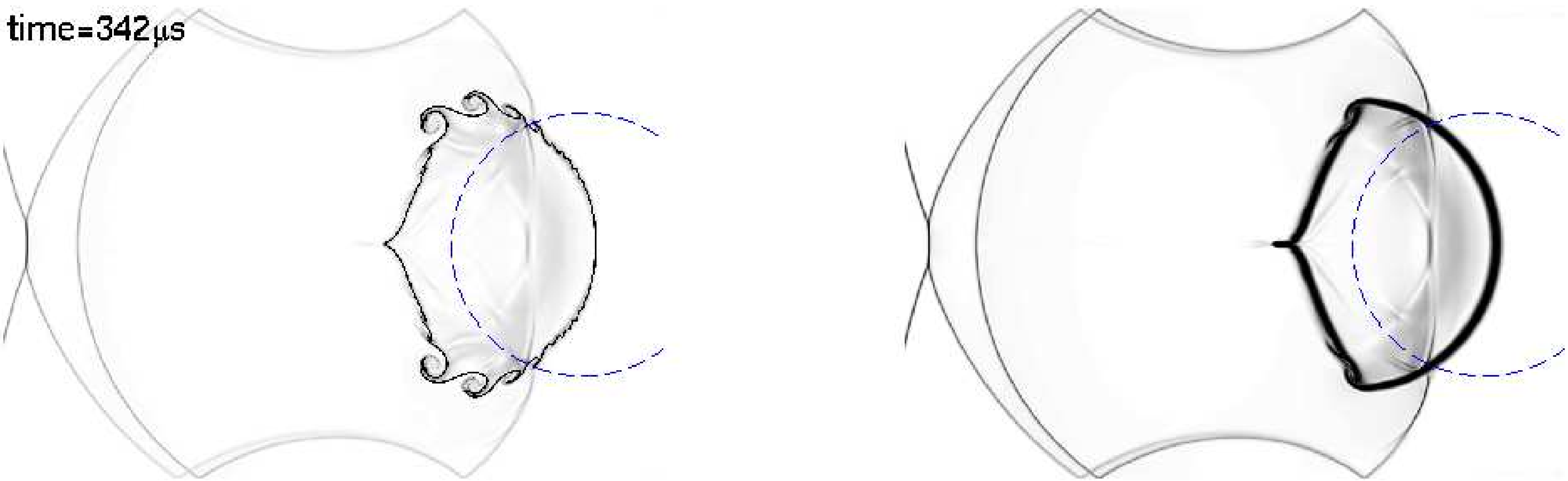
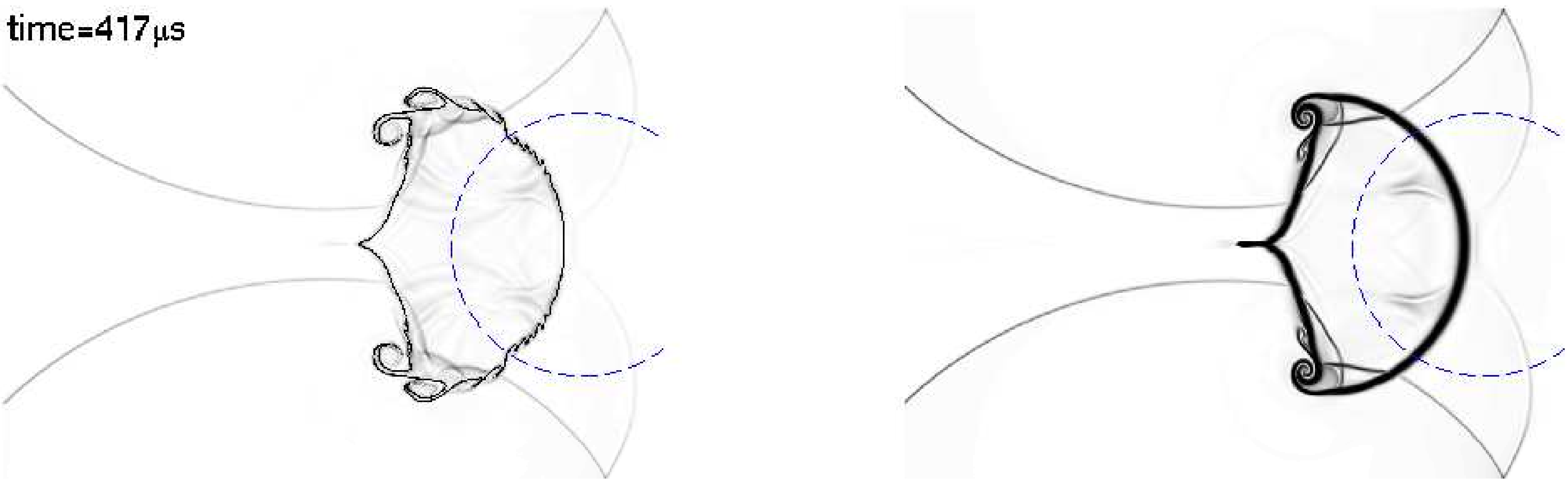
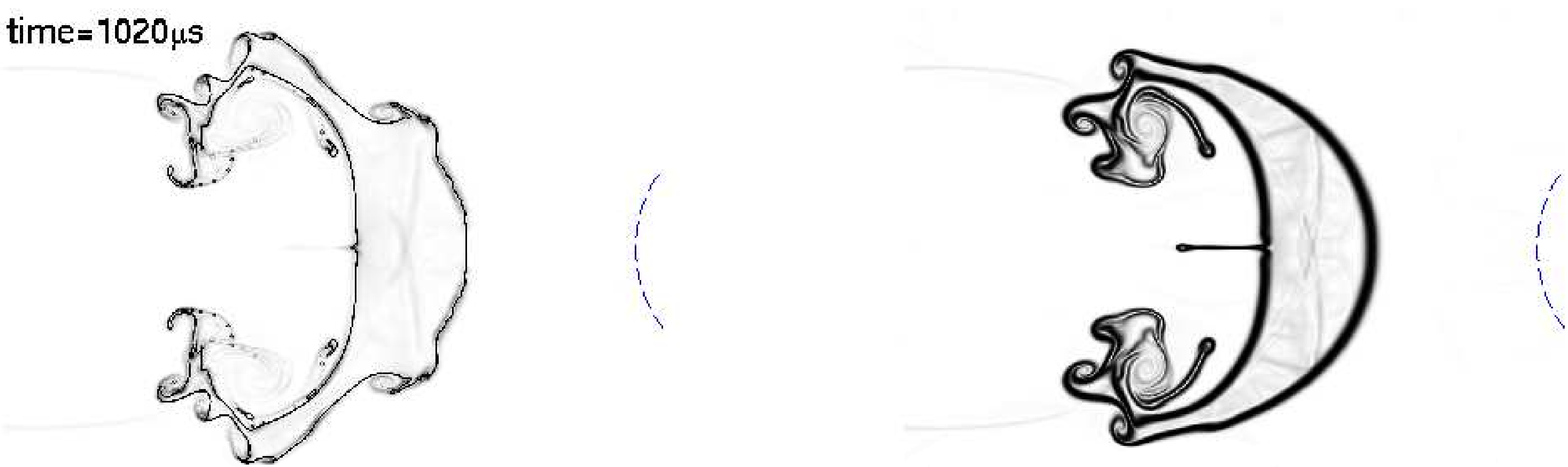


Shock-Bubble Interaction



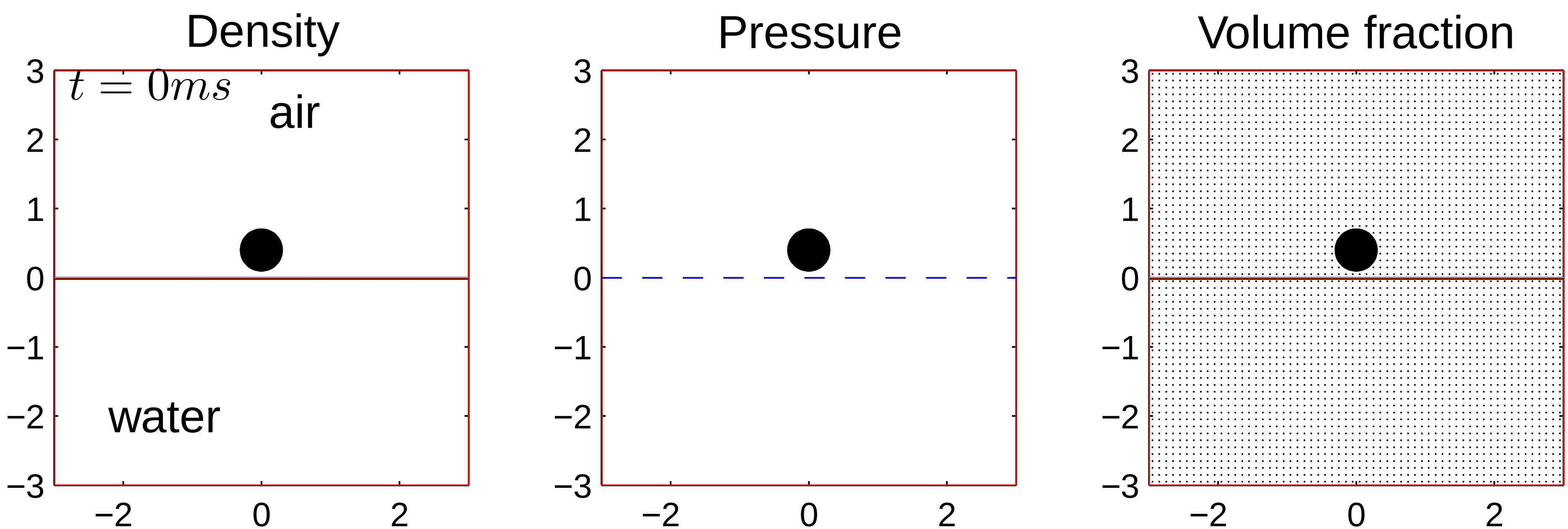
Shock-Bubble Interaction

- Small moving irregular cells: stability & accuracy

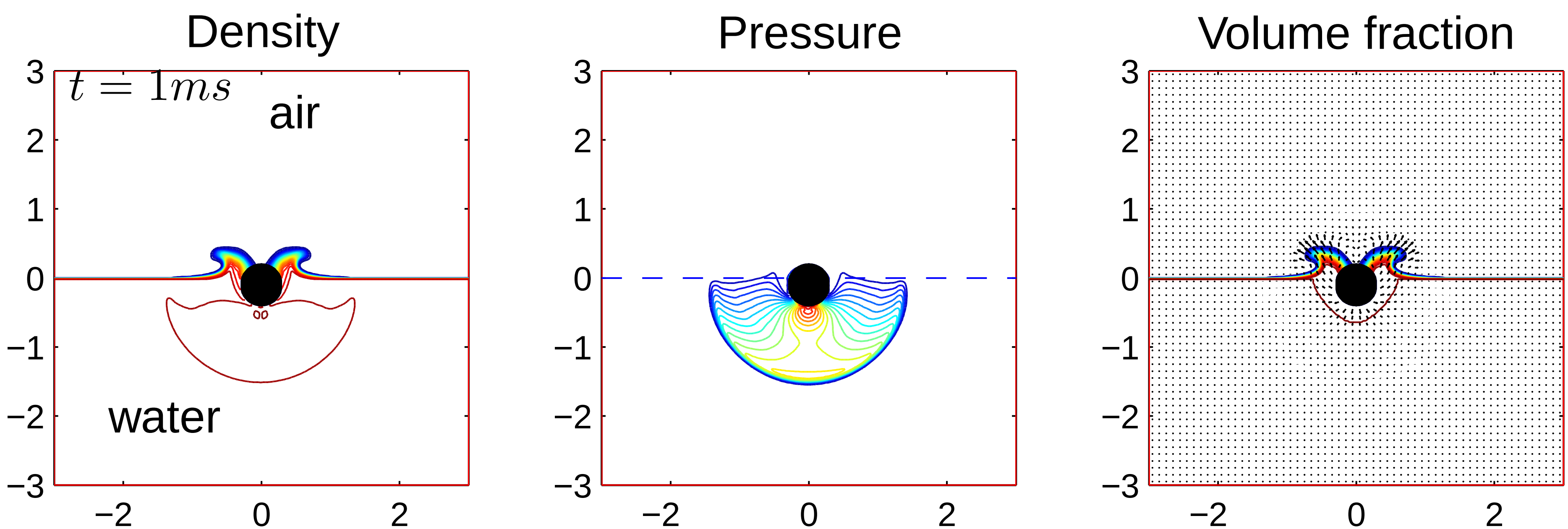


Falling Rigid Object in Water Tank

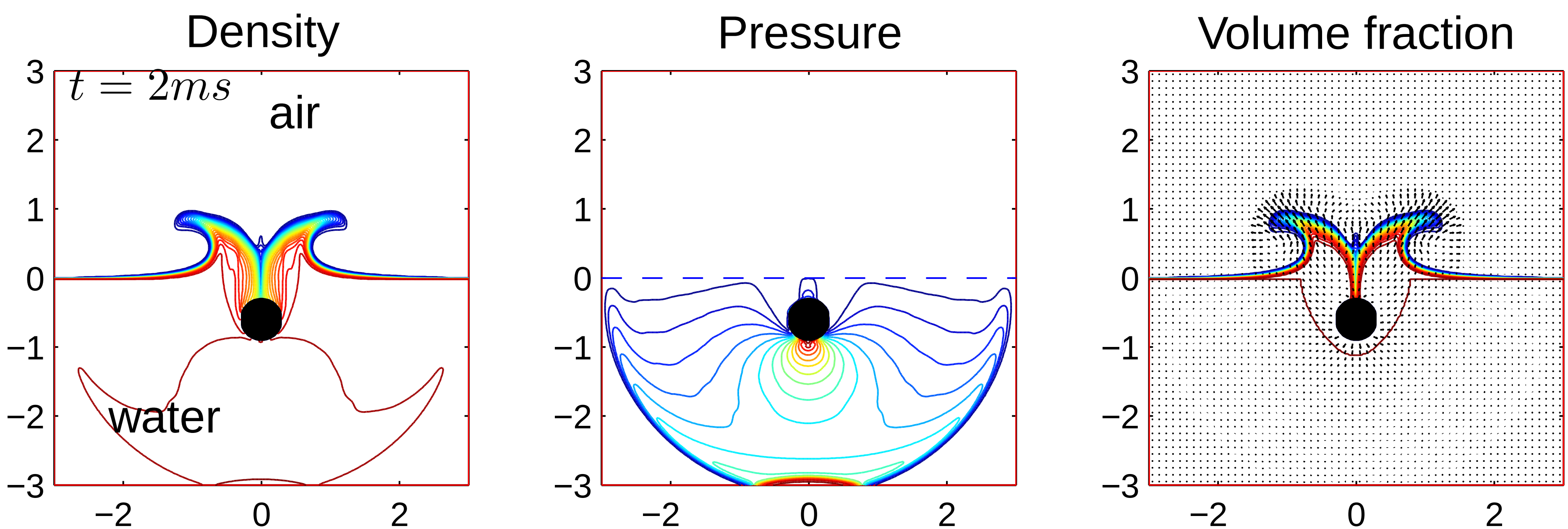
- Moving boundary tracking & interface capturing



Falling Rigid Object in Water Tank

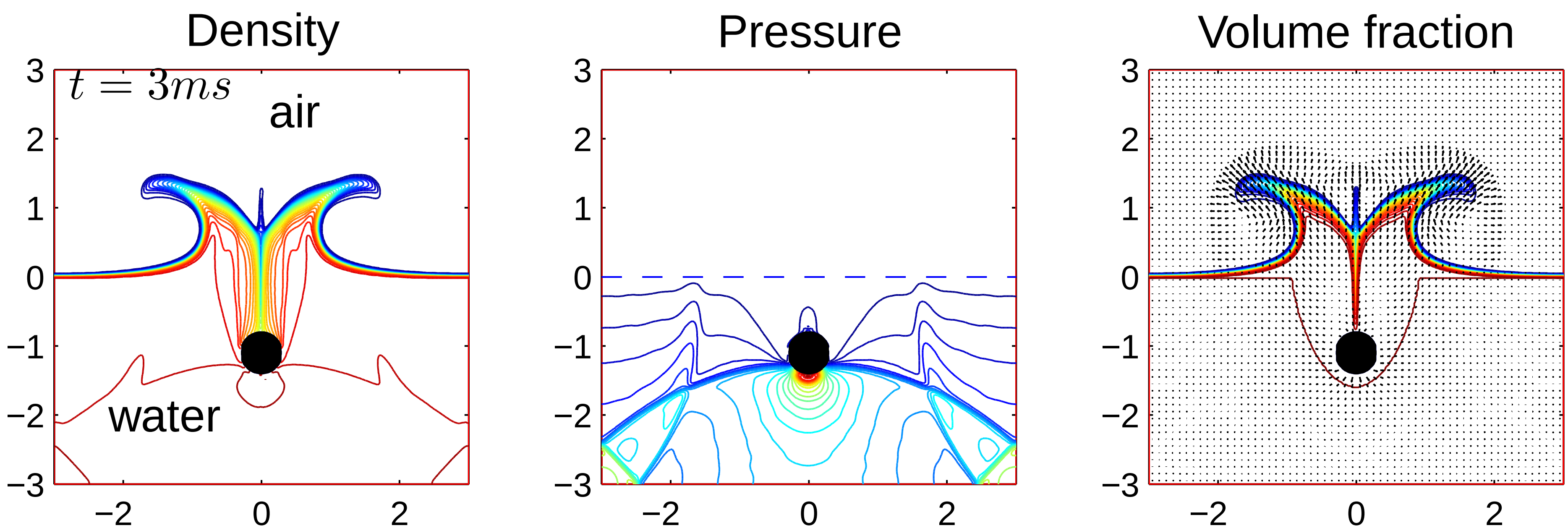


Falling Rigid Object in Water Tank

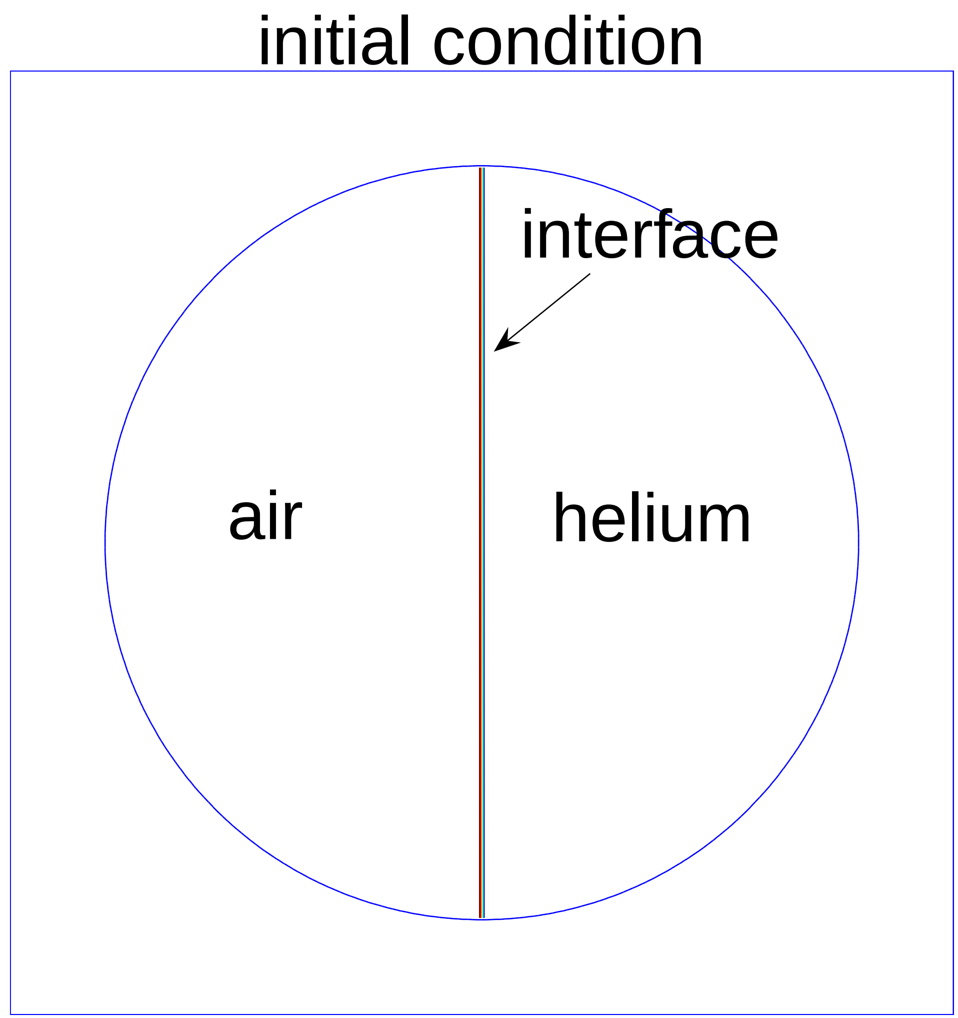


Falling Rigid Object in Water Tank

- Small moving irregular cells: stability & accuracy

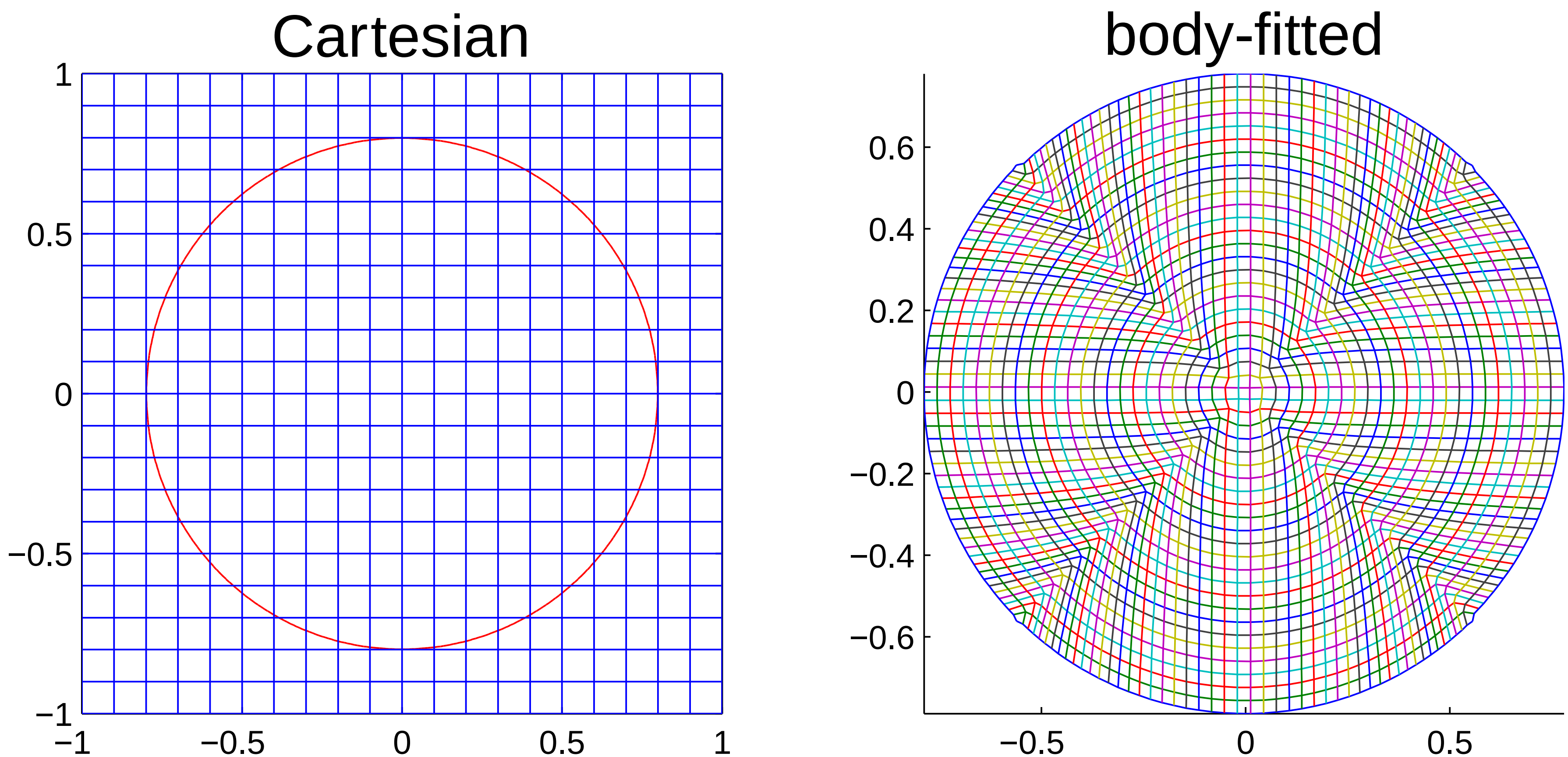


Moving Cylindrical Vessel



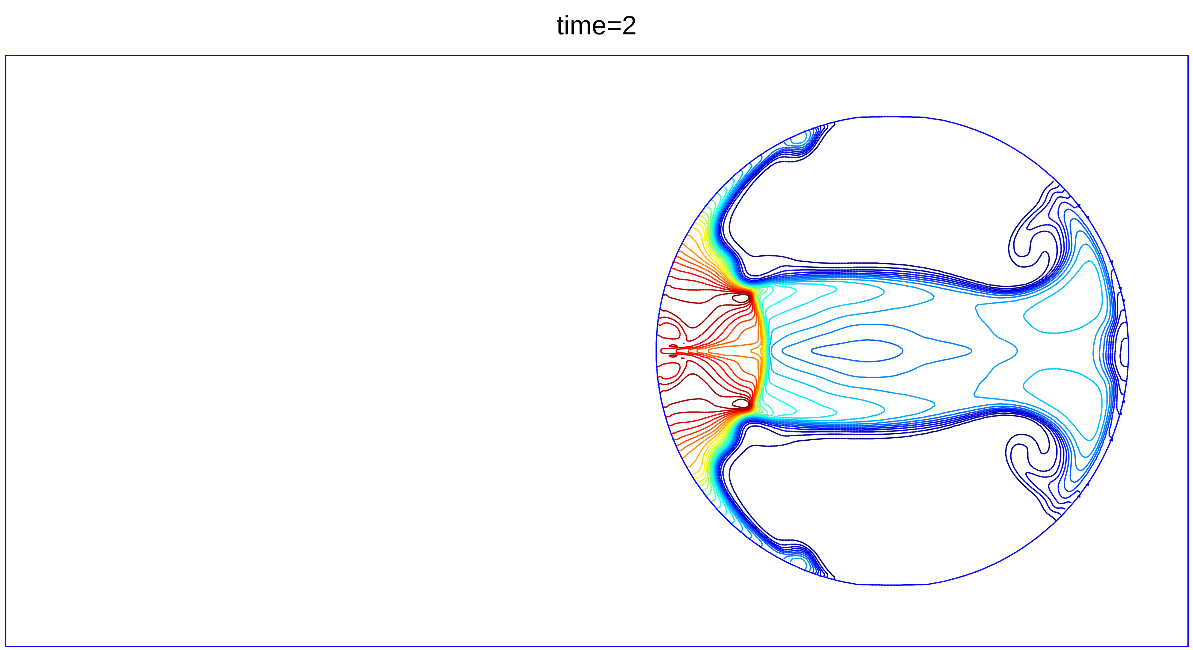
Moving Cylindrical Vessel

- Two sample grid systems used in computation



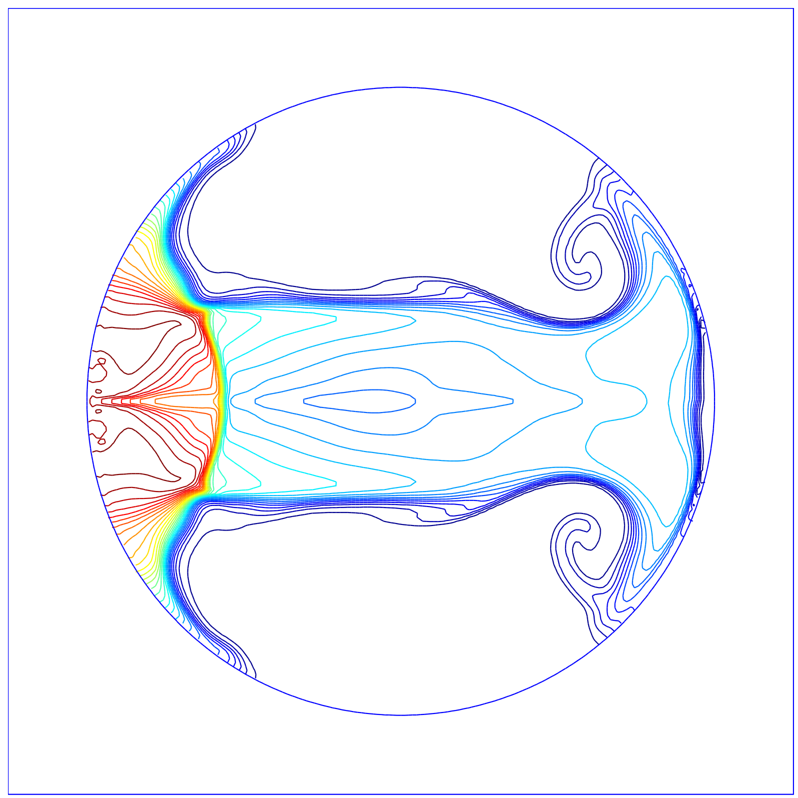
Moving Cylindrical Vessel

Cartesian grid results with embedded moving boundary



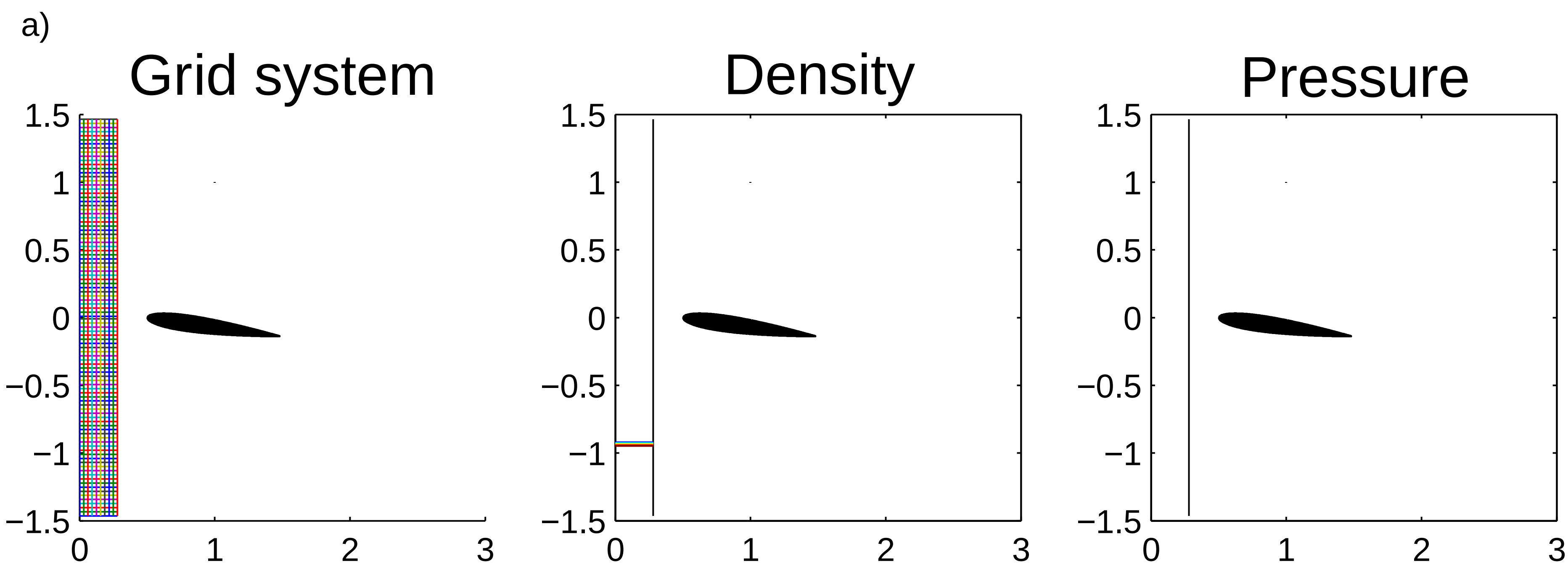
Moving Cylindrical Vessel

Cartesian grid results with embedded stationary boundary



NACA0012 over heavier gas

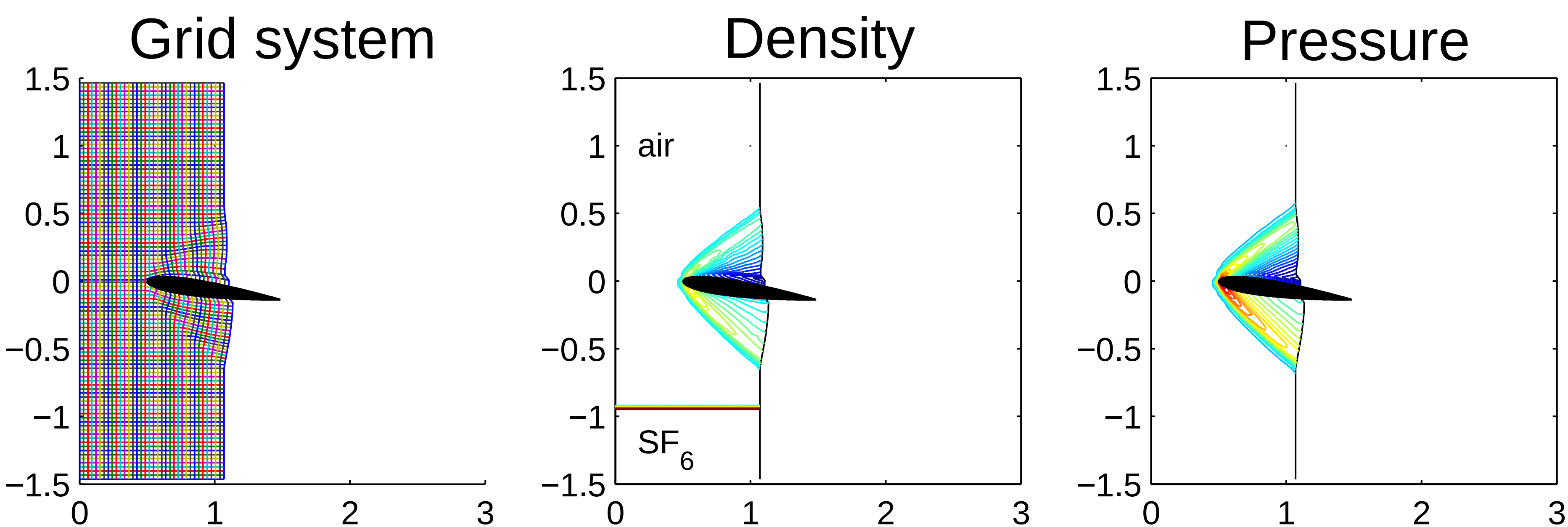
- Automatic time-marching grid



NACA0012 over heavier gas

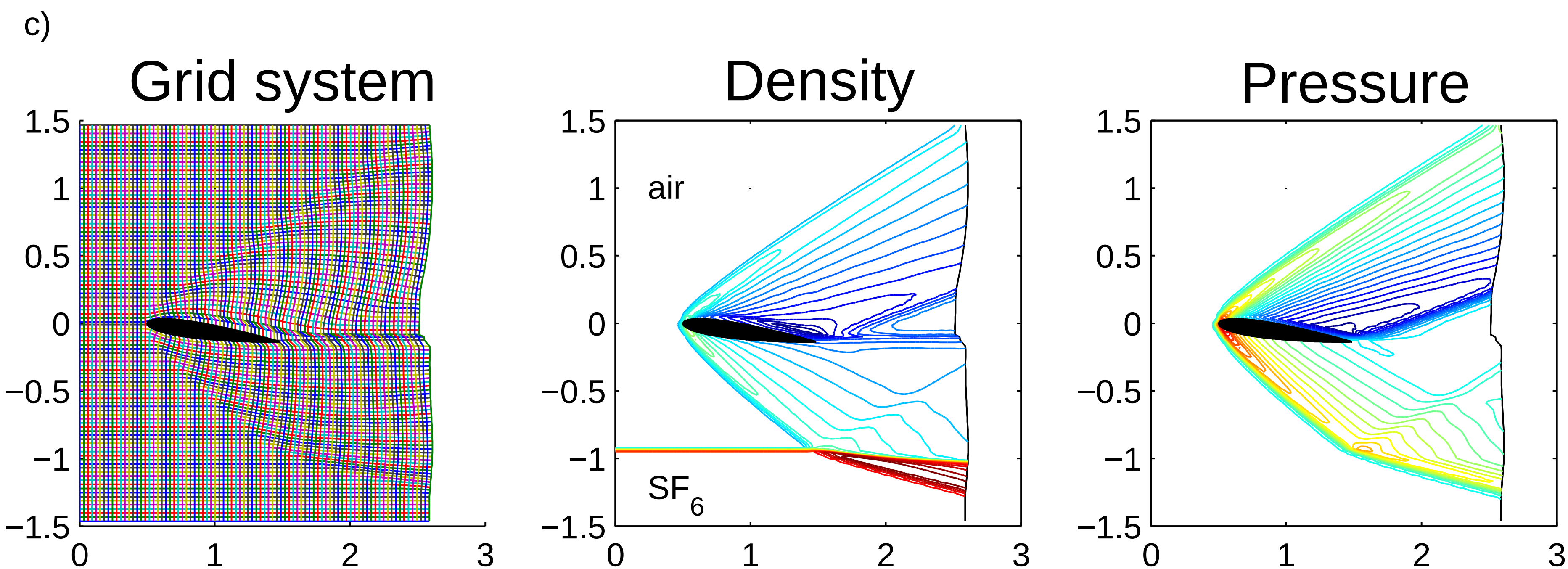
- Automatic time-marching grid

b)

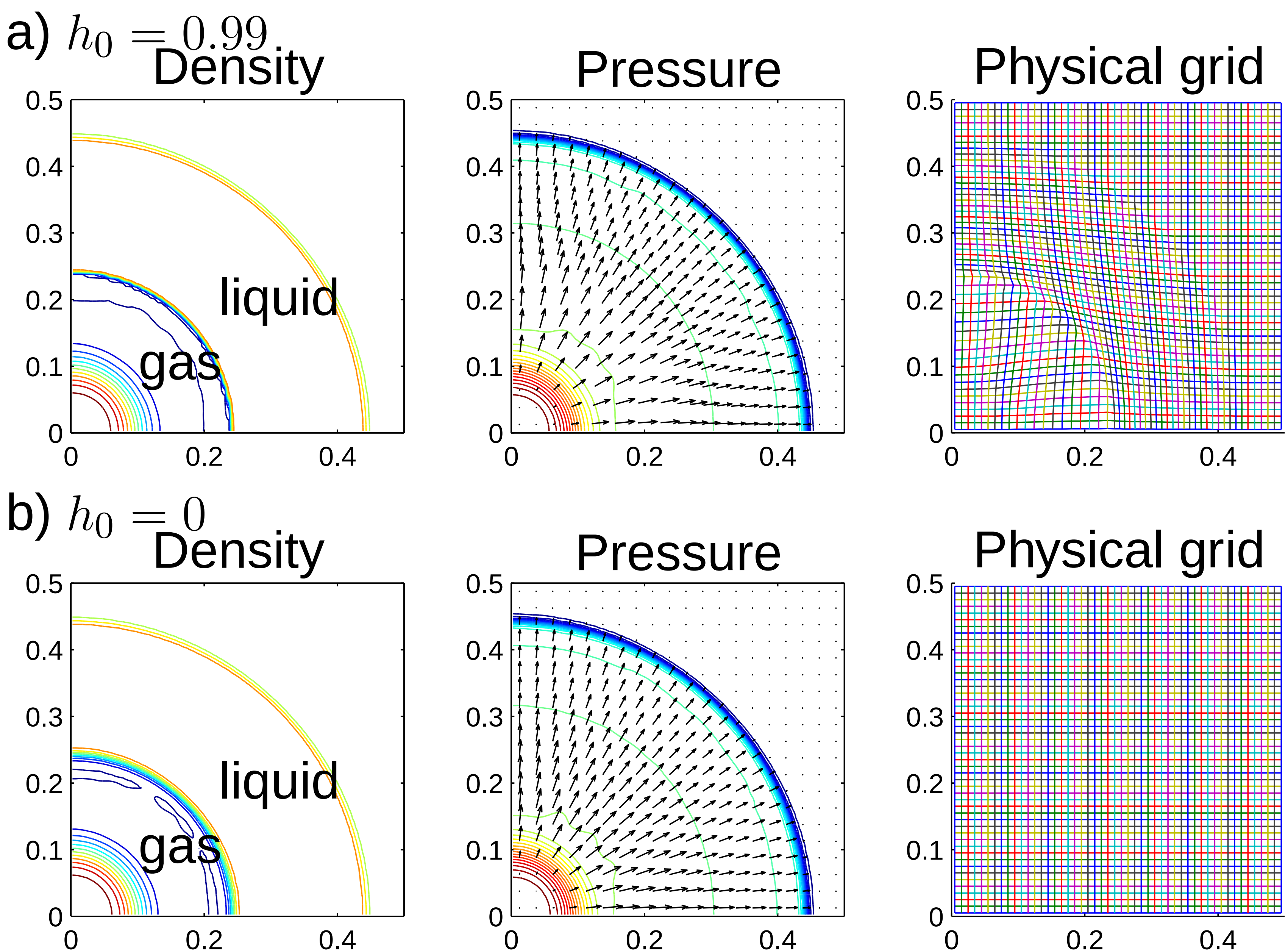


NACA0012 over heavier gas

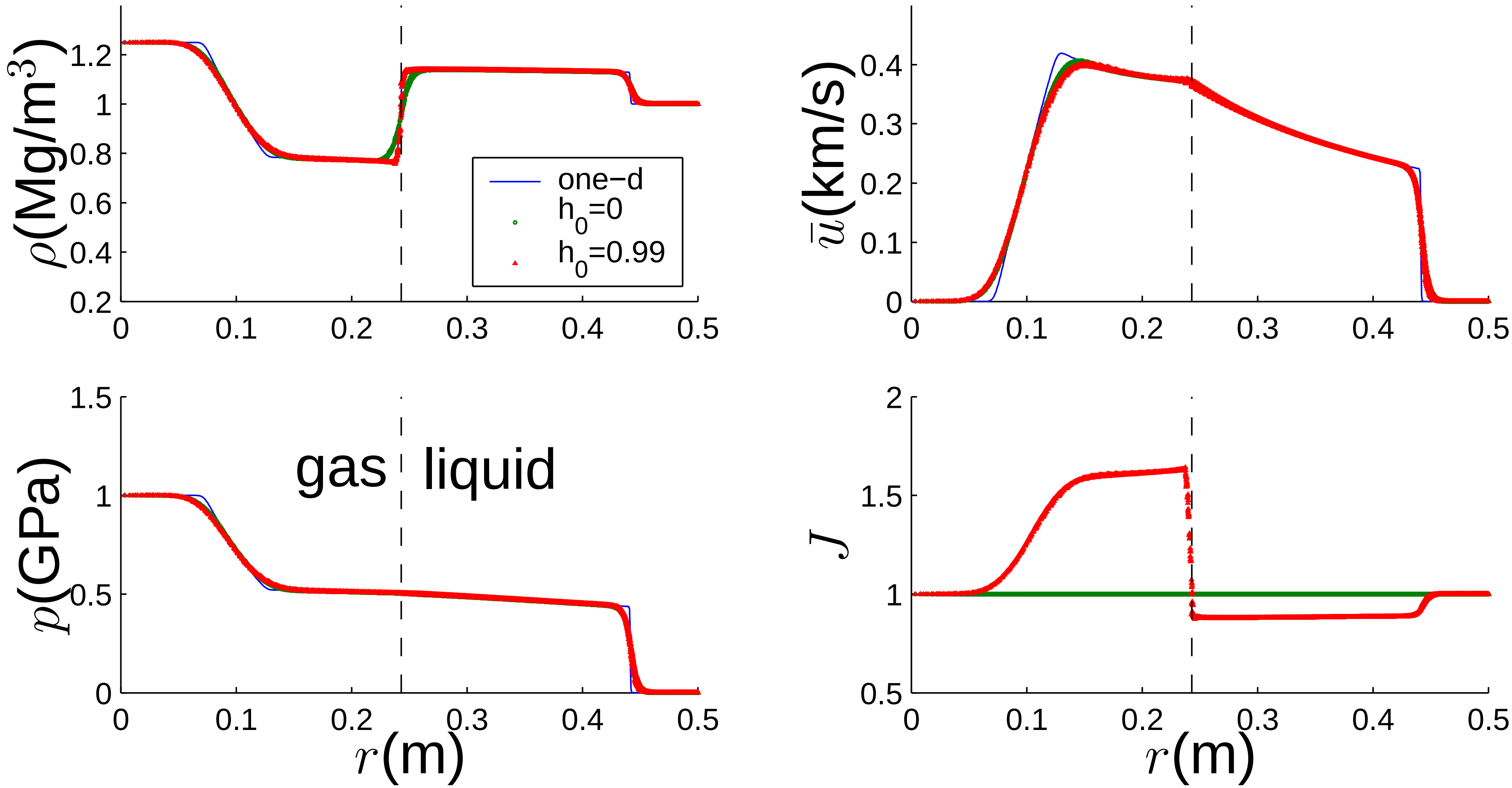
- Automatic time-marching grid



Radially Symmetric Problem

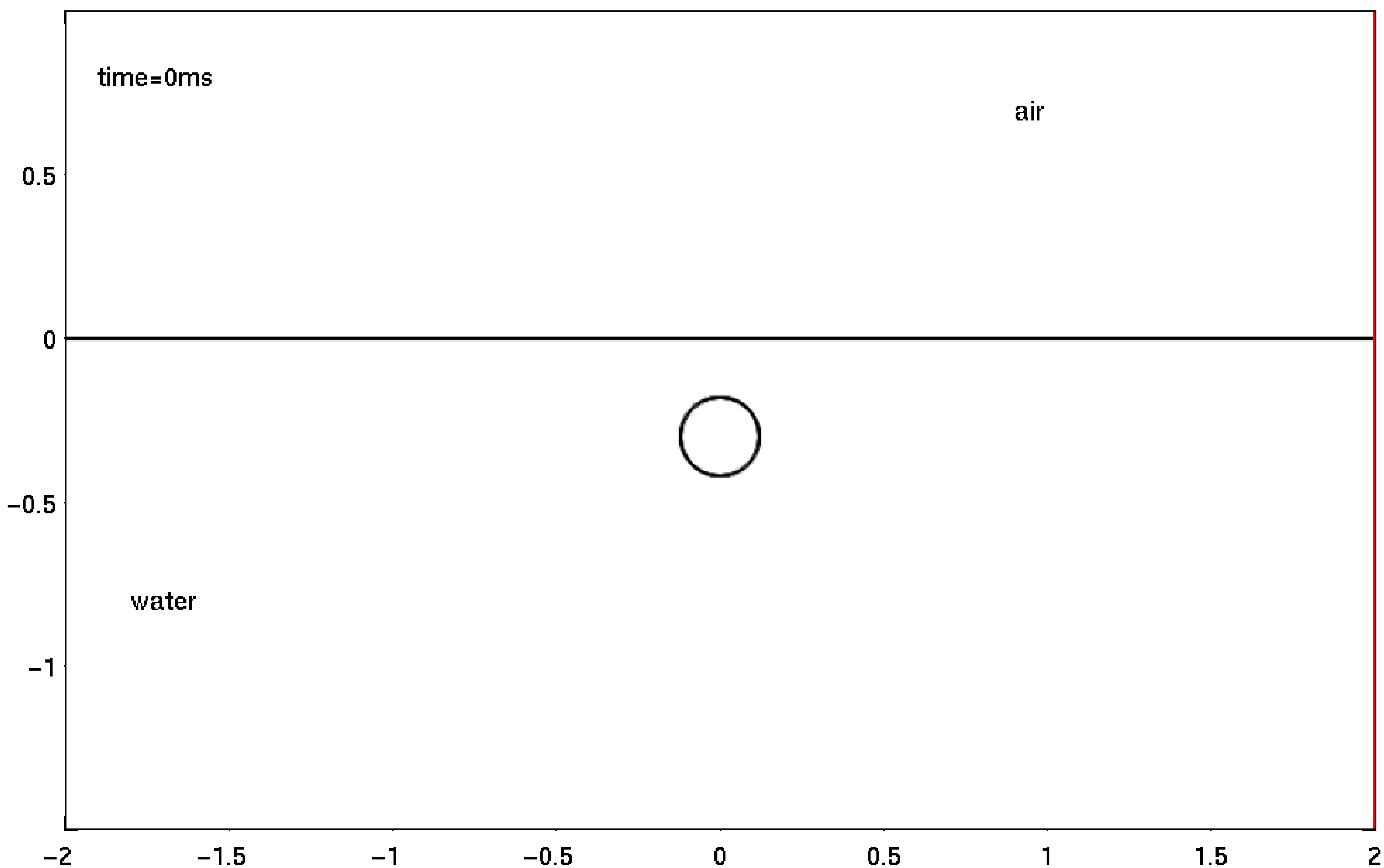


Radially Symmetric Prob. (Cont.)



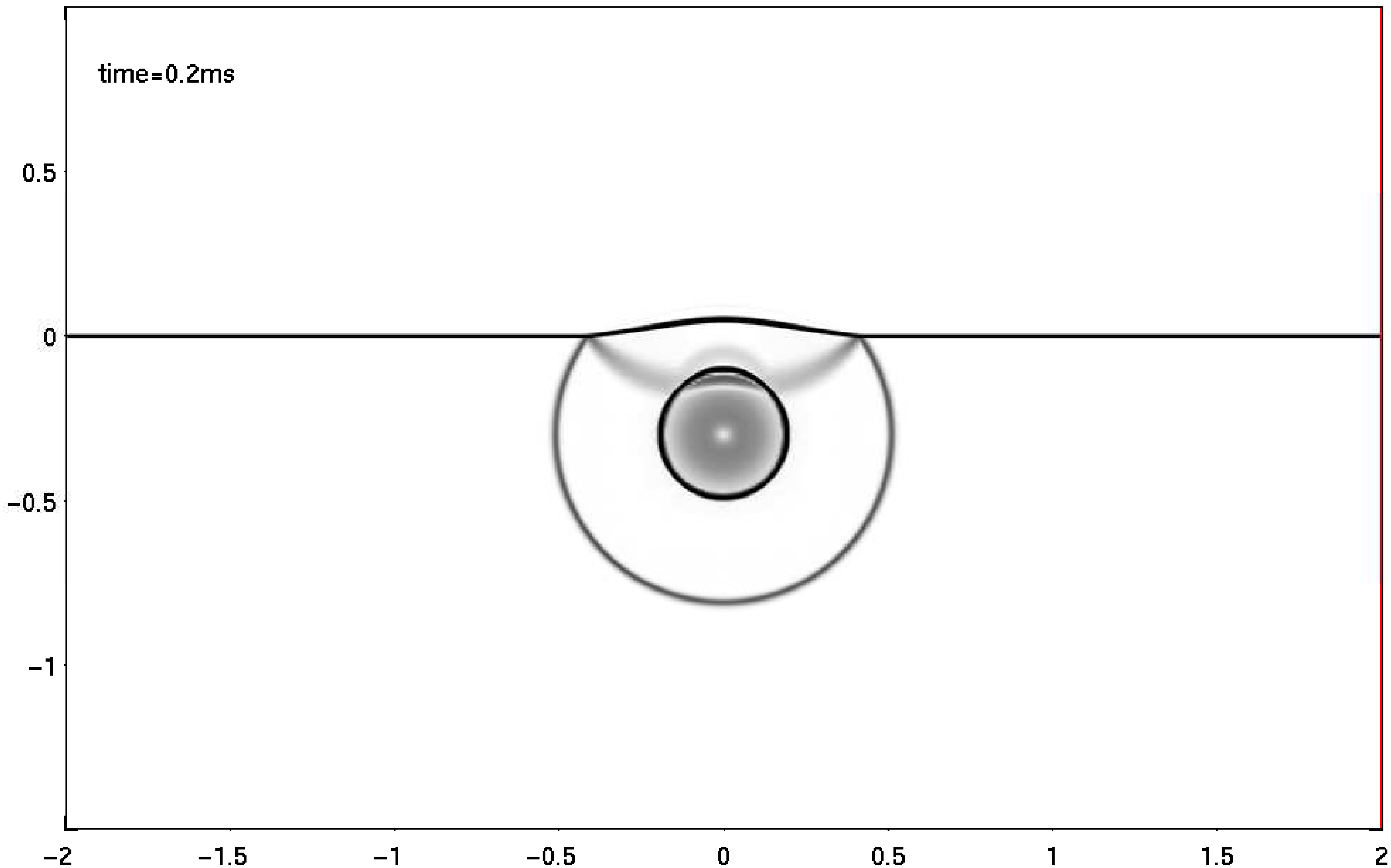
Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



Underwater Explosions

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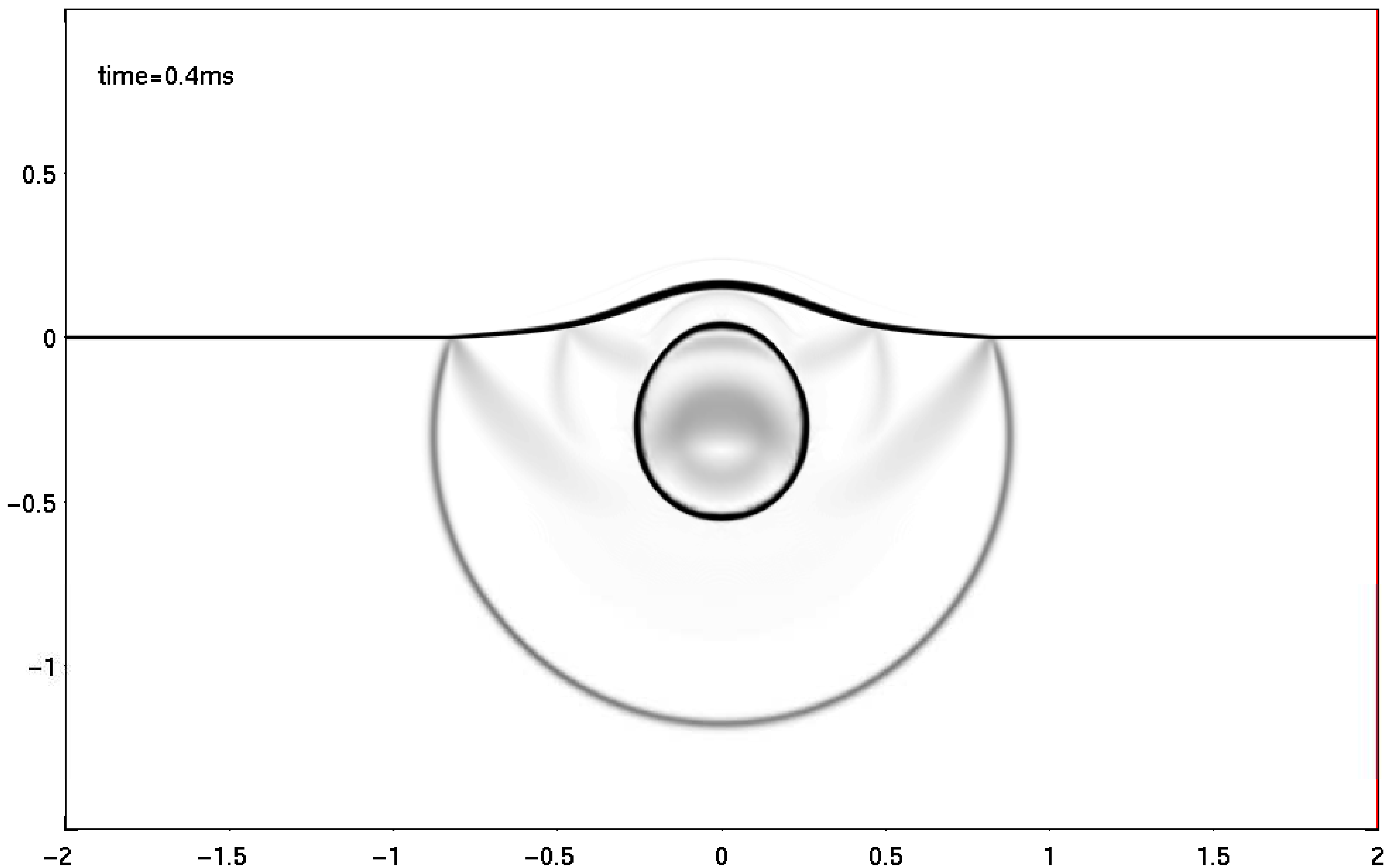
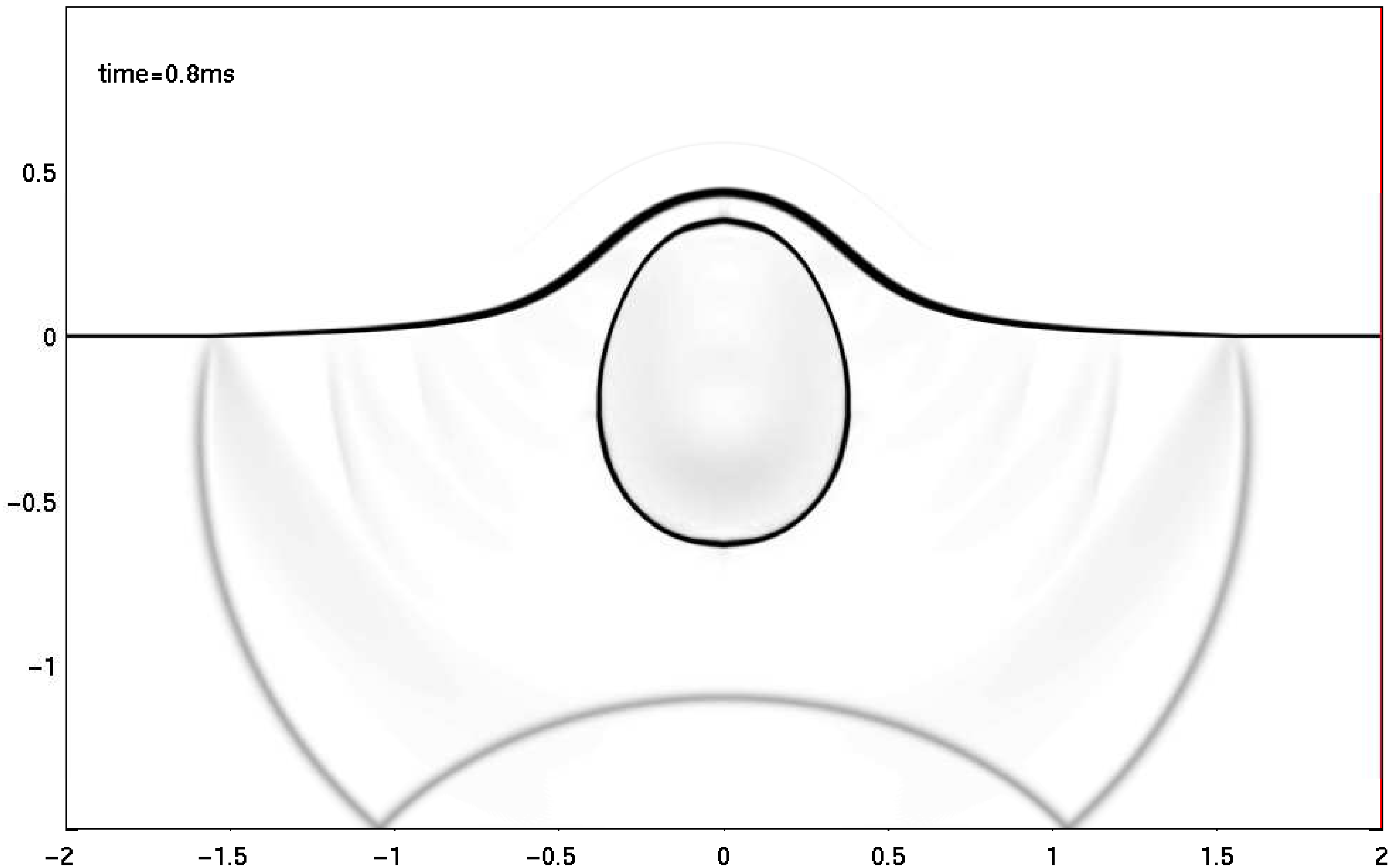


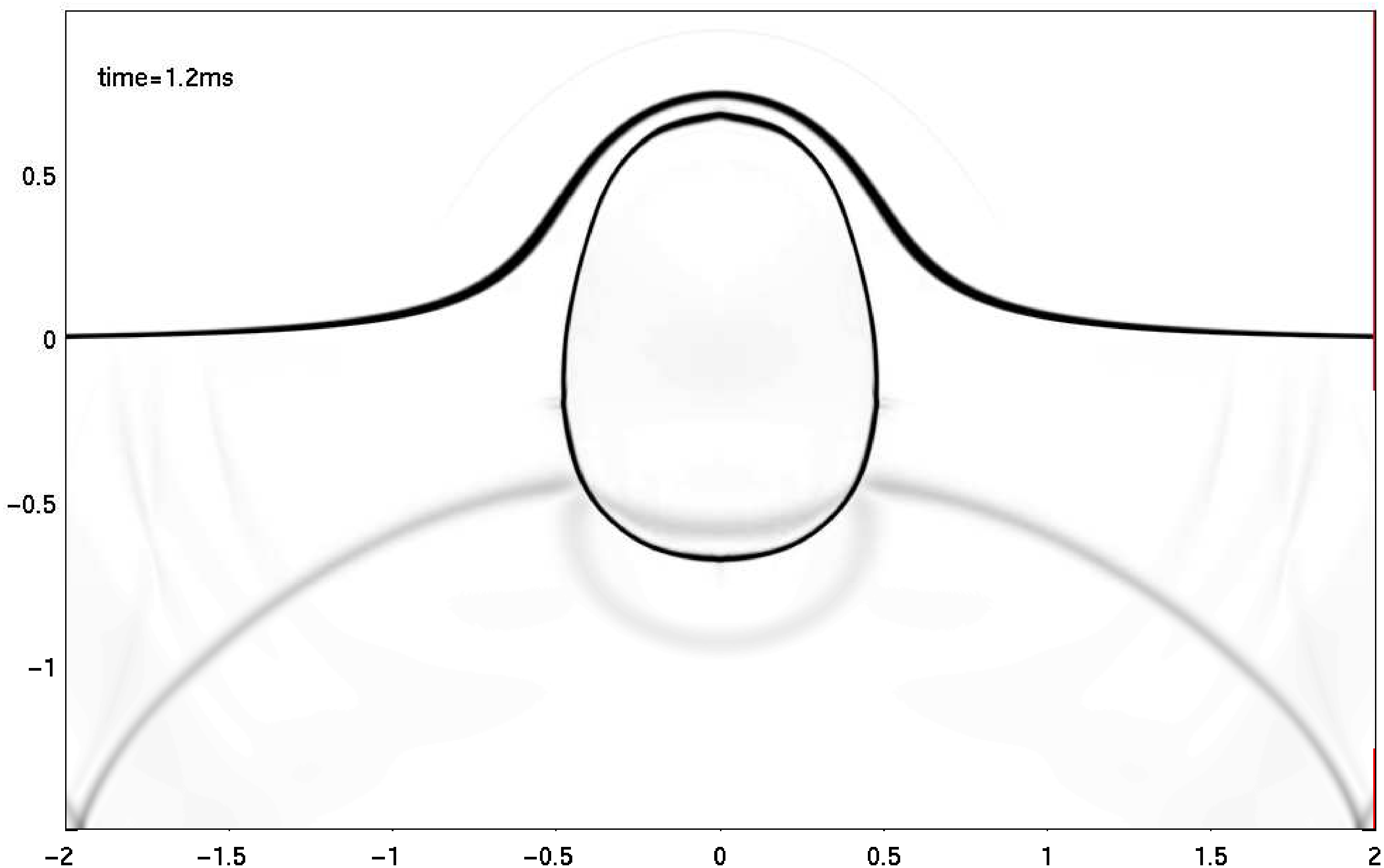
Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid

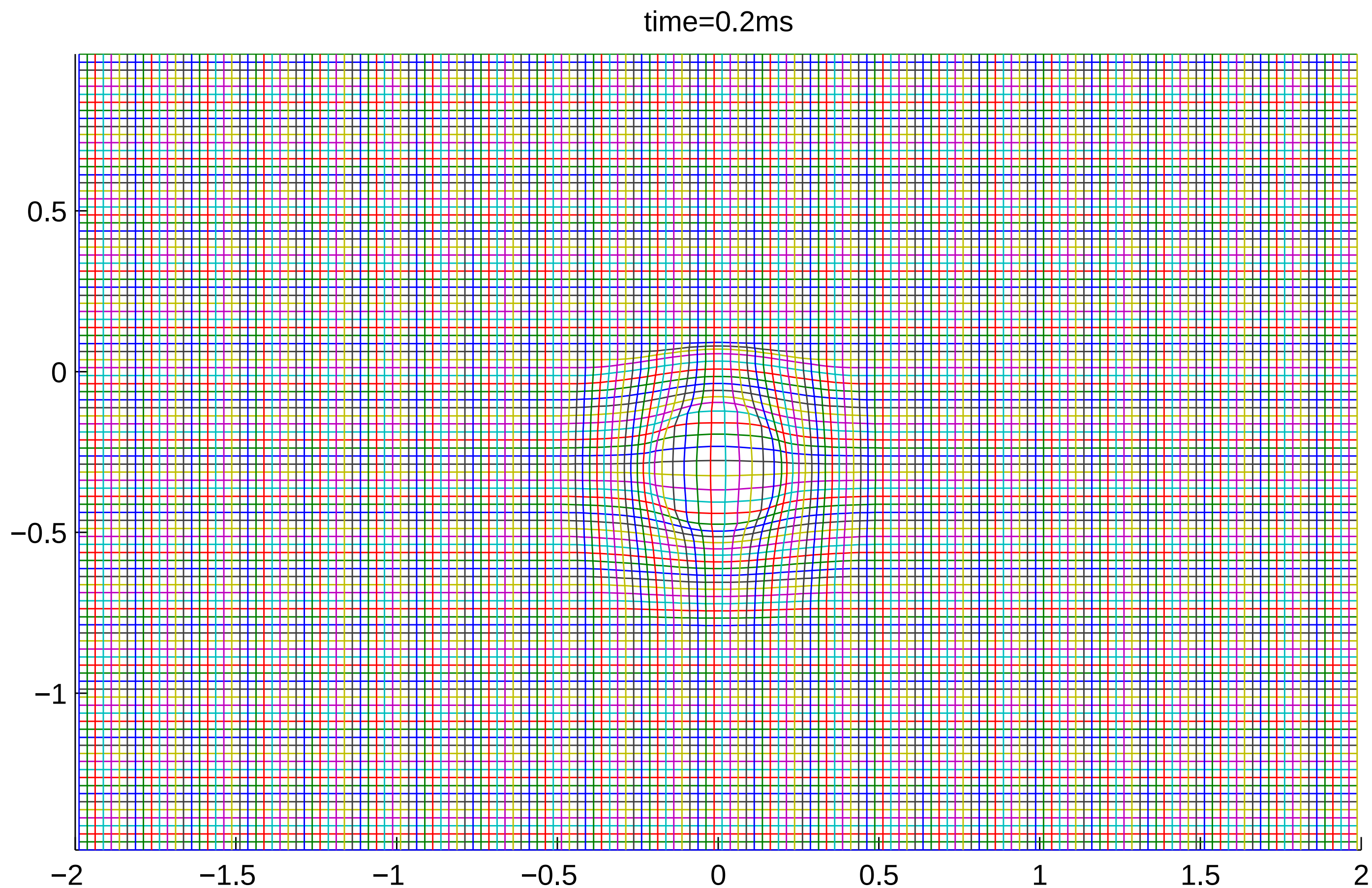


Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$

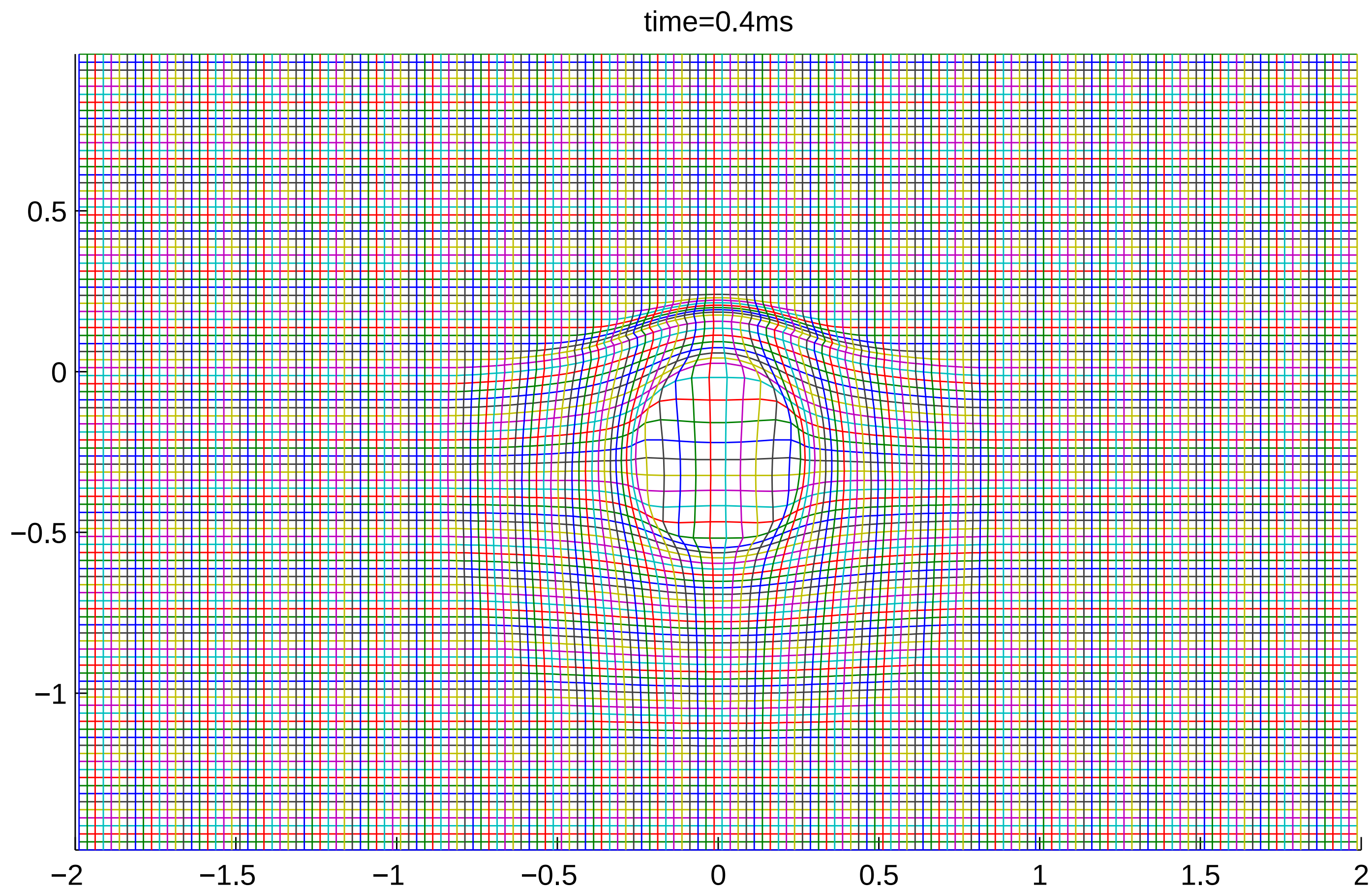
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



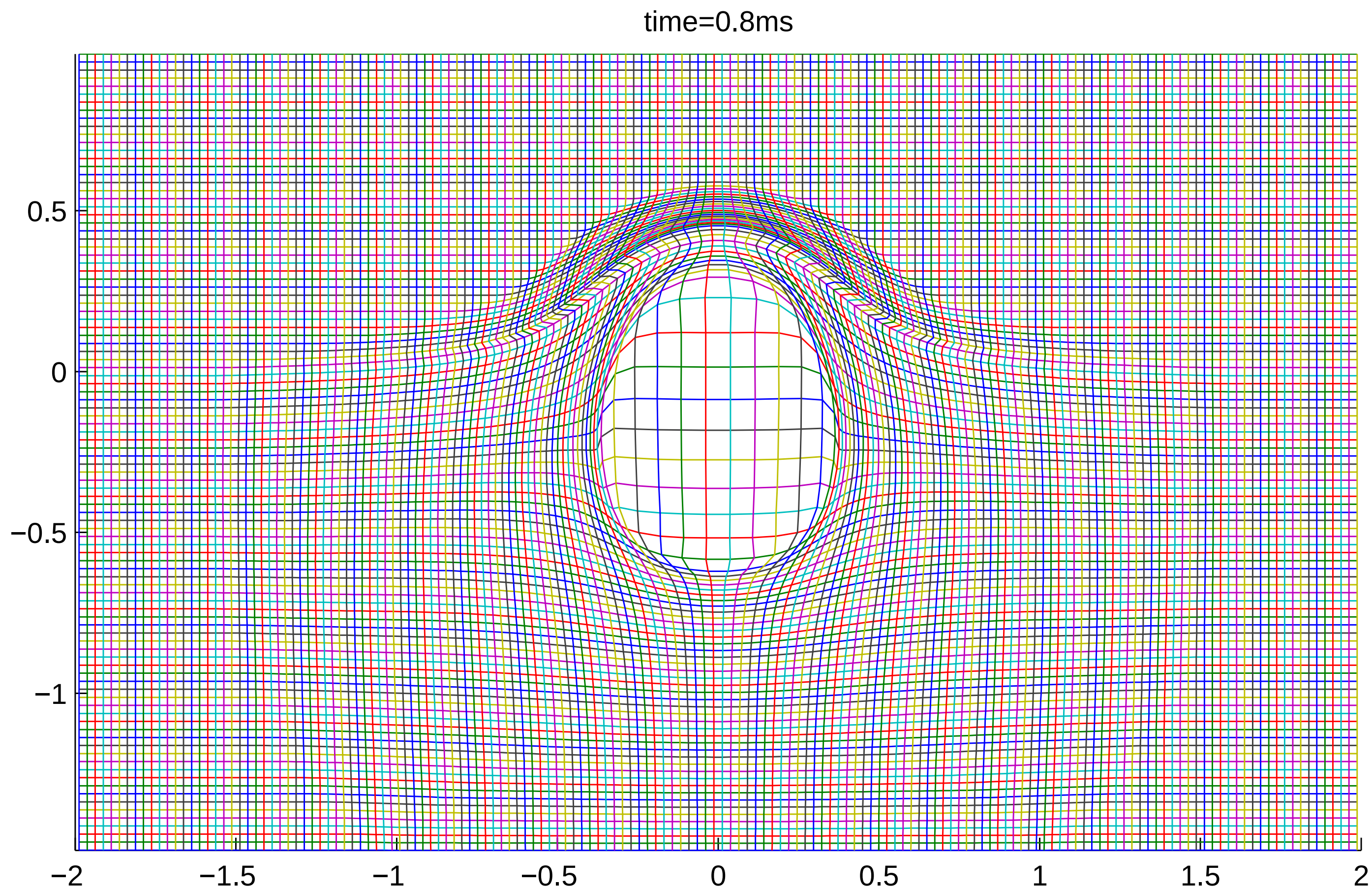
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



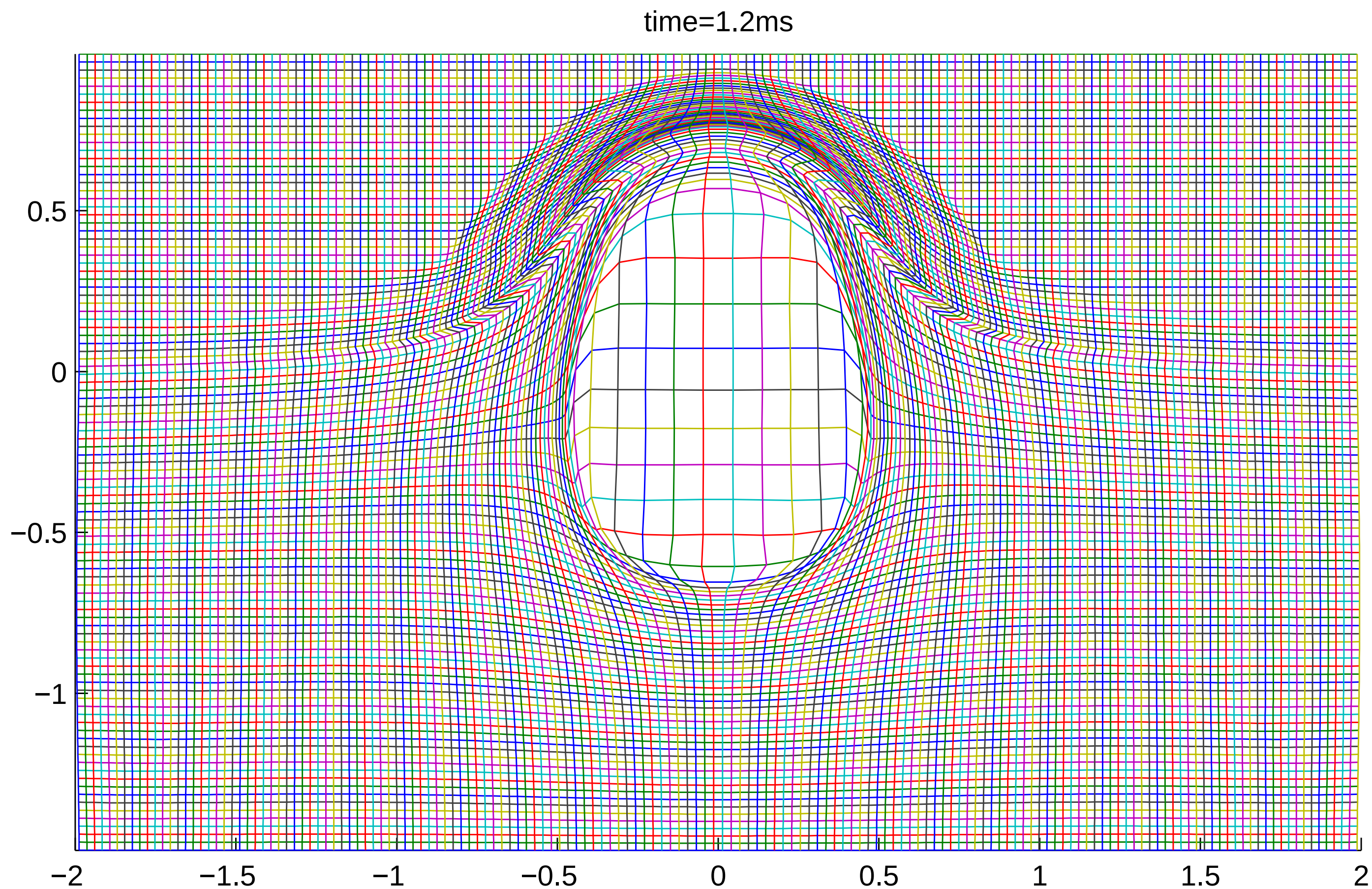
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



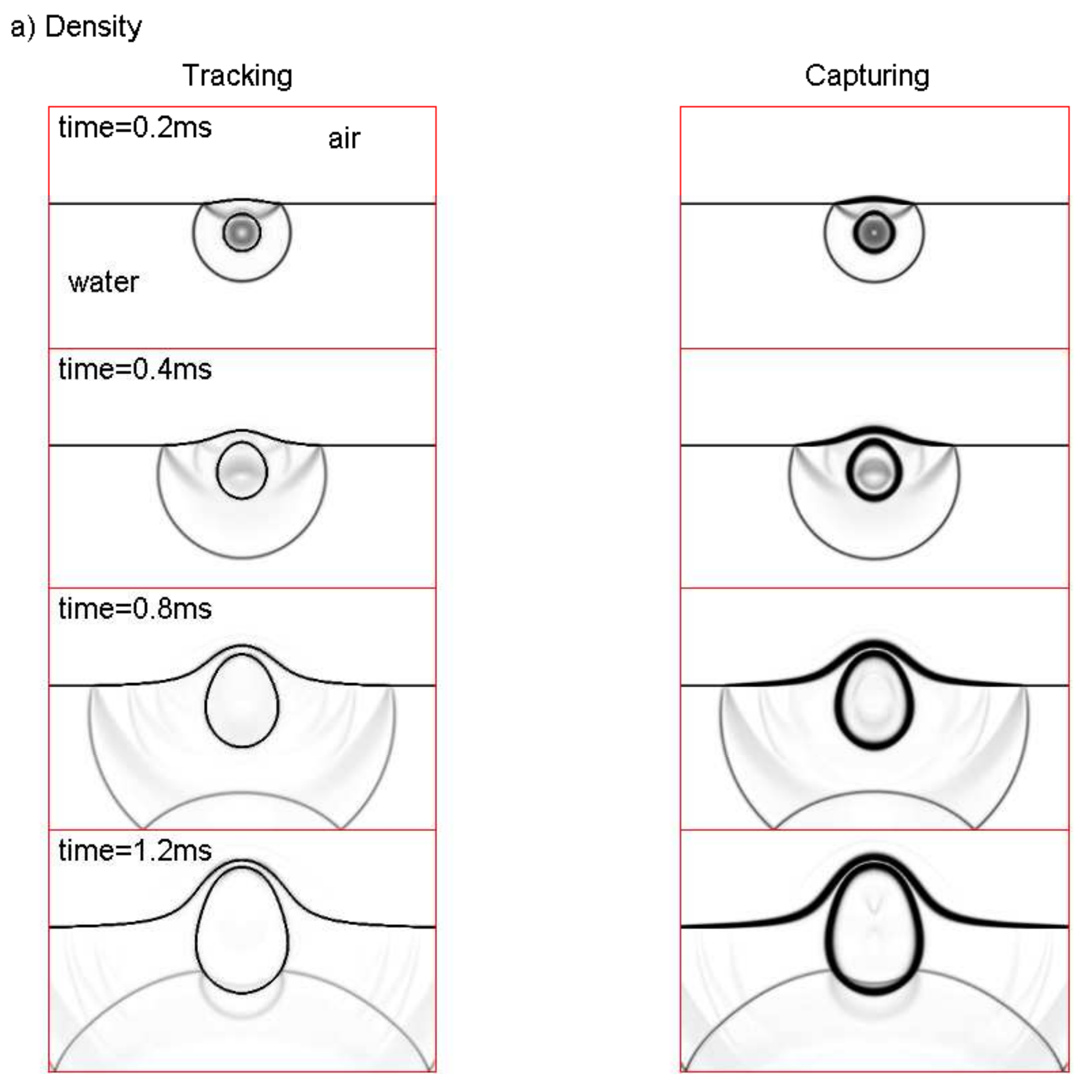
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



Underwater Explosions (Cont.)

- Volume tracking & interface capturing results

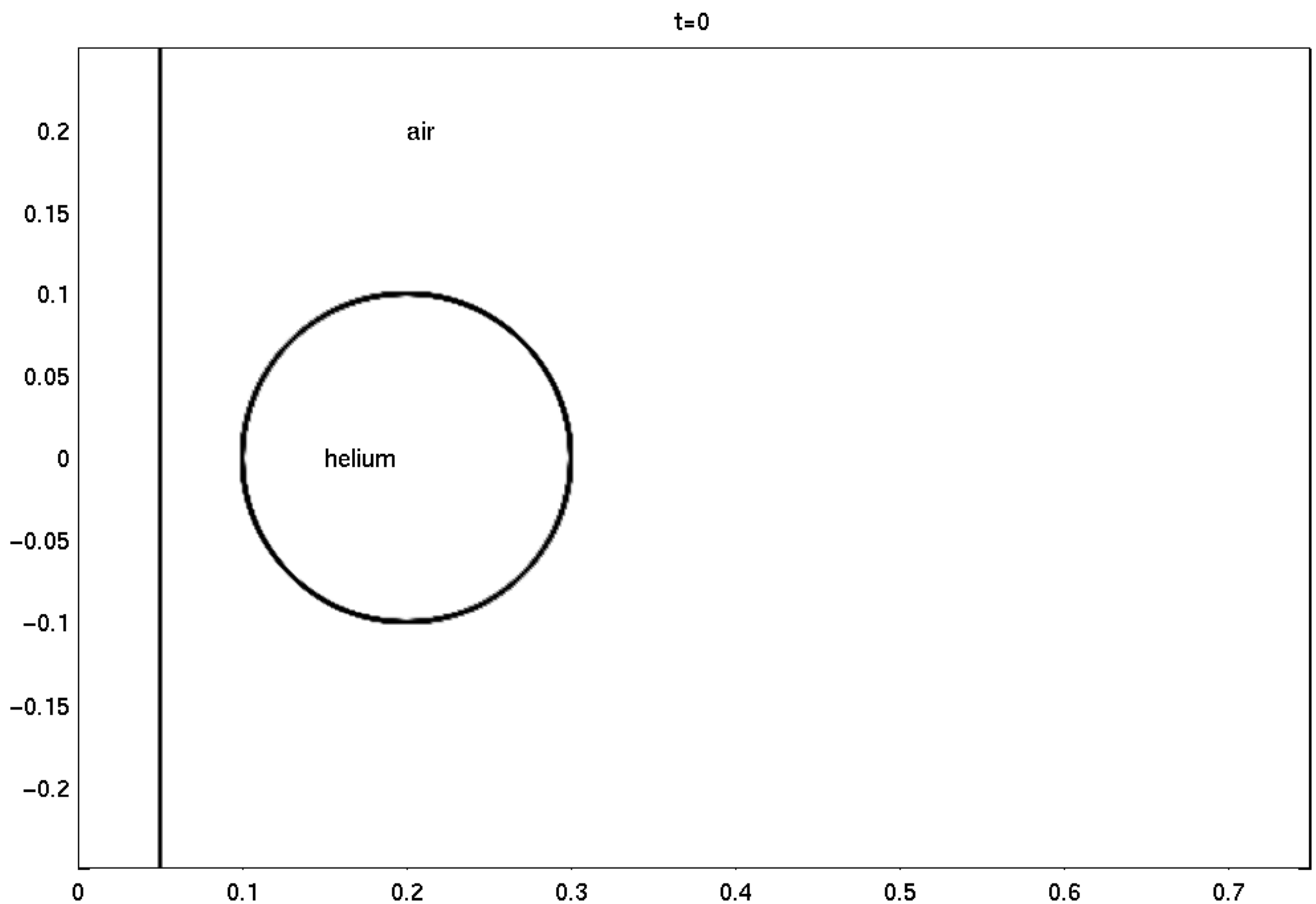


Underwater Explosions (Cont.)

- Generalized curvilinear grid: single bubble animation
- Cartesian grid: multiple bubble animation

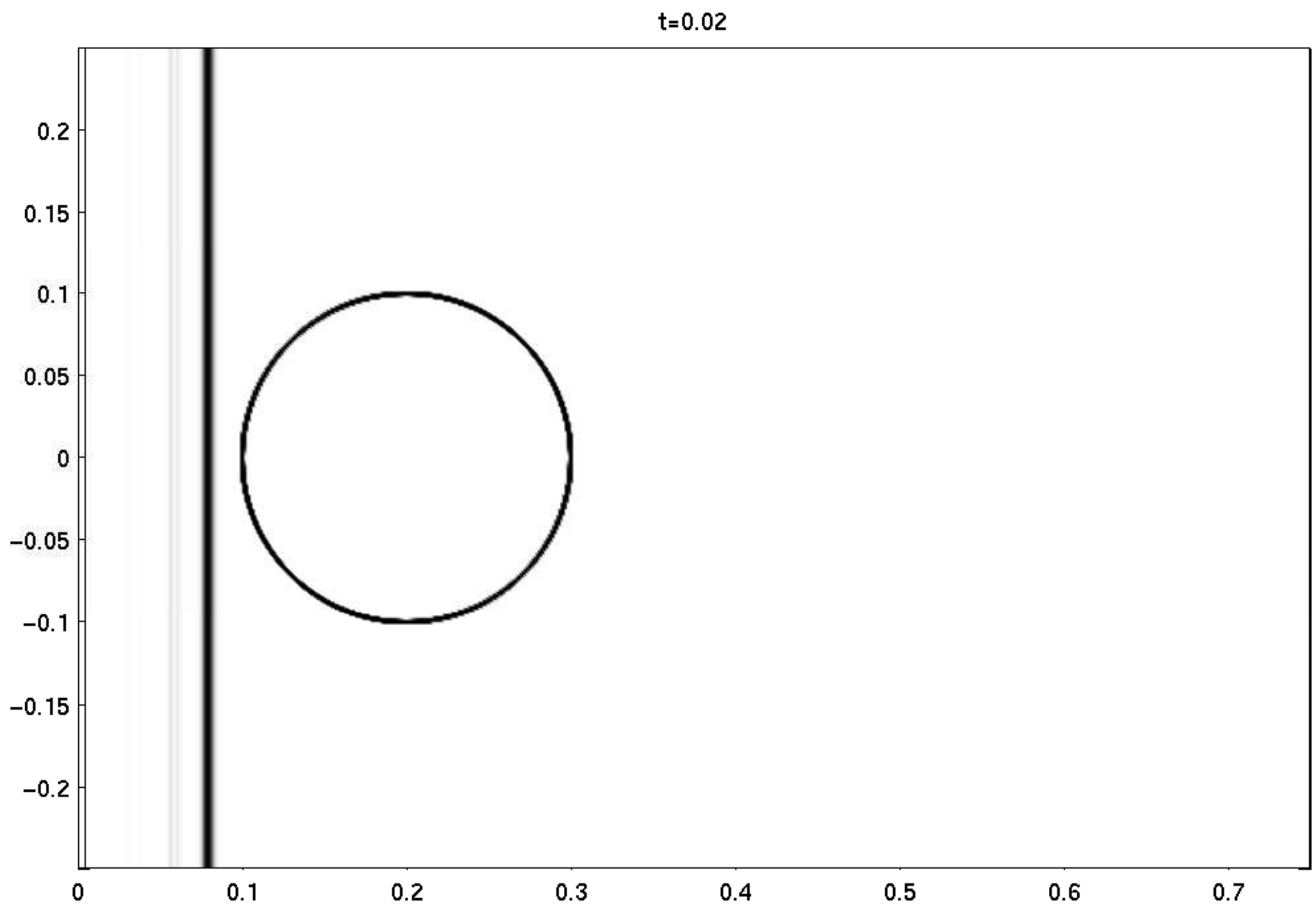
Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



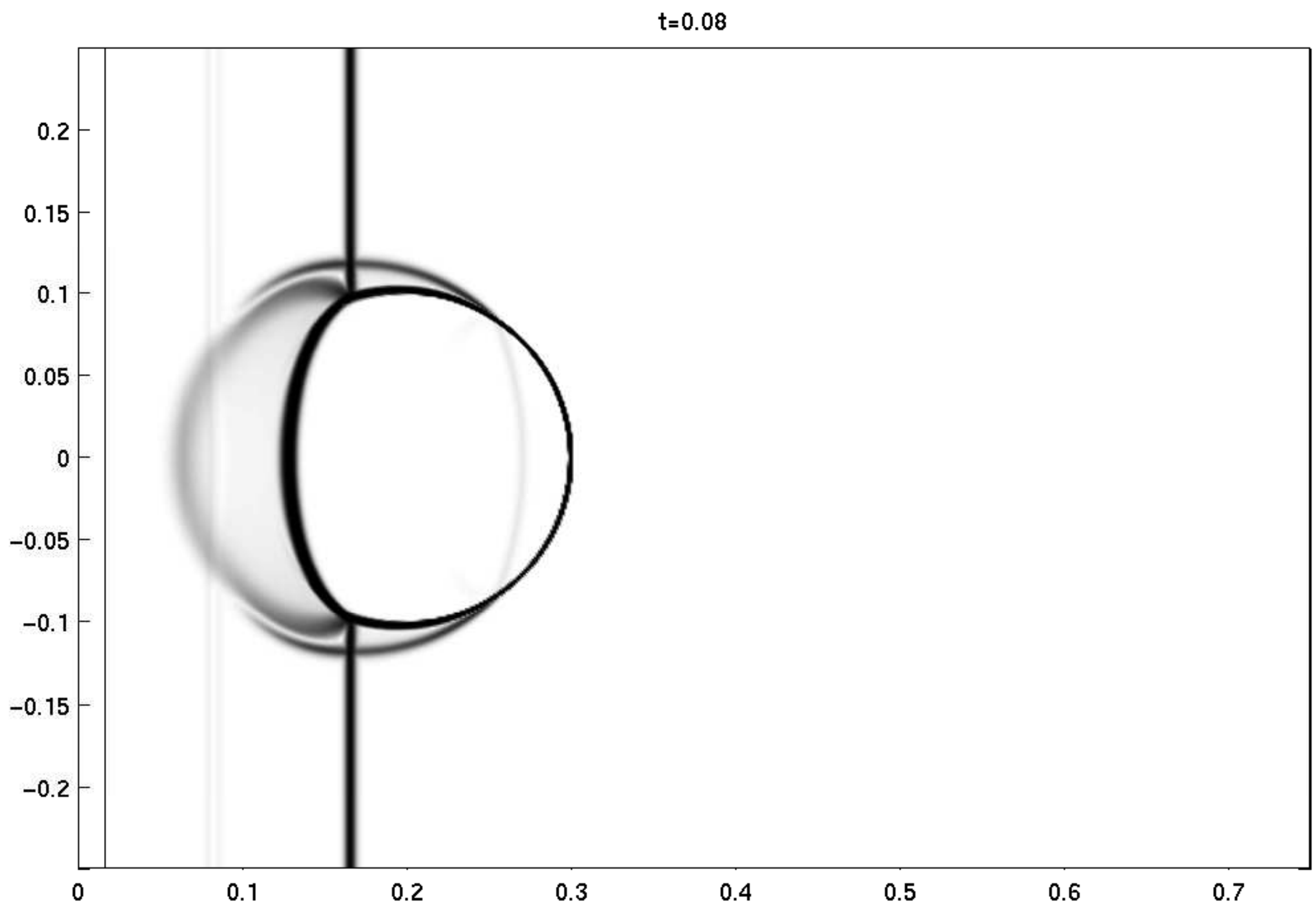
Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



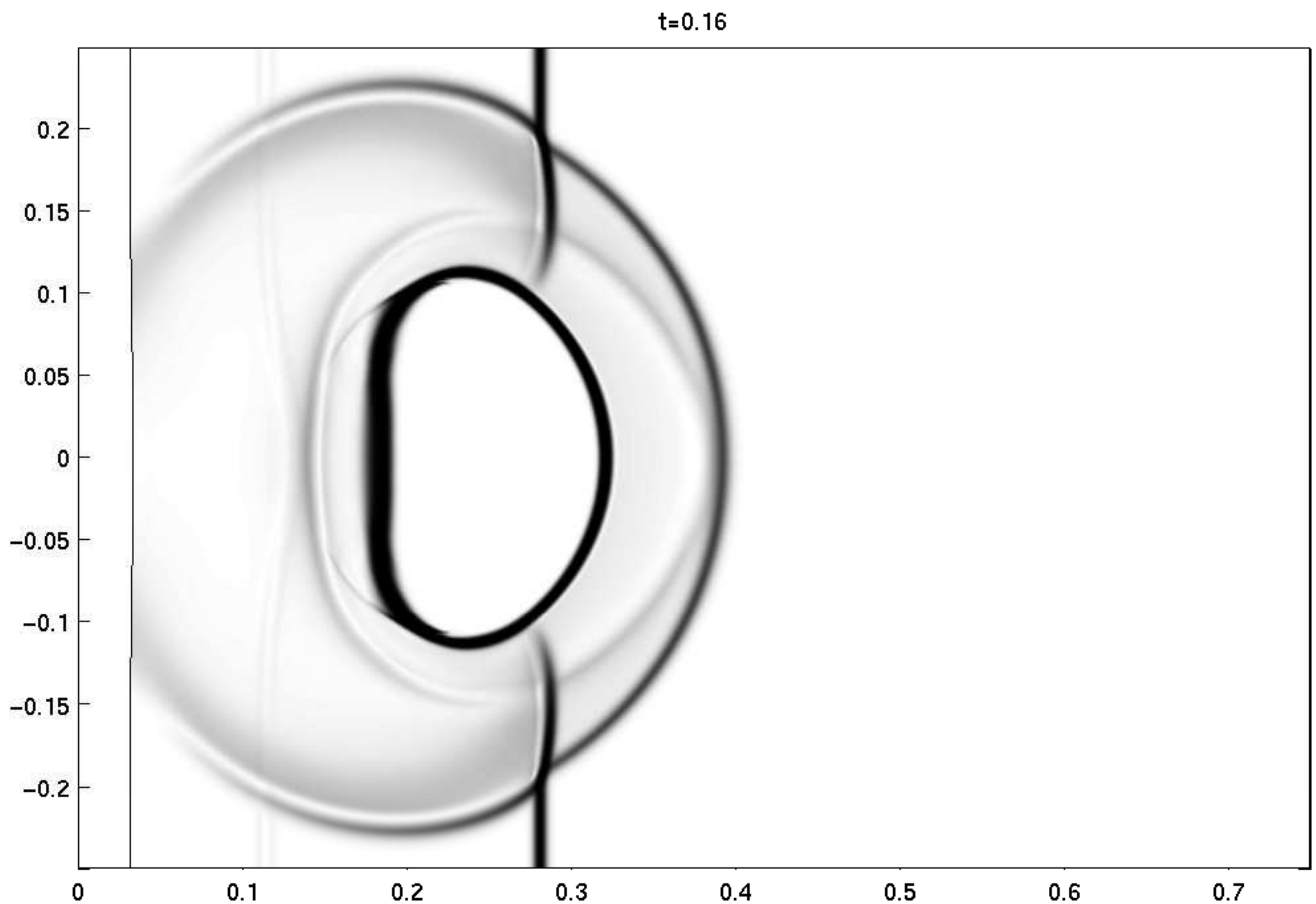
Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



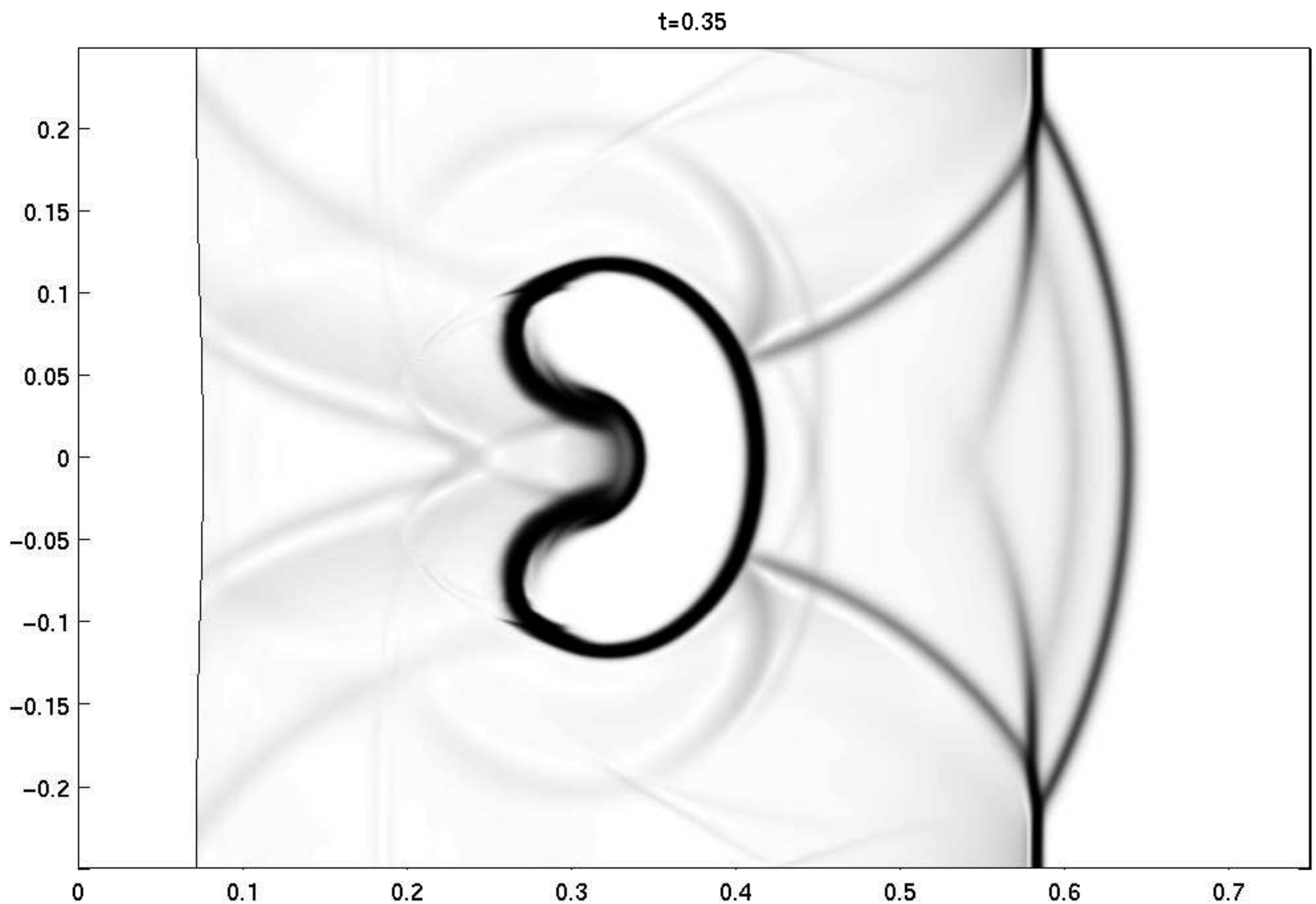
Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Helium) (Cont.)

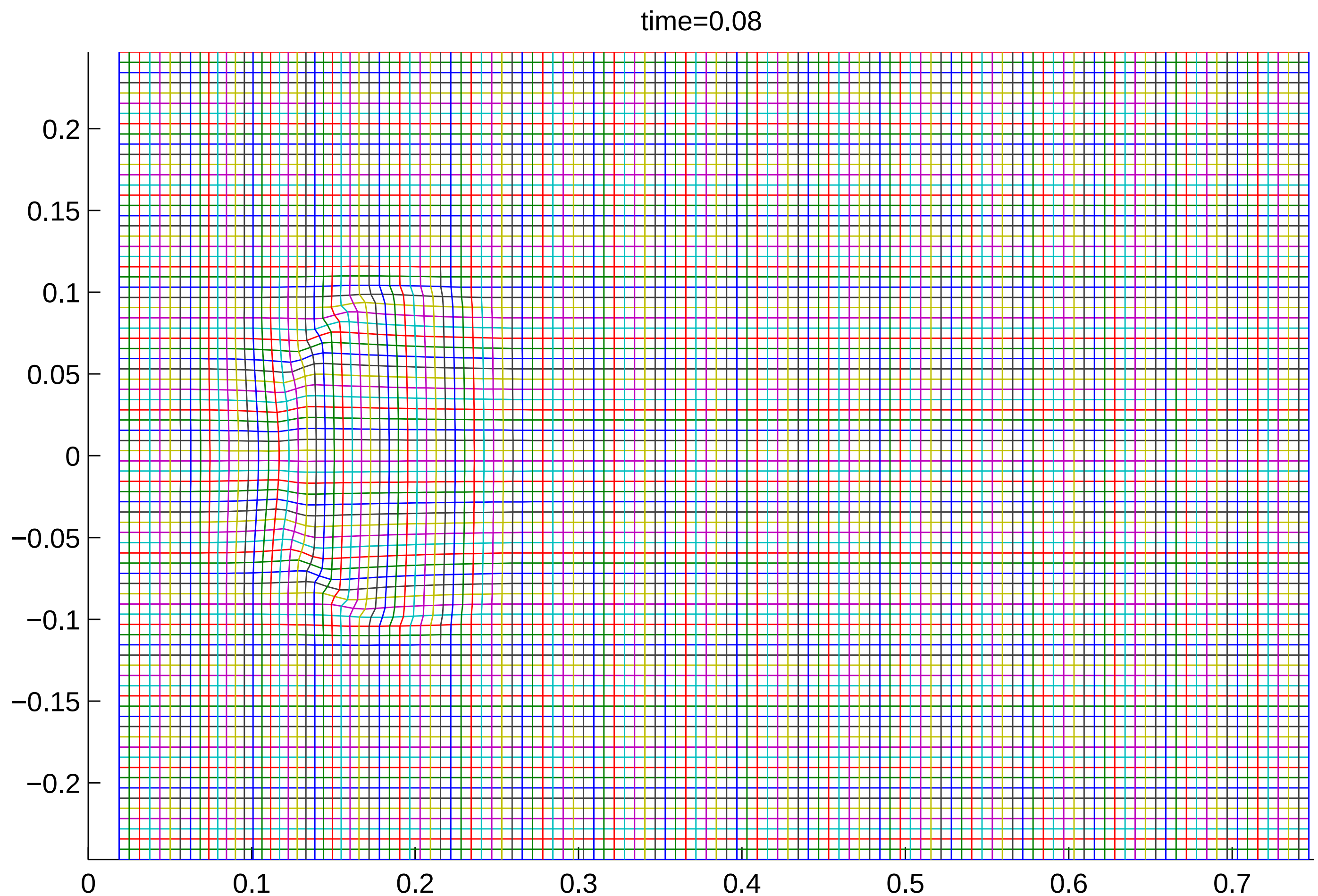
- Grid system (coarsen by factor 5) with $h_0 = 0.5$

Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$

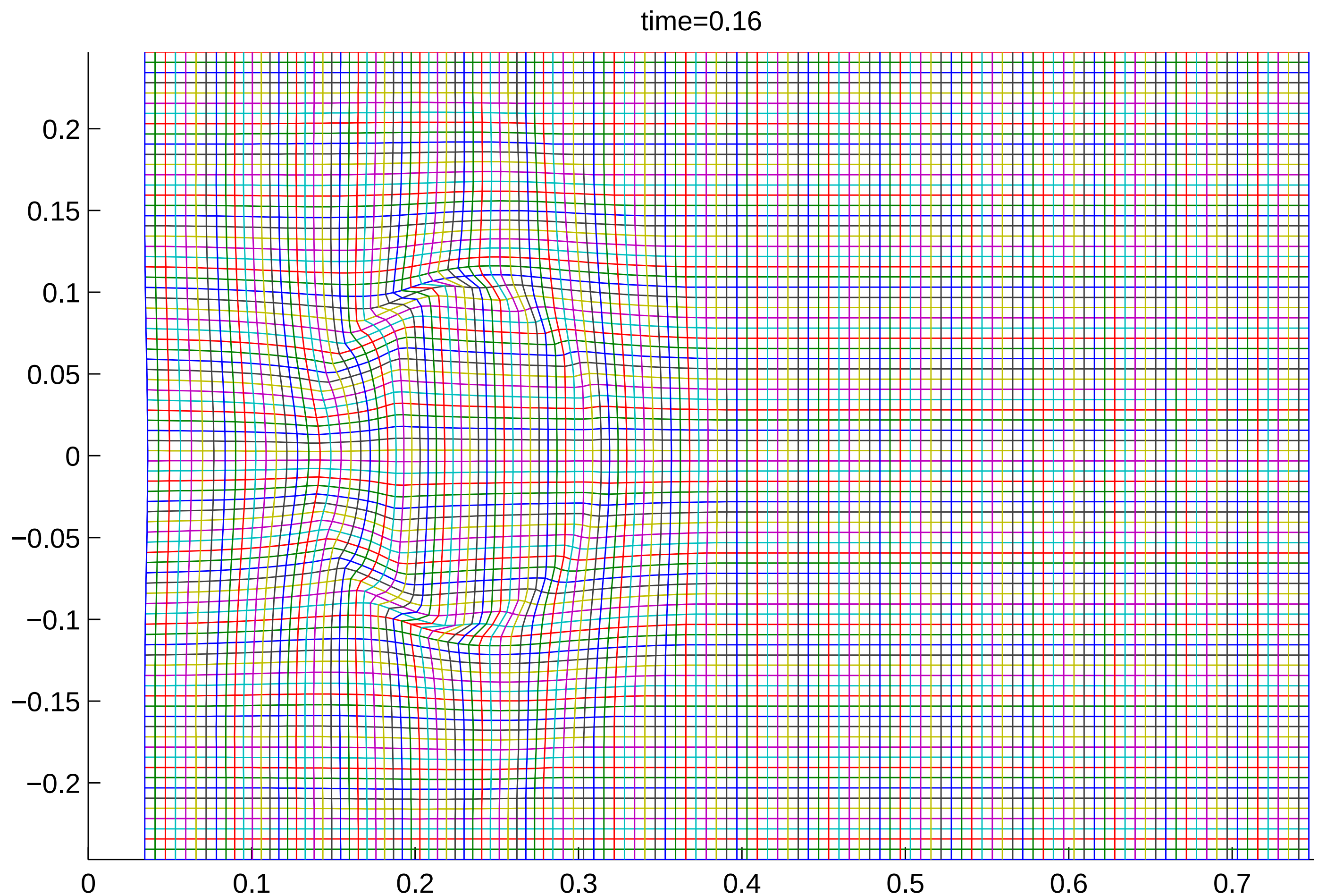
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



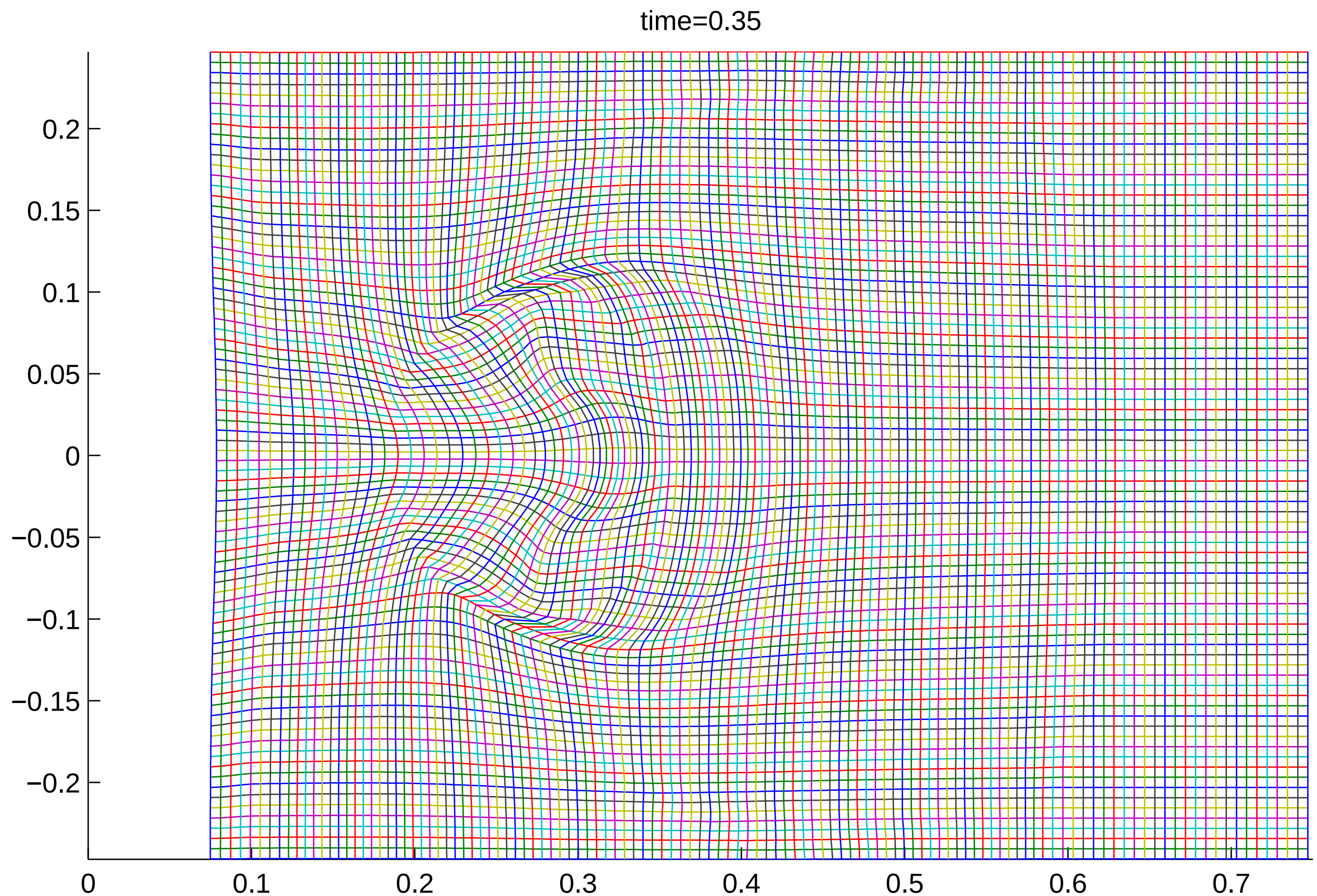
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



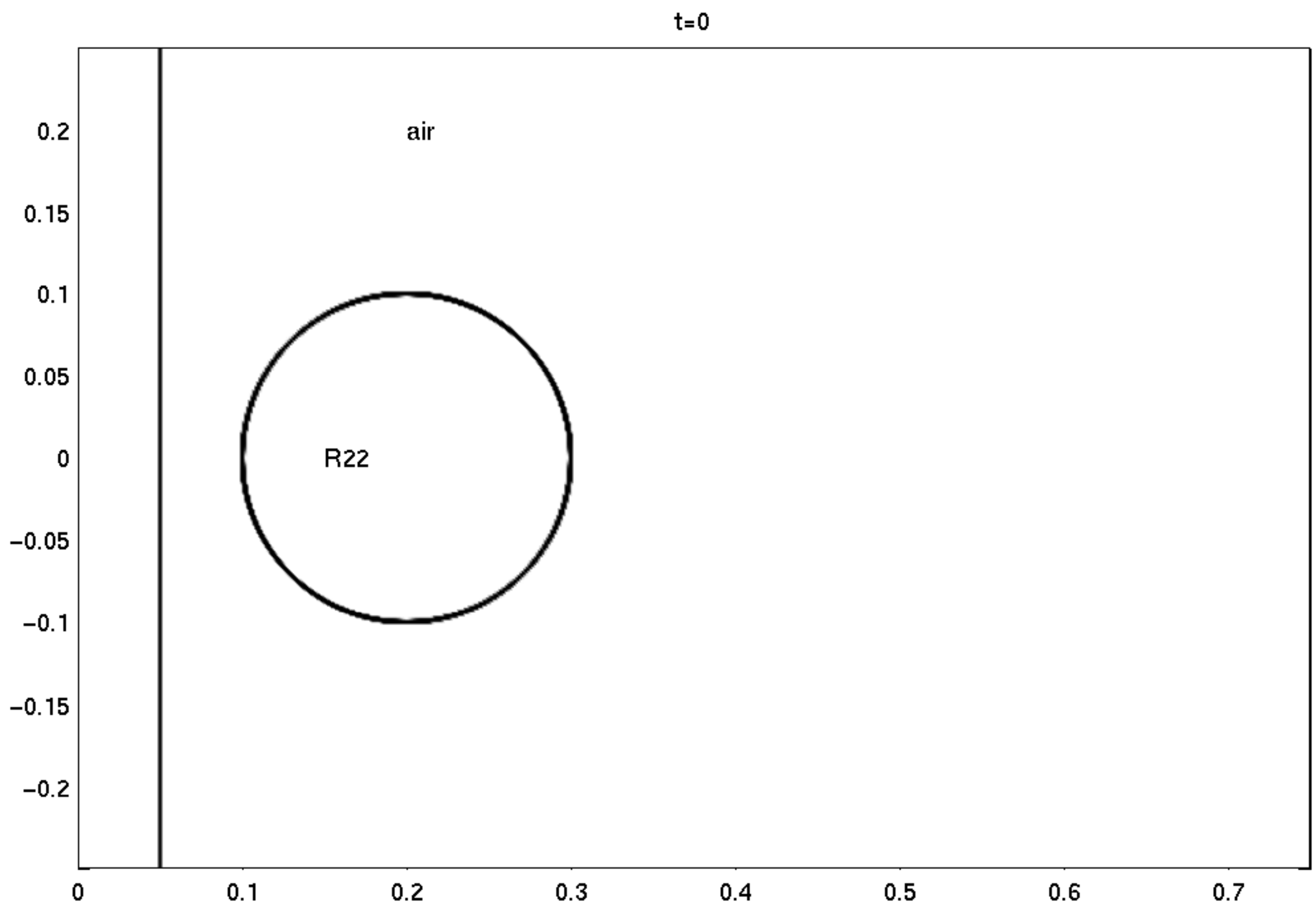
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



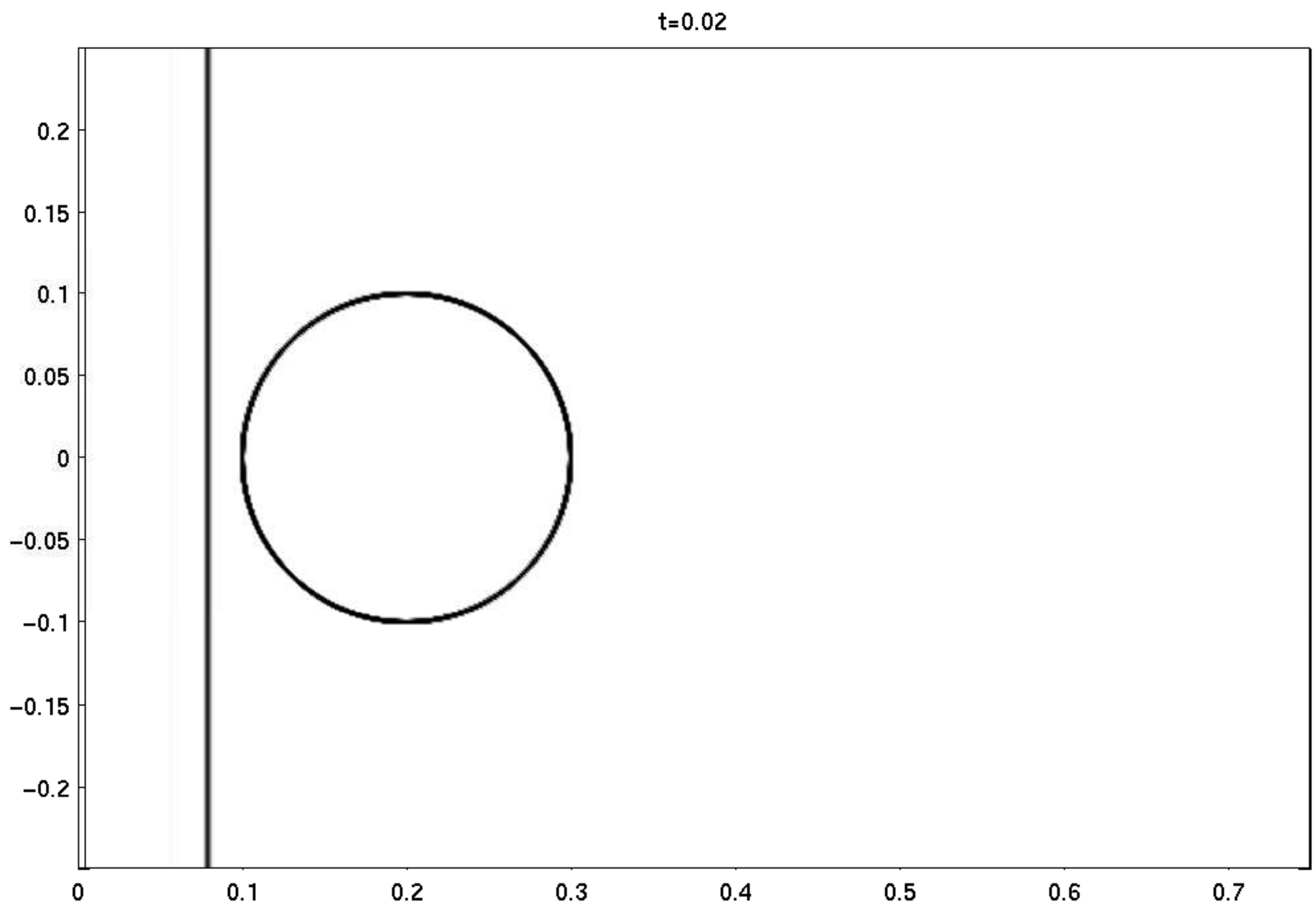
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



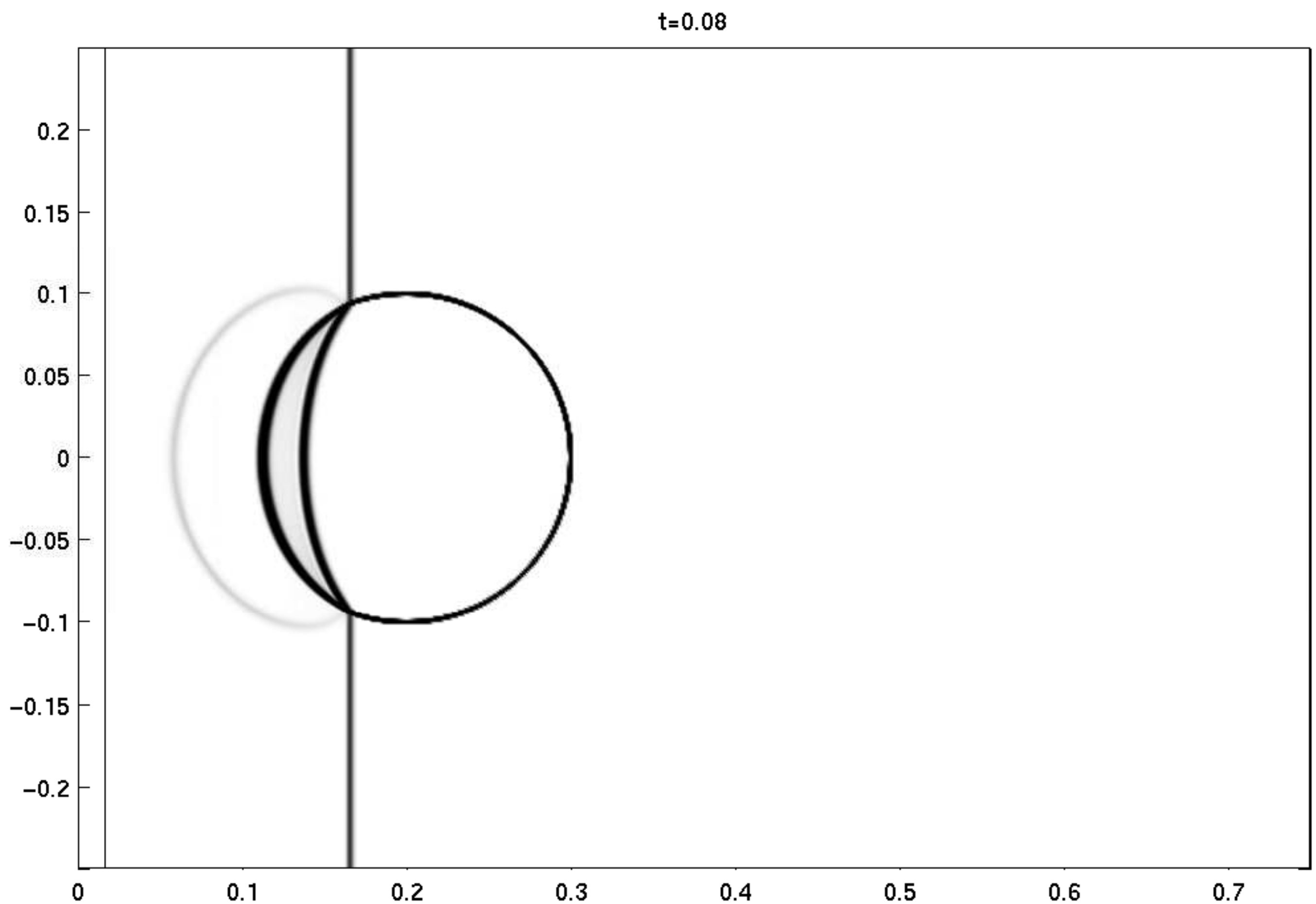
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



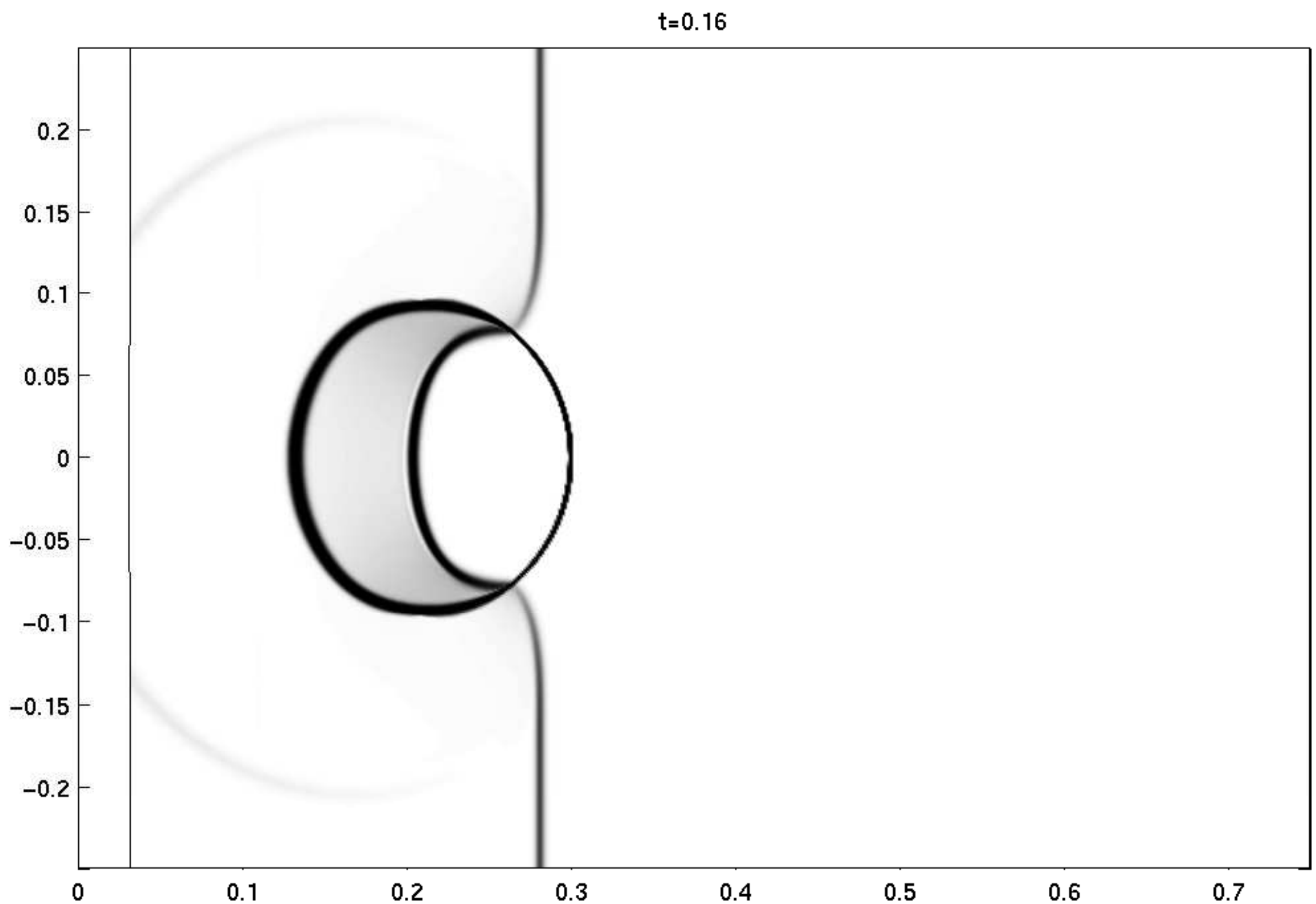
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



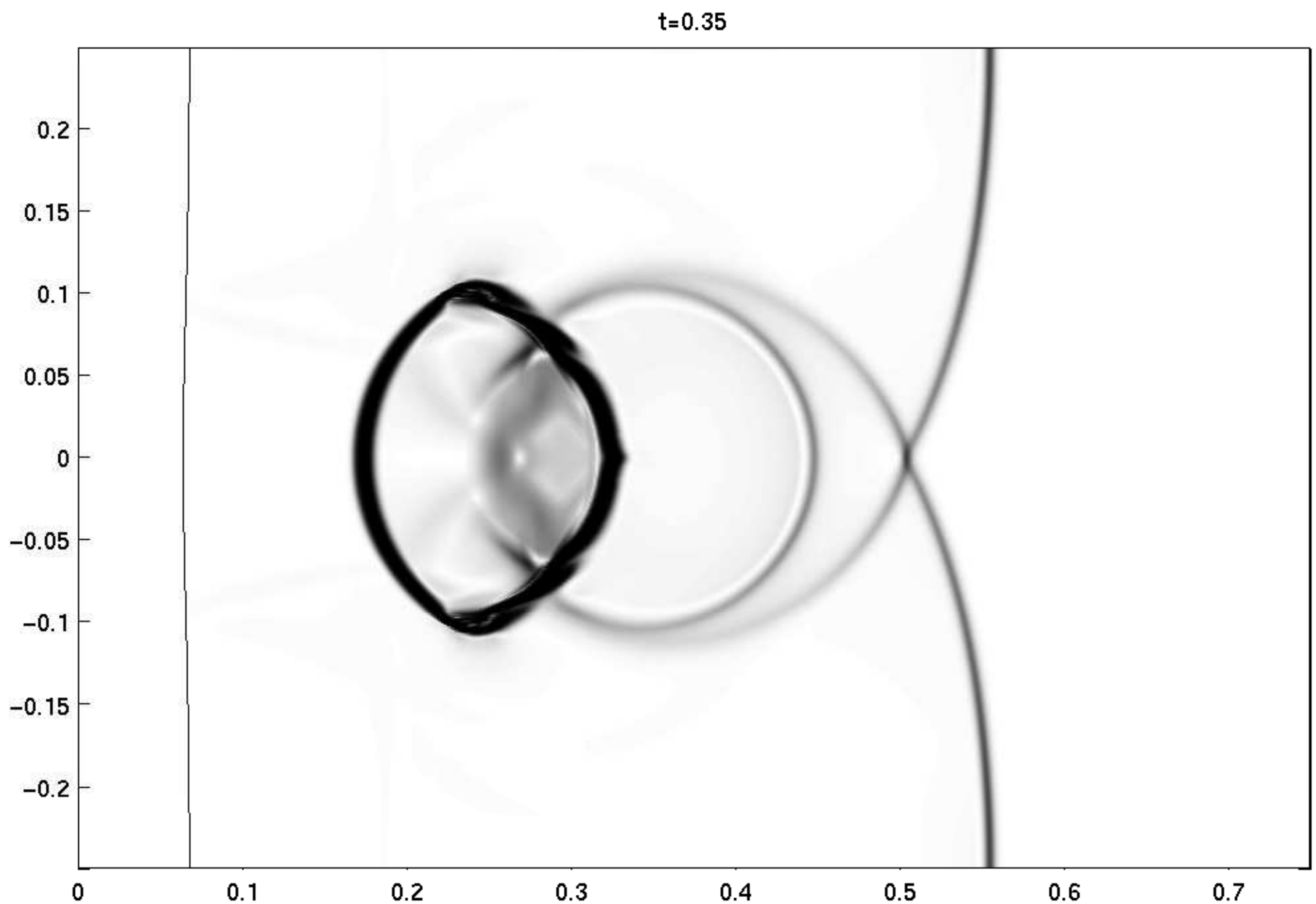
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$

Shock-Bubble (R22) (Cont.)

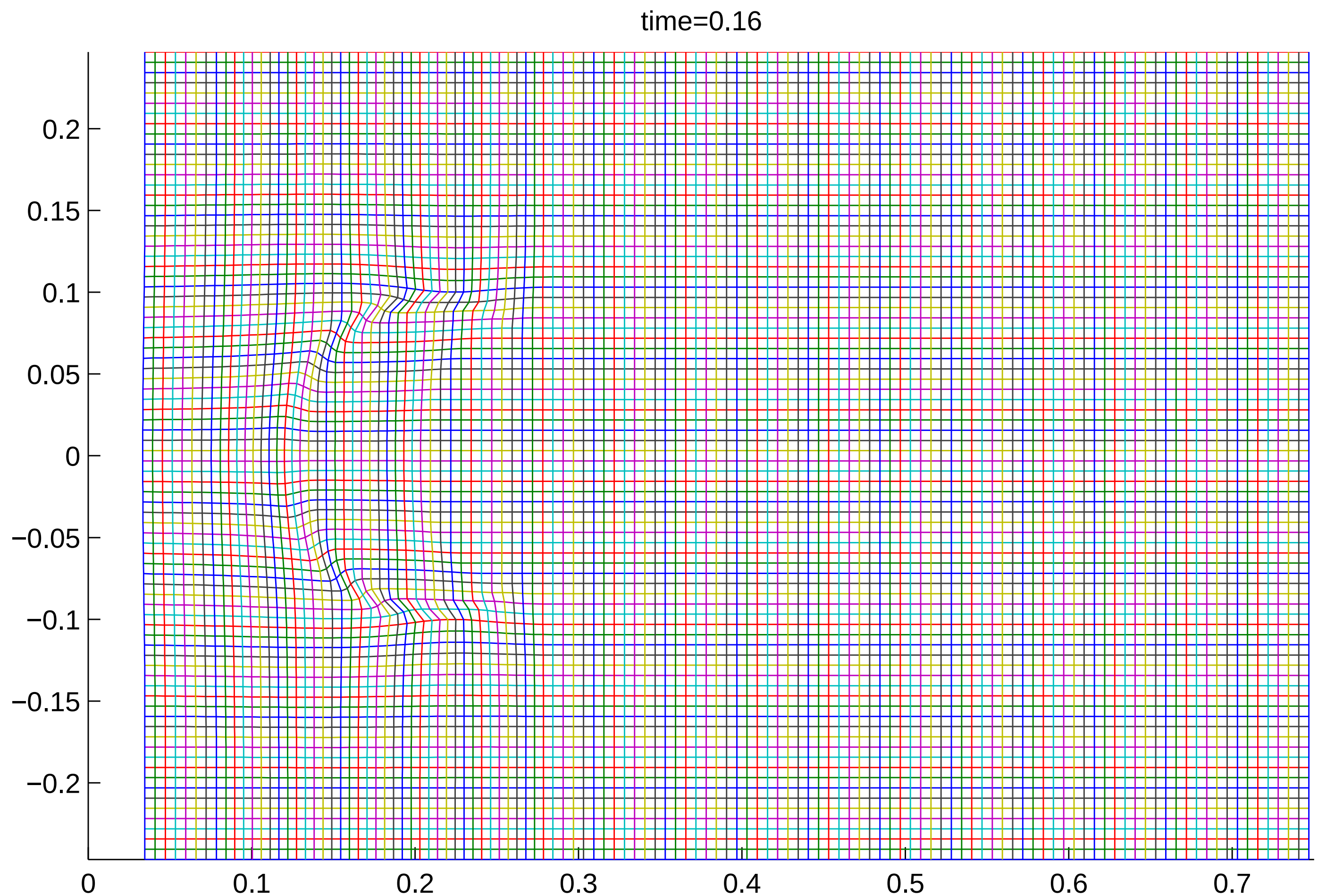
- Grid system (coarsen by factor 5) with $h_0 = 0.5$

Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$

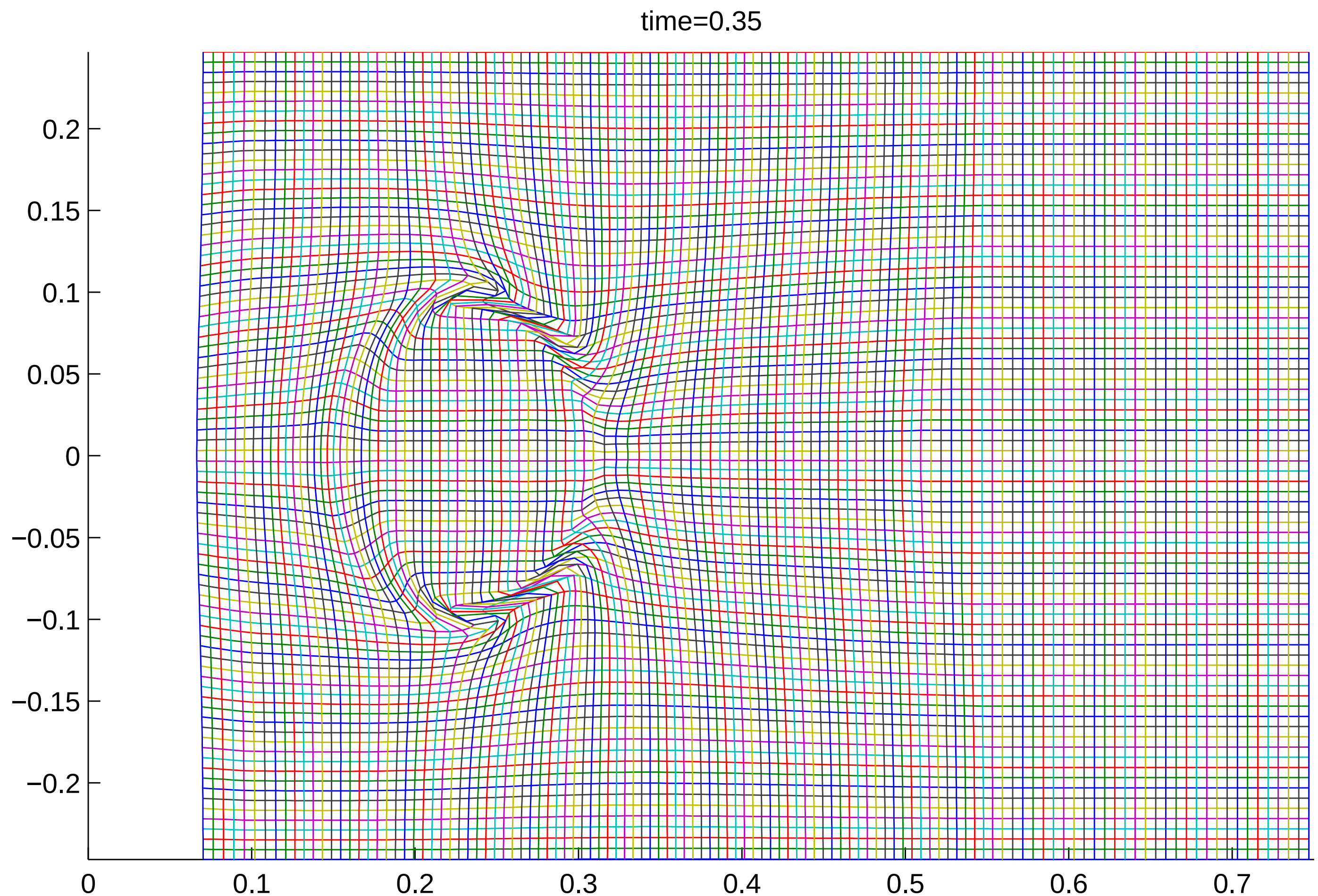
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



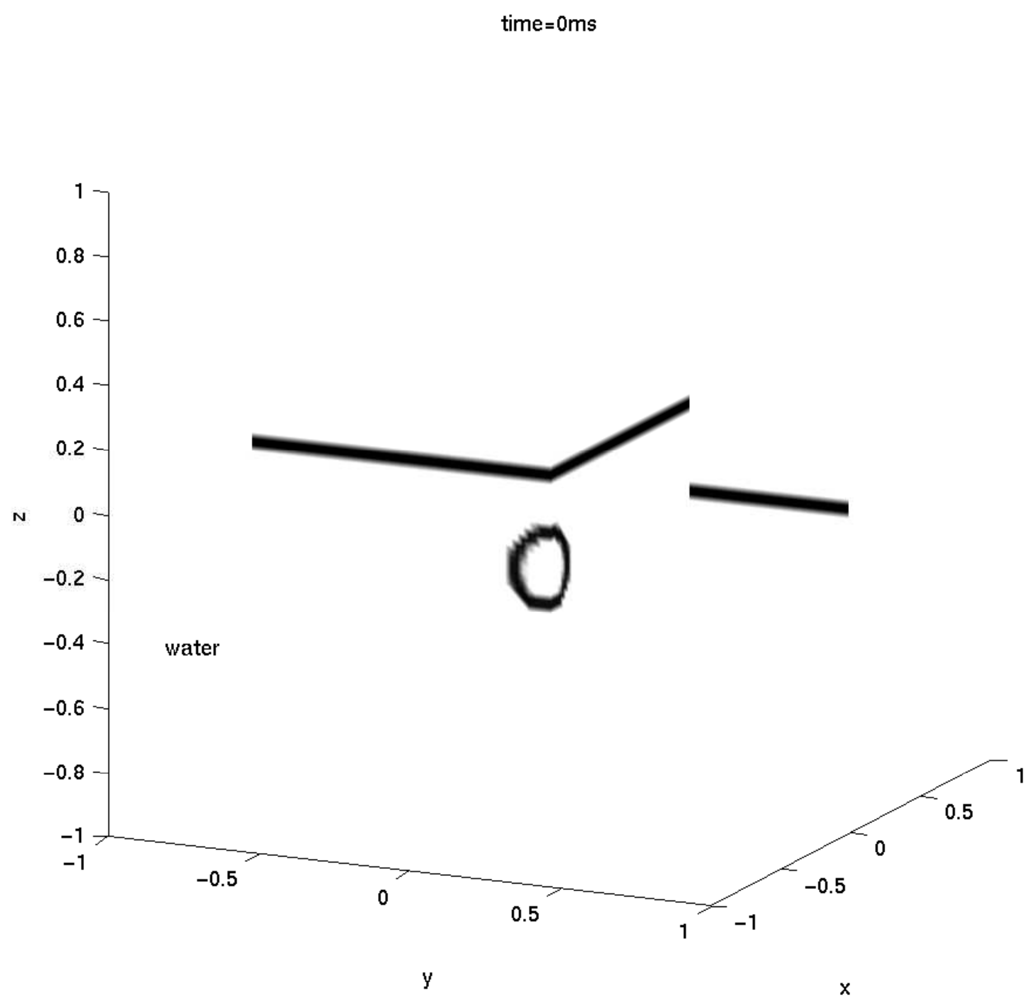
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



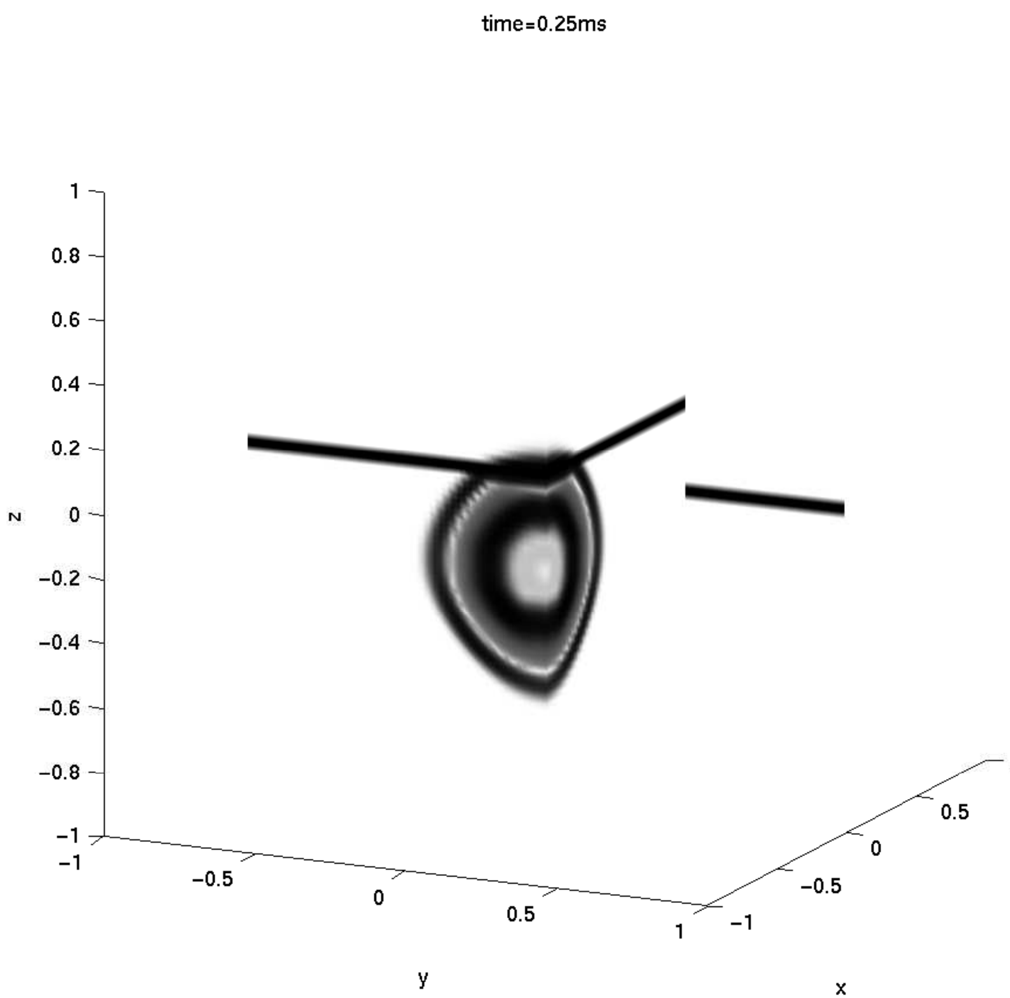
Underwater Explosions

- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



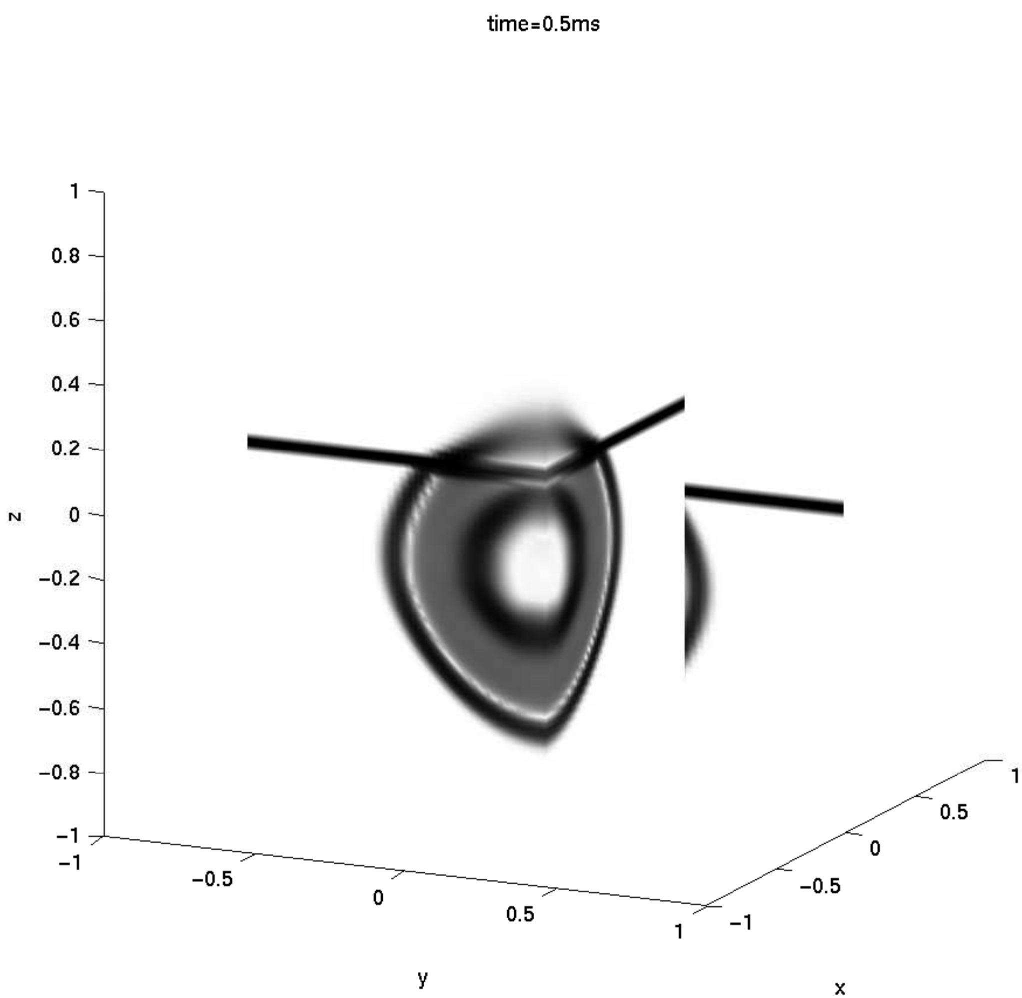
Underwater Explosions

- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



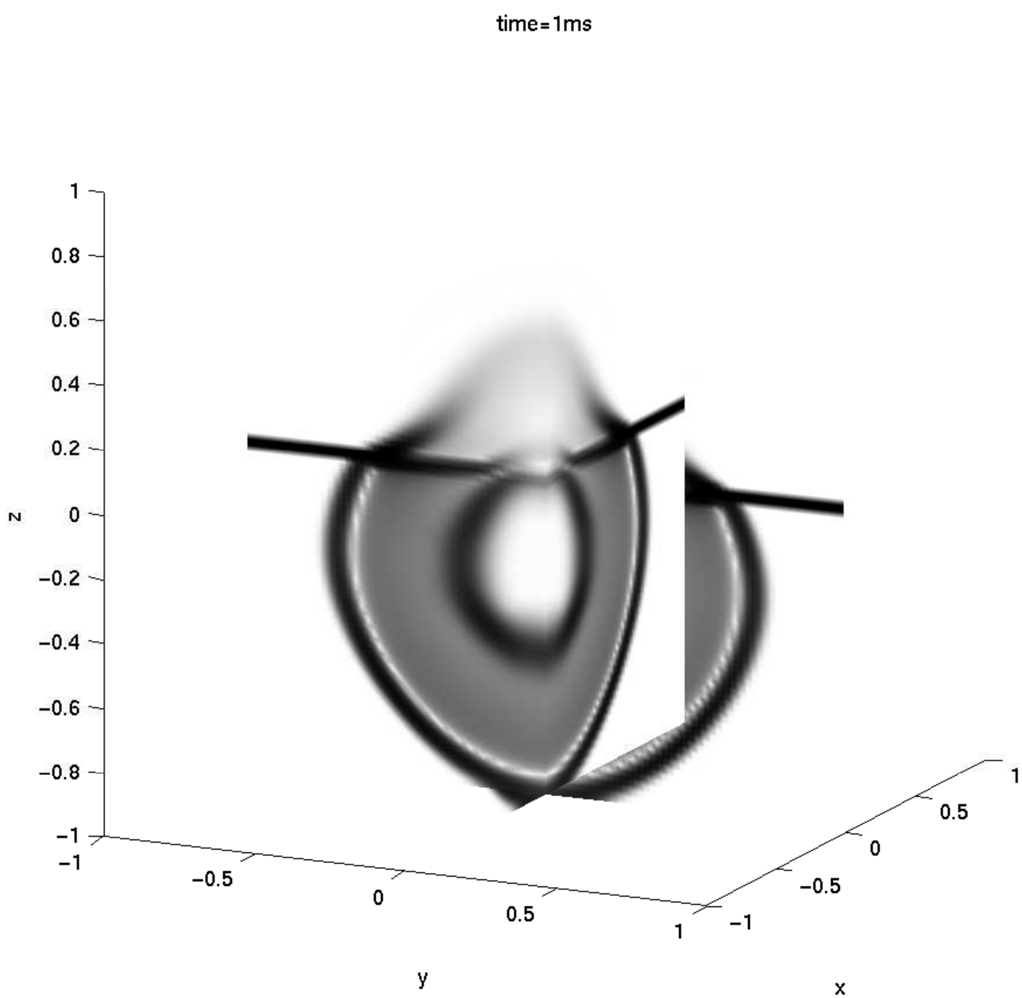
Underwater Explosions

- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



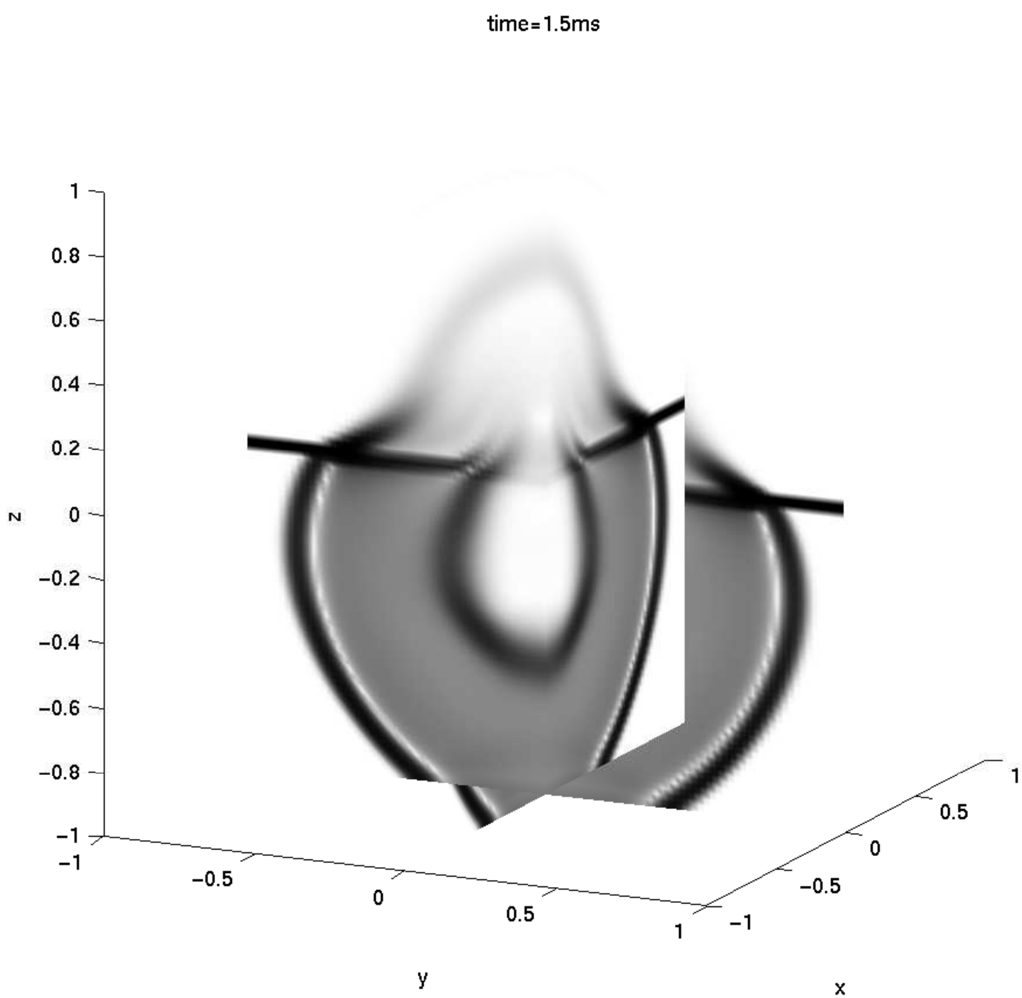
Underwater Explosions

- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



Underwater Explosions

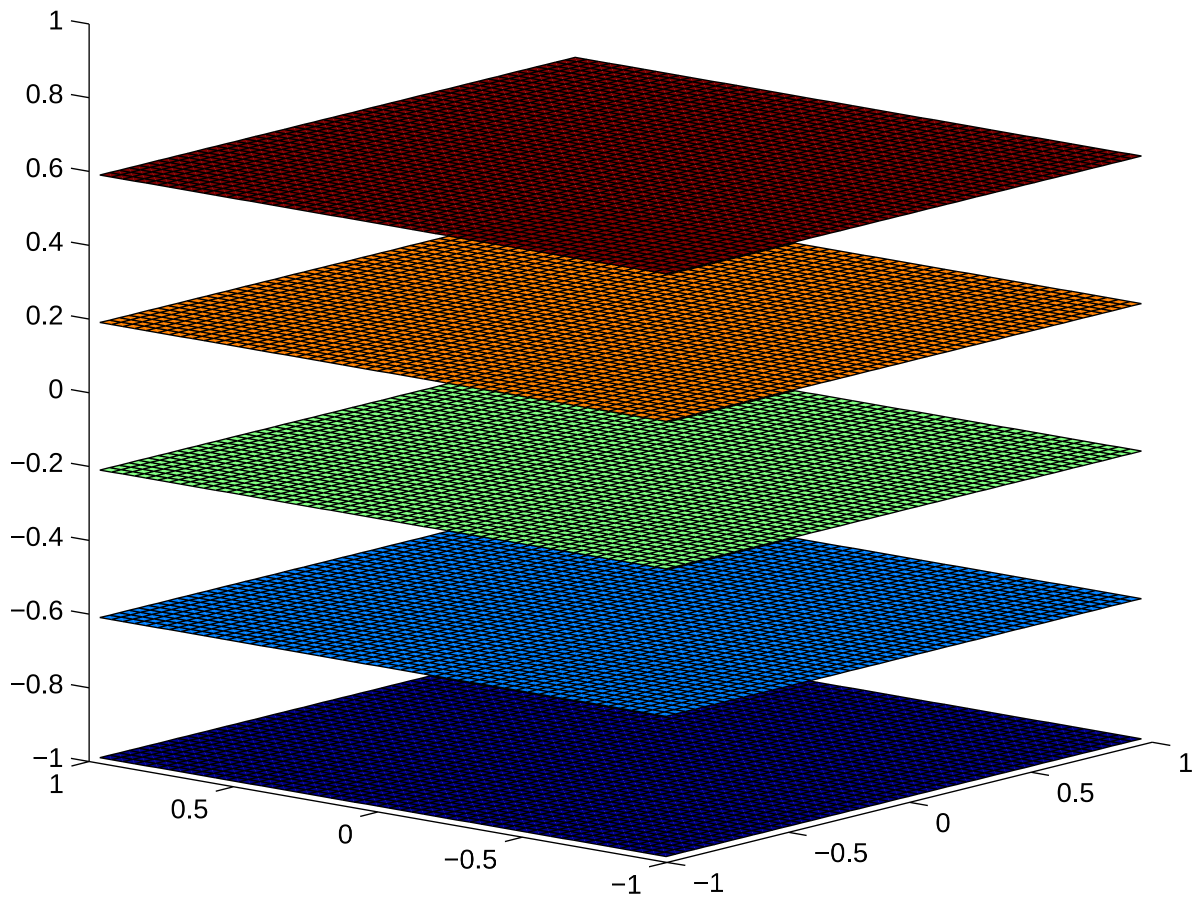
- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

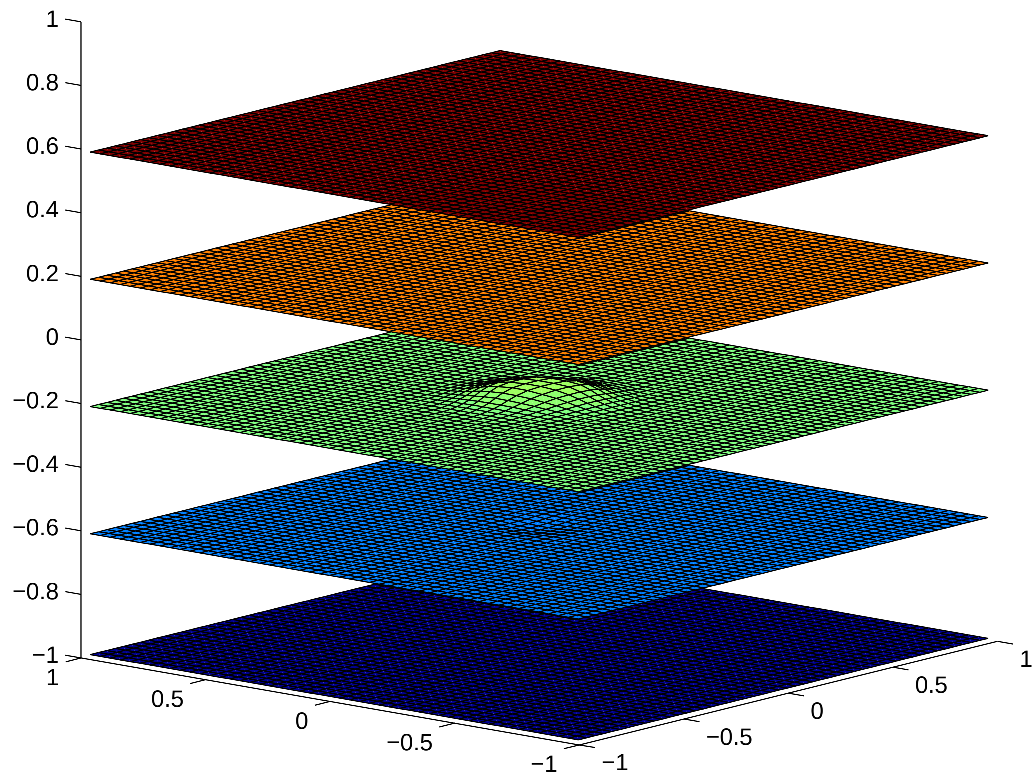
time $= 0$



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

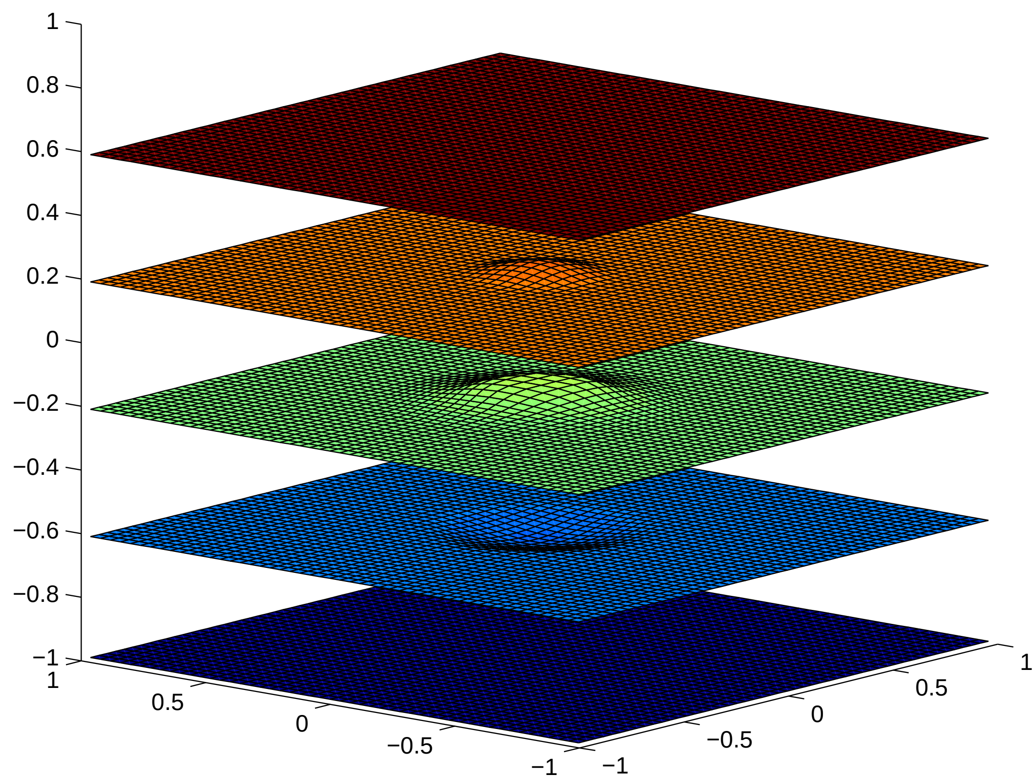
time $= 0.25$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

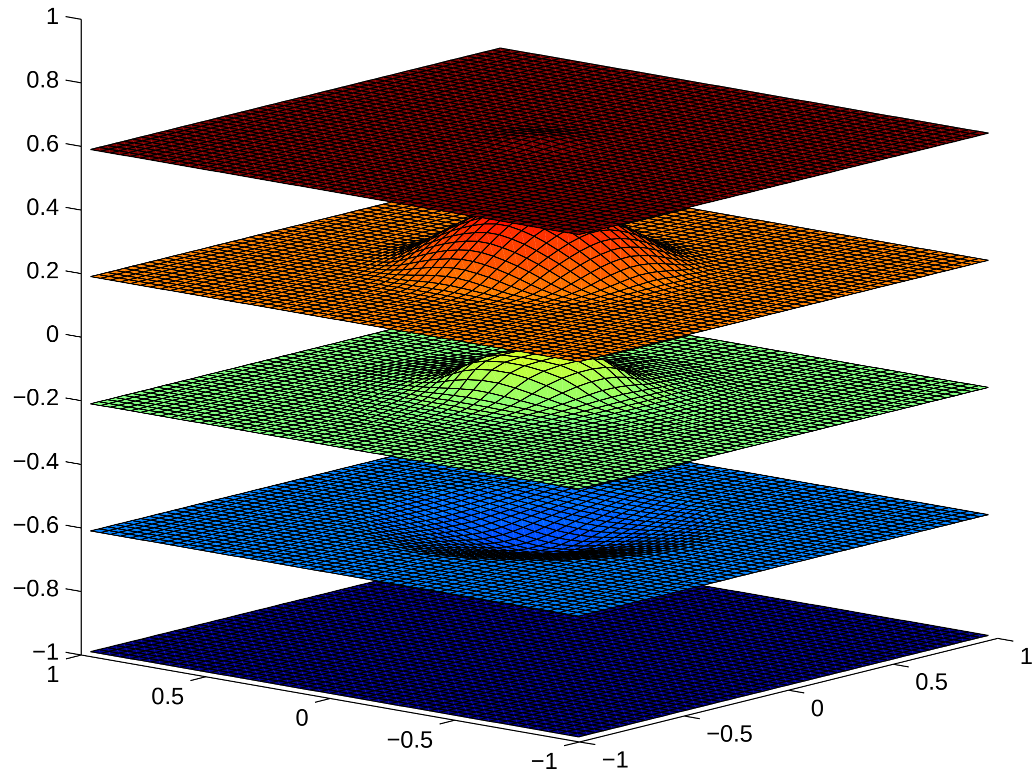
time $= 0.5$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

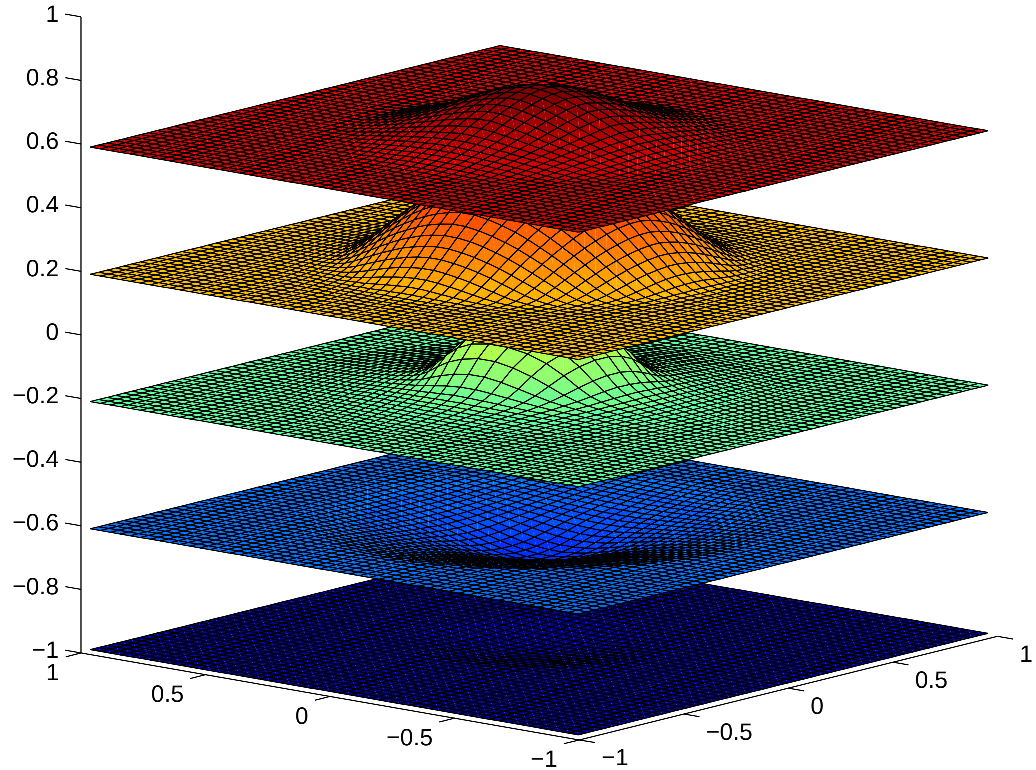
time $= 1.0$ms



3D Underwater Explosions (Cont.)

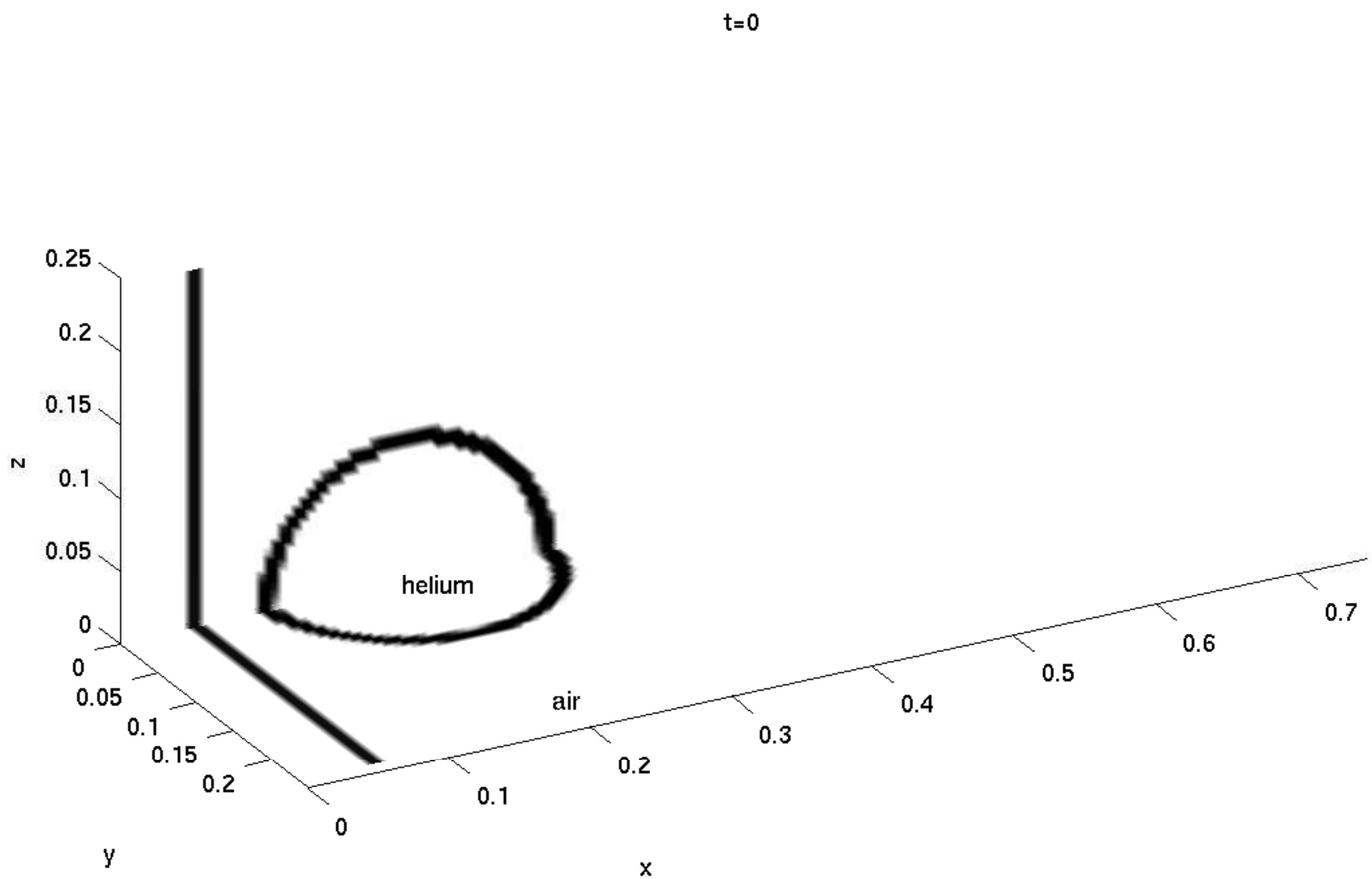
- Grid system (coarsen by factor 2) with $h_0 = 0.6$

time $= 1.5$ms



3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



3D Shock-Bubble (Helium)

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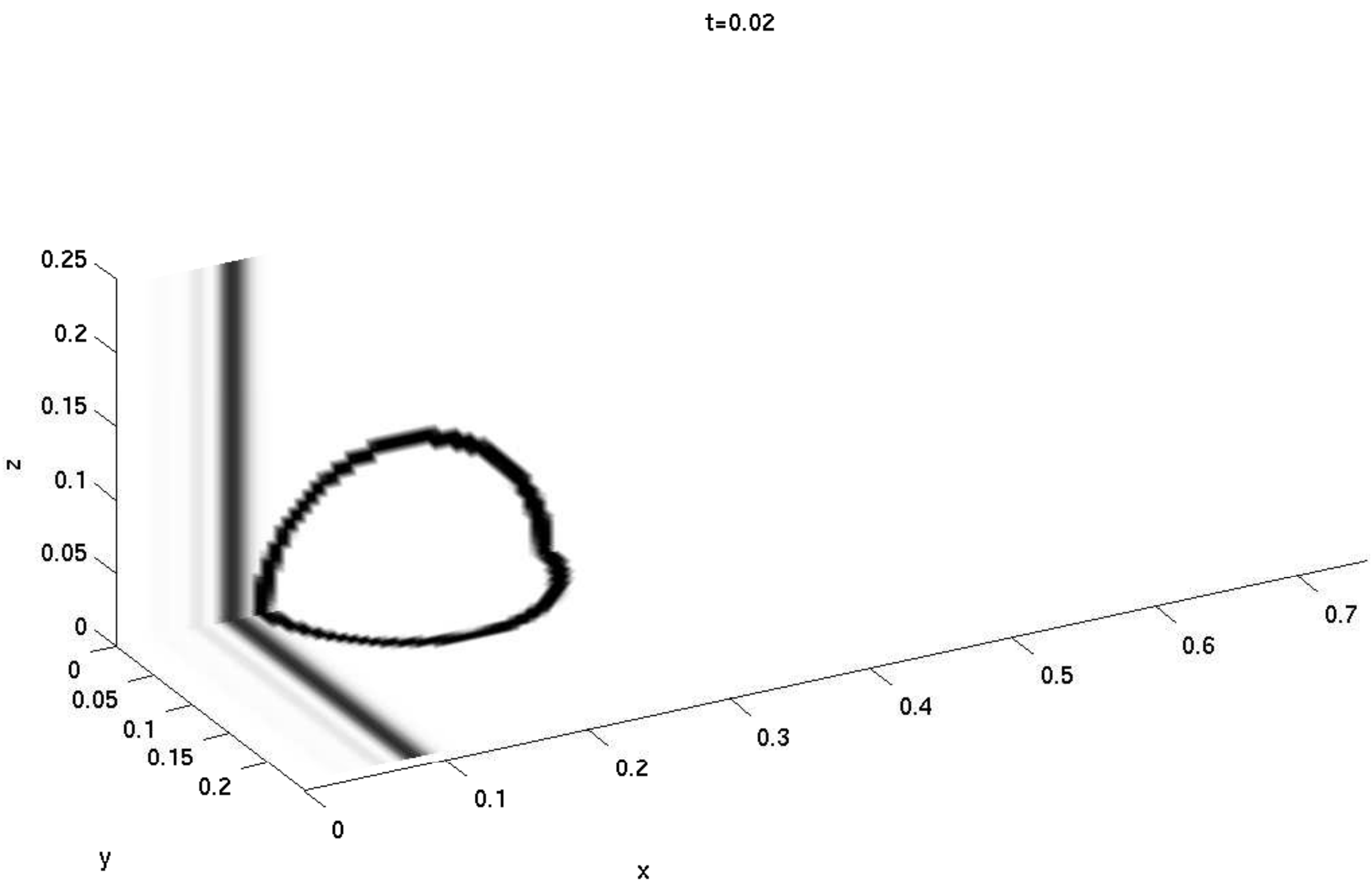

3D Shock-Bubble (Helium)

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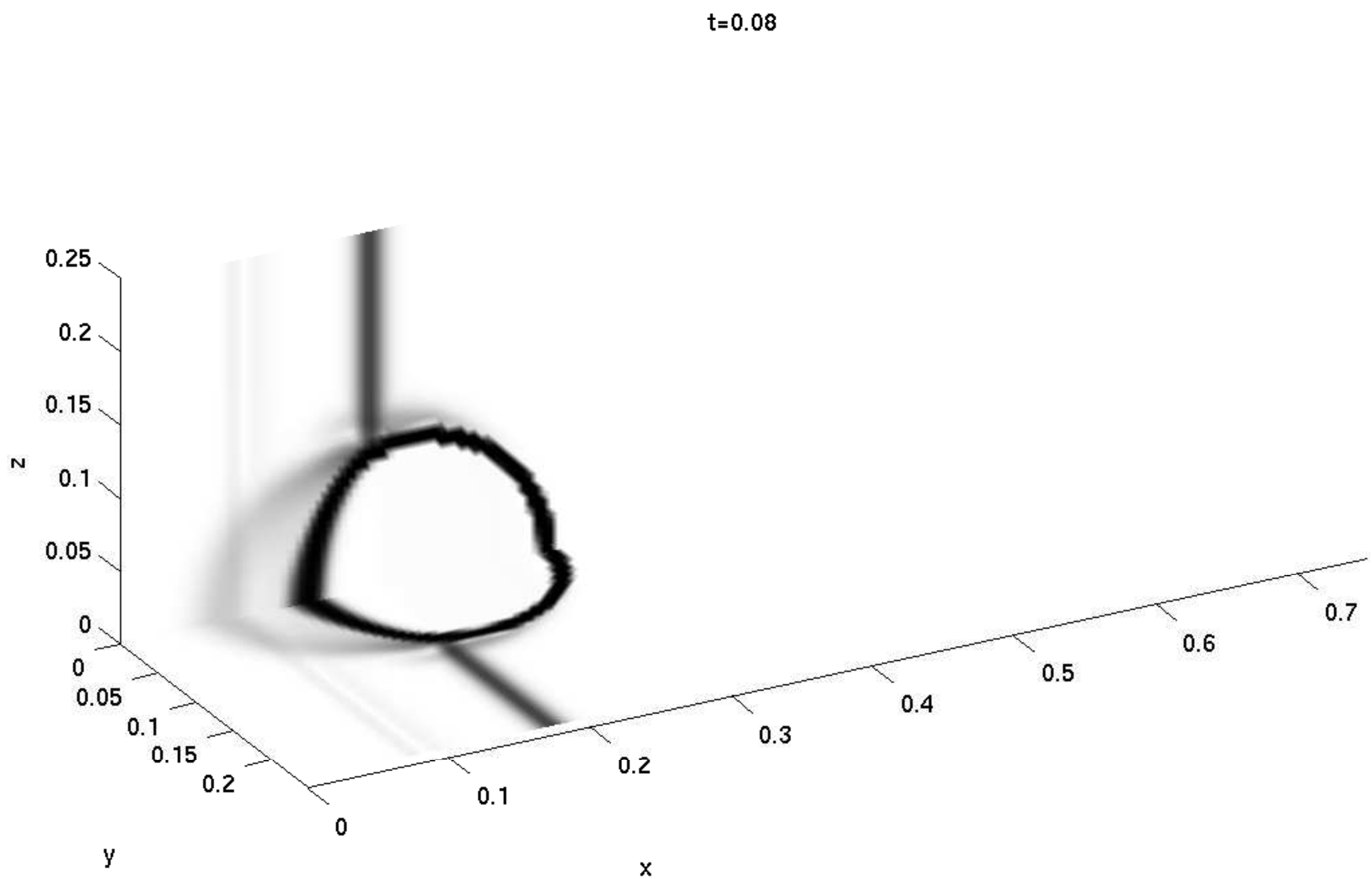
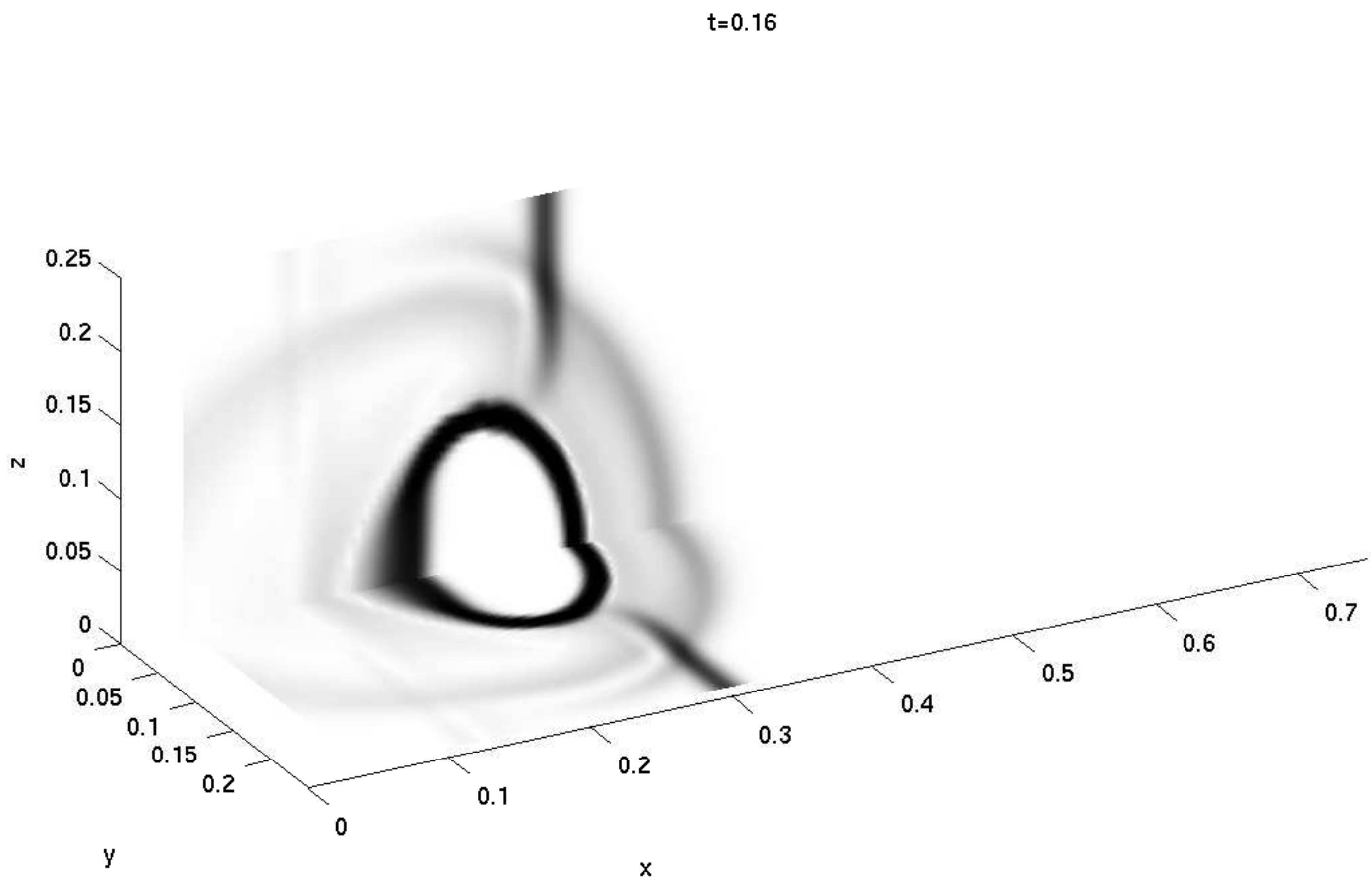


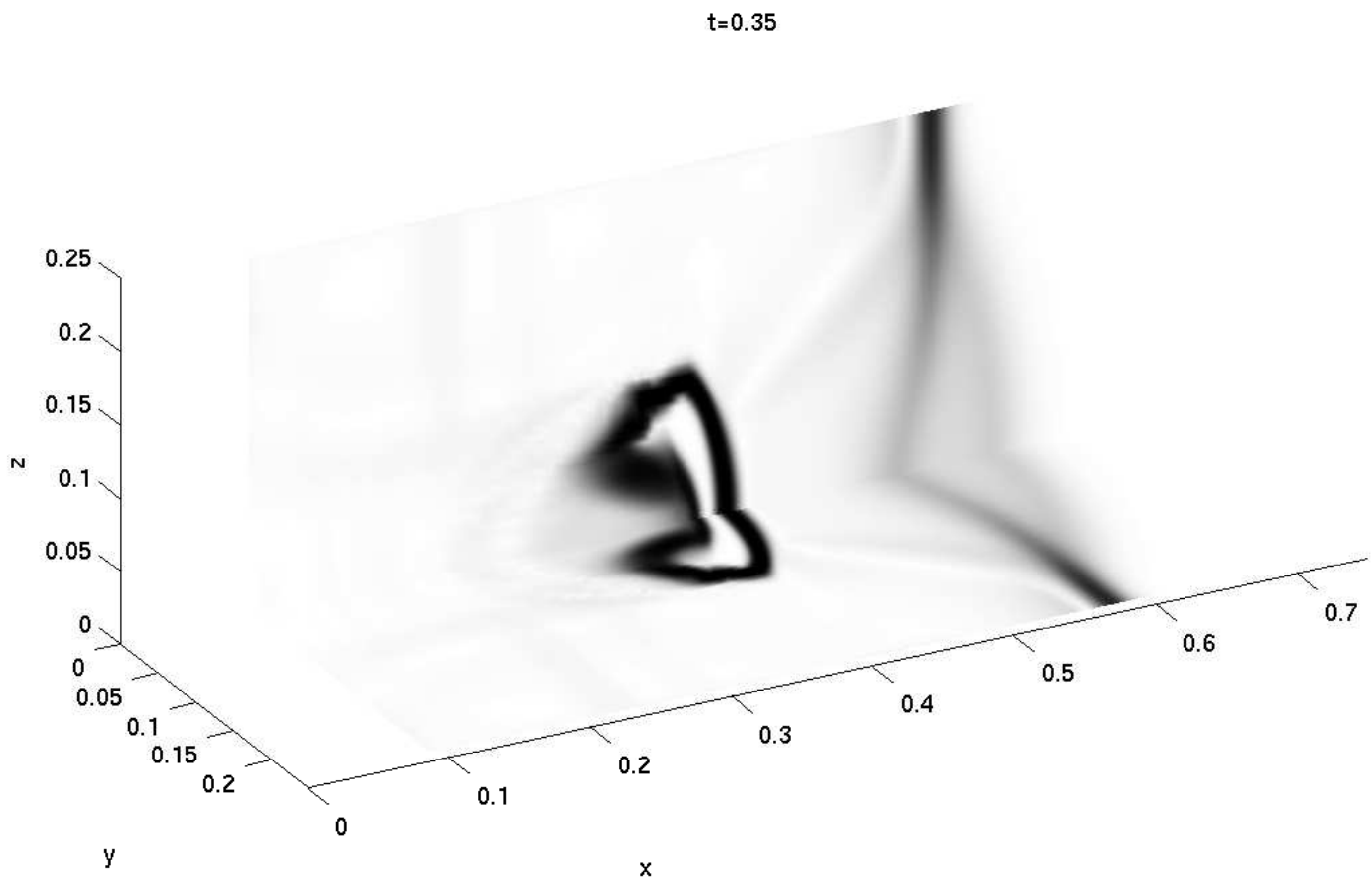
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



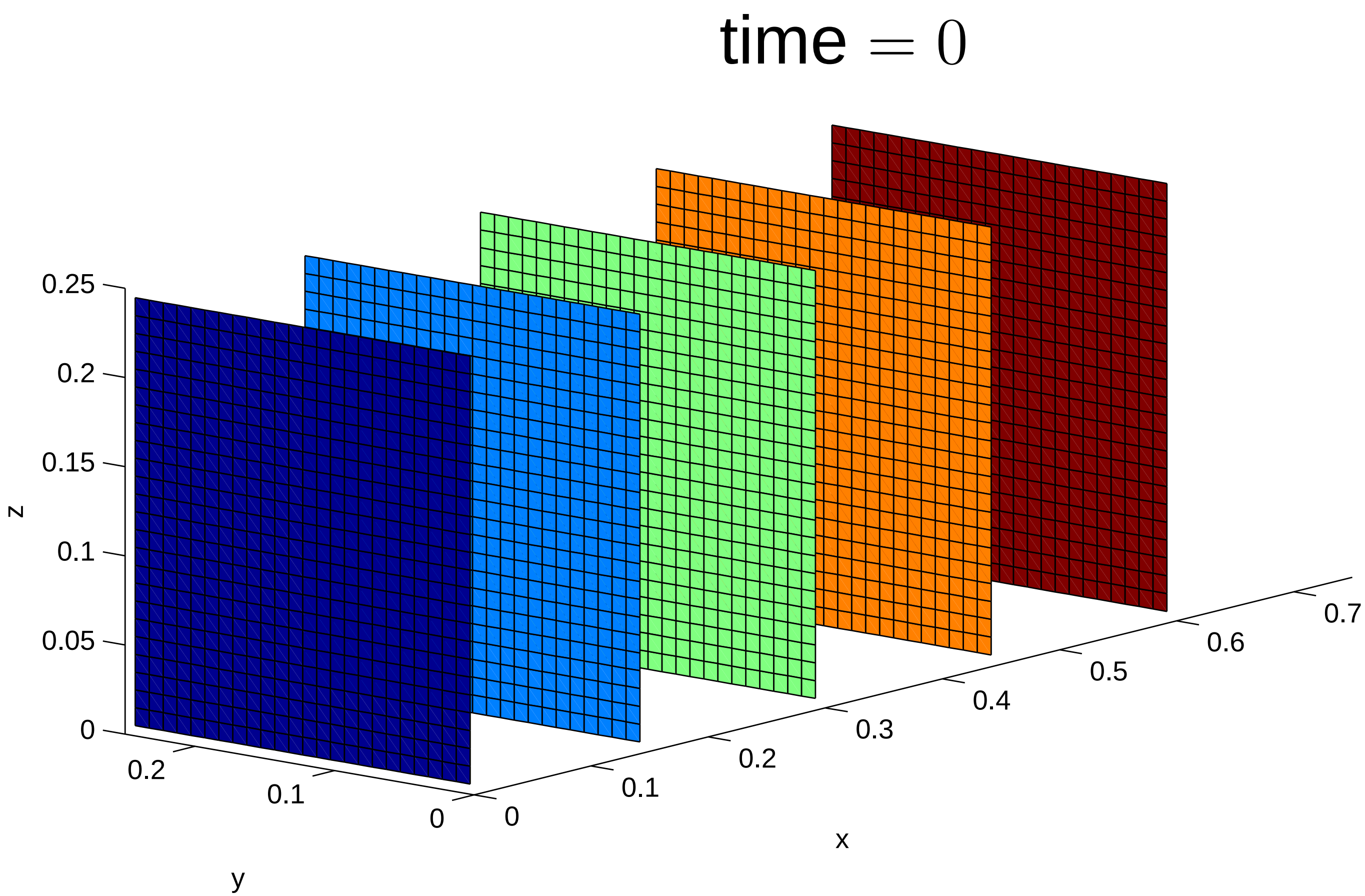
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



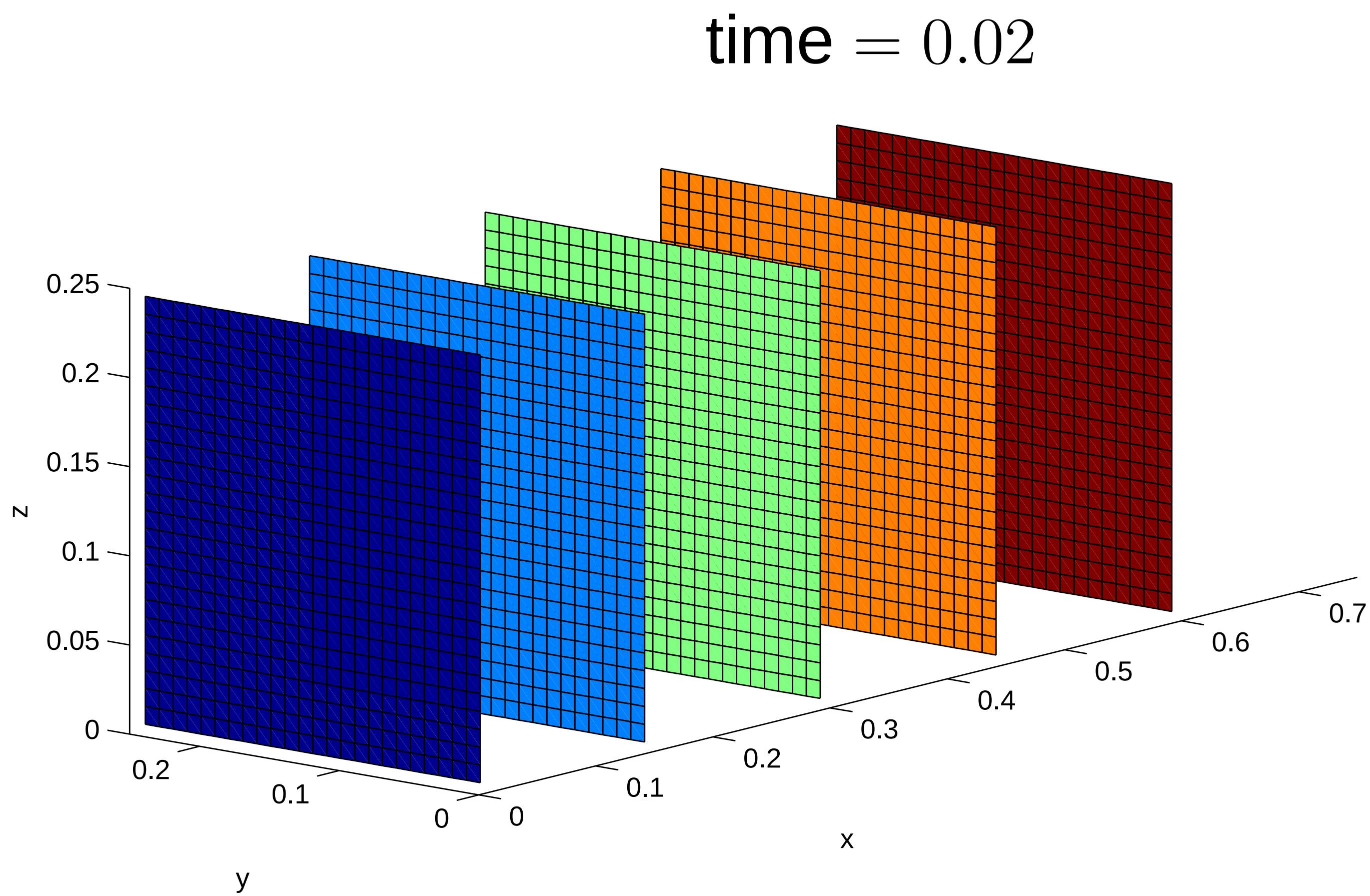
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



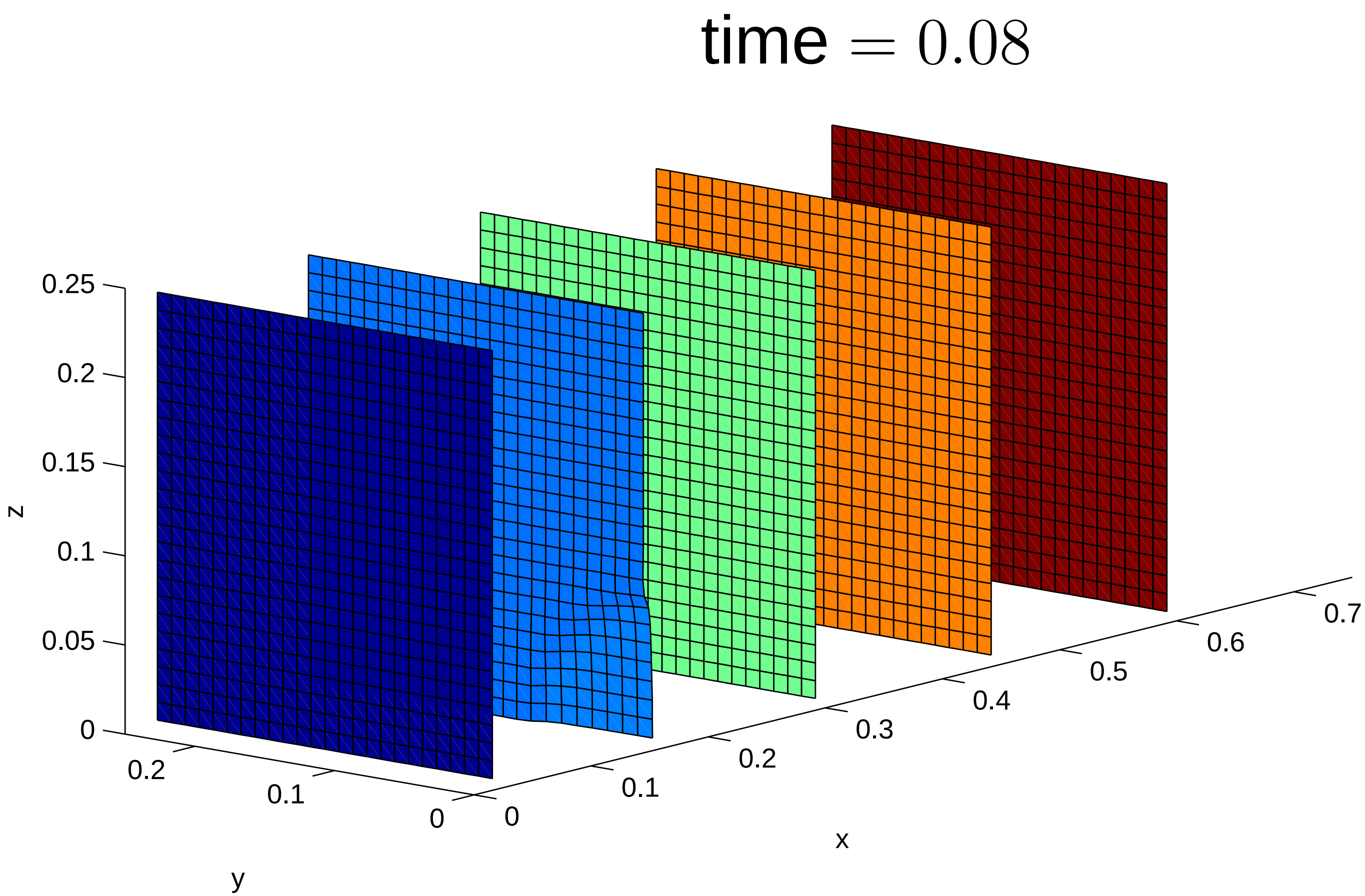
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



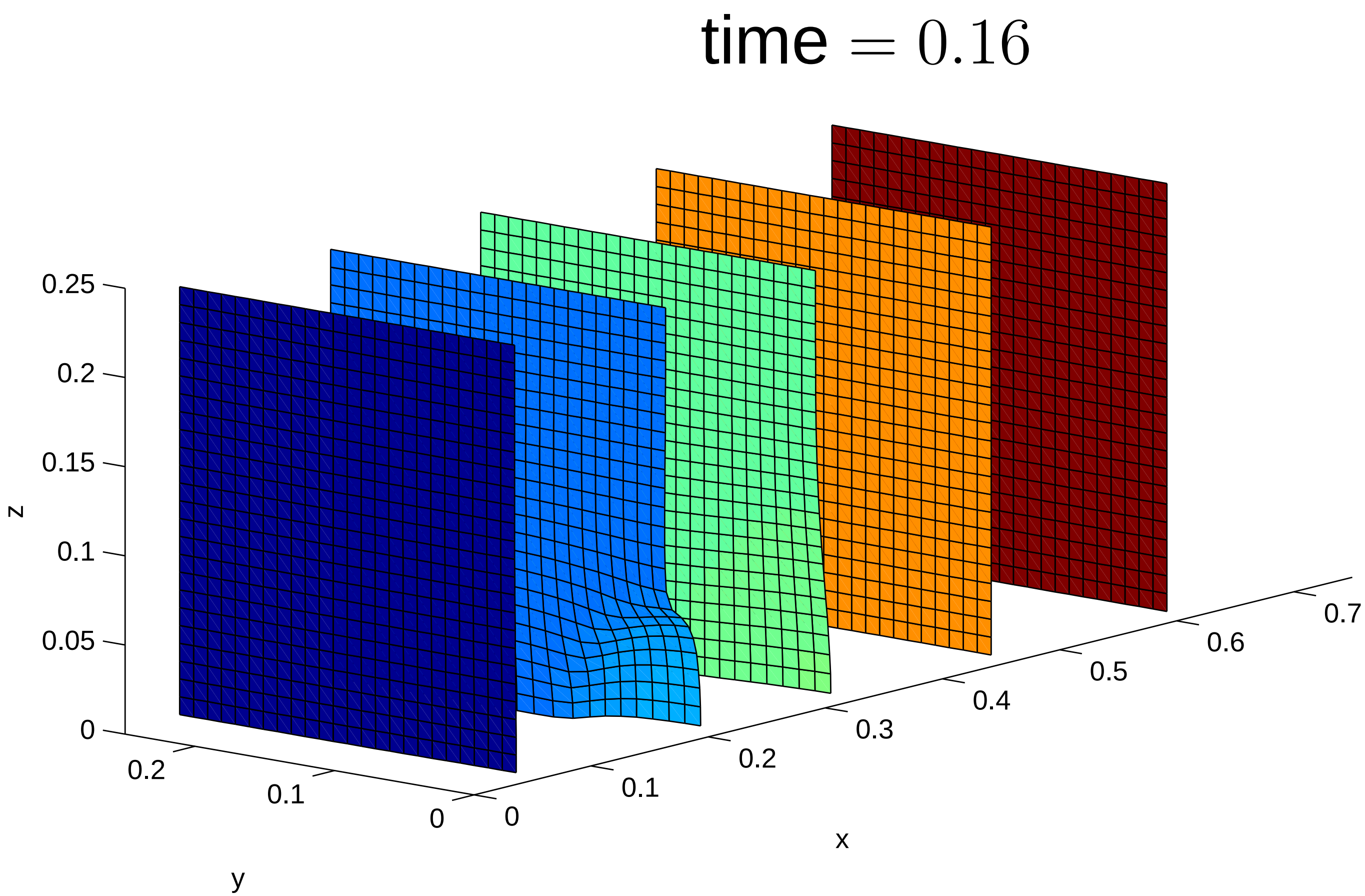
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



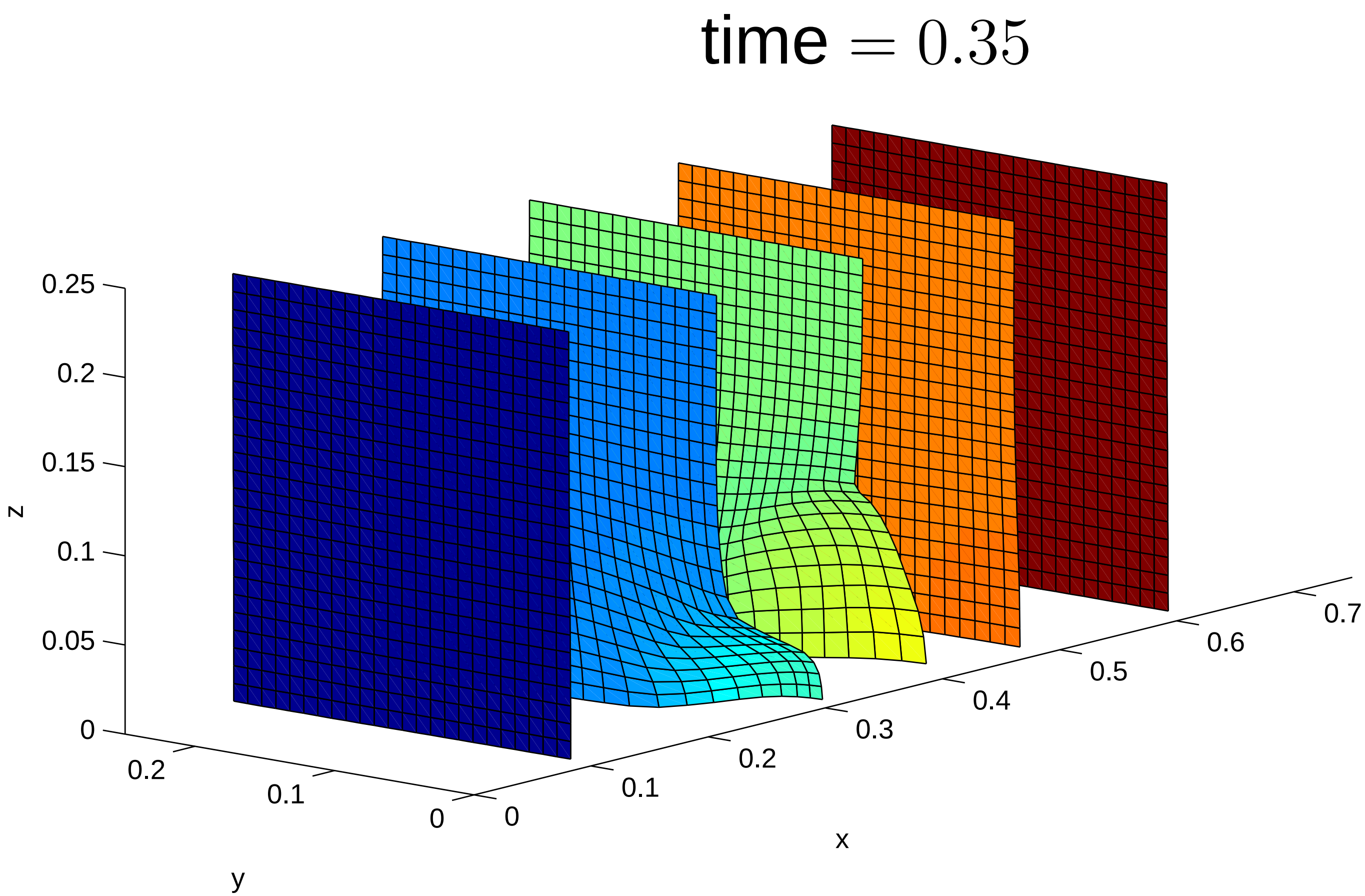
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



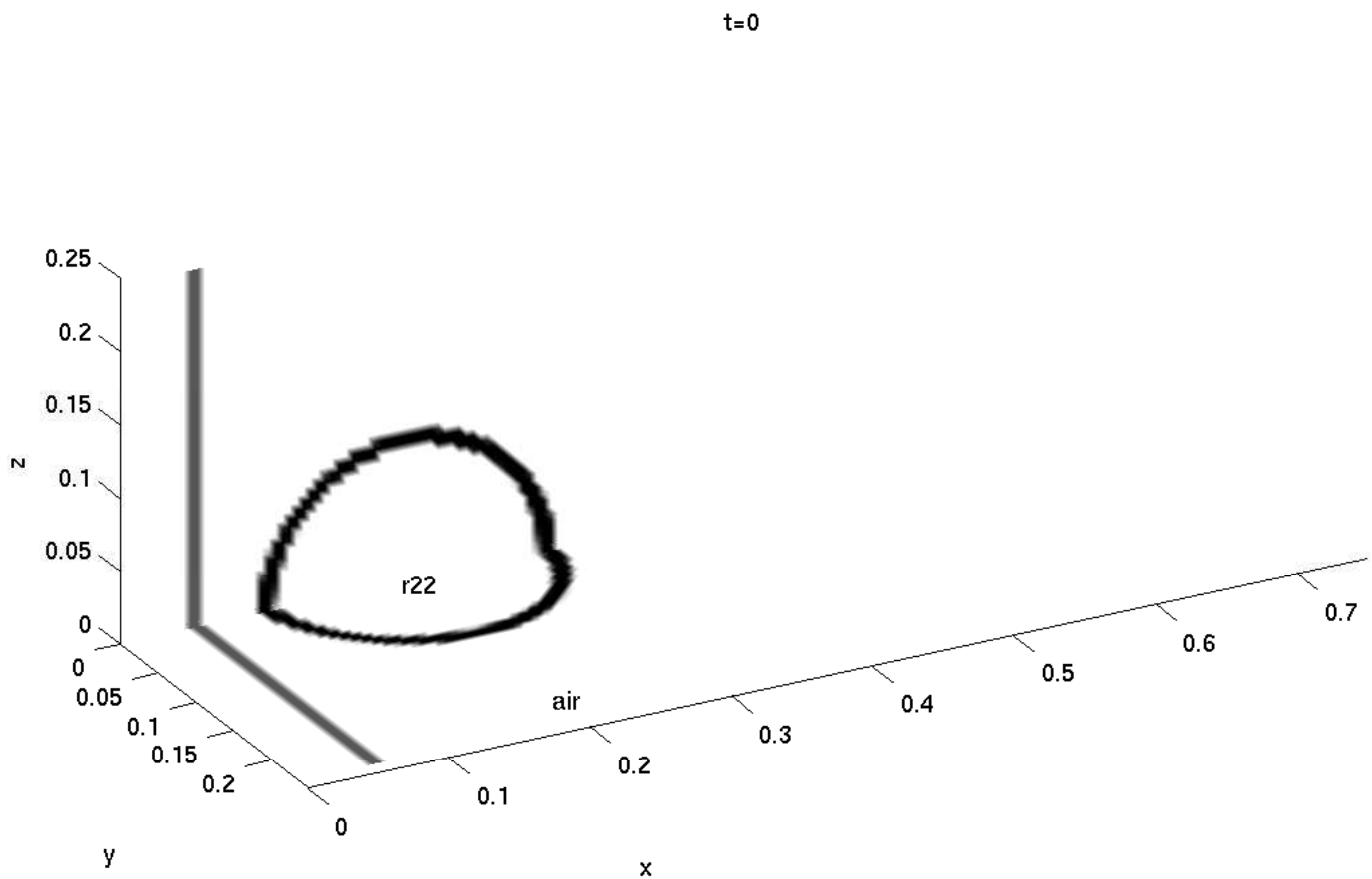
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



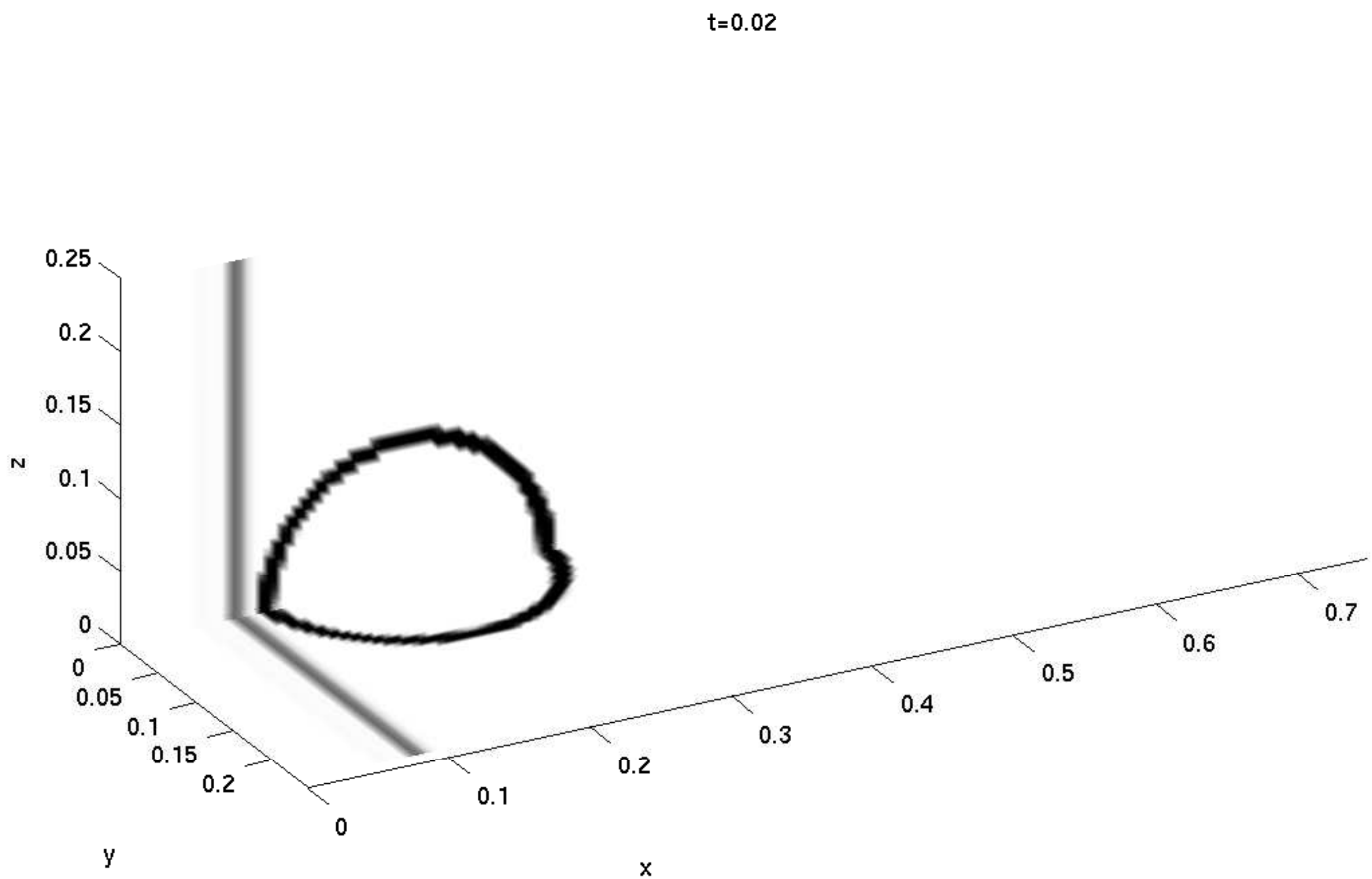
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



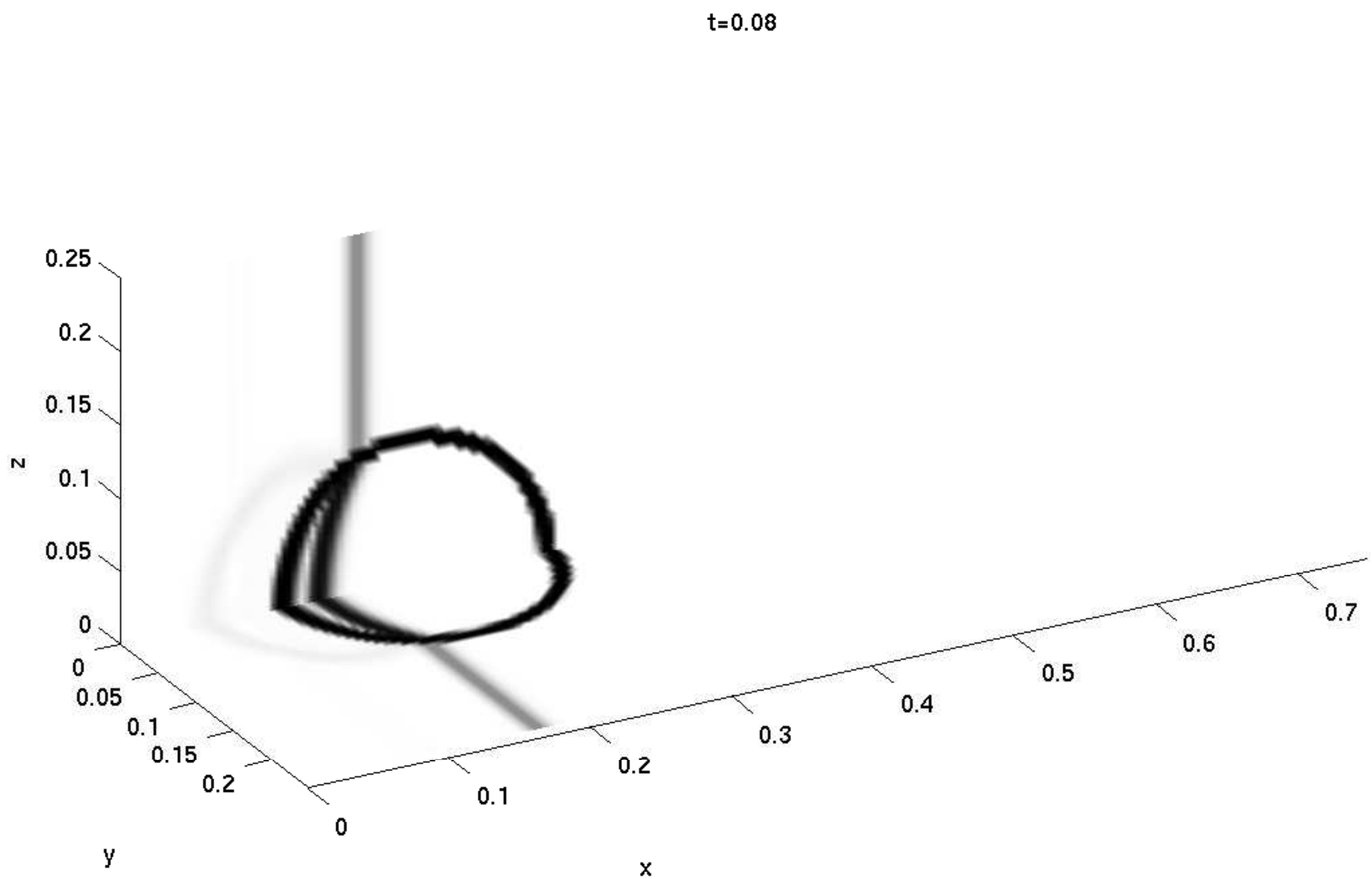
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



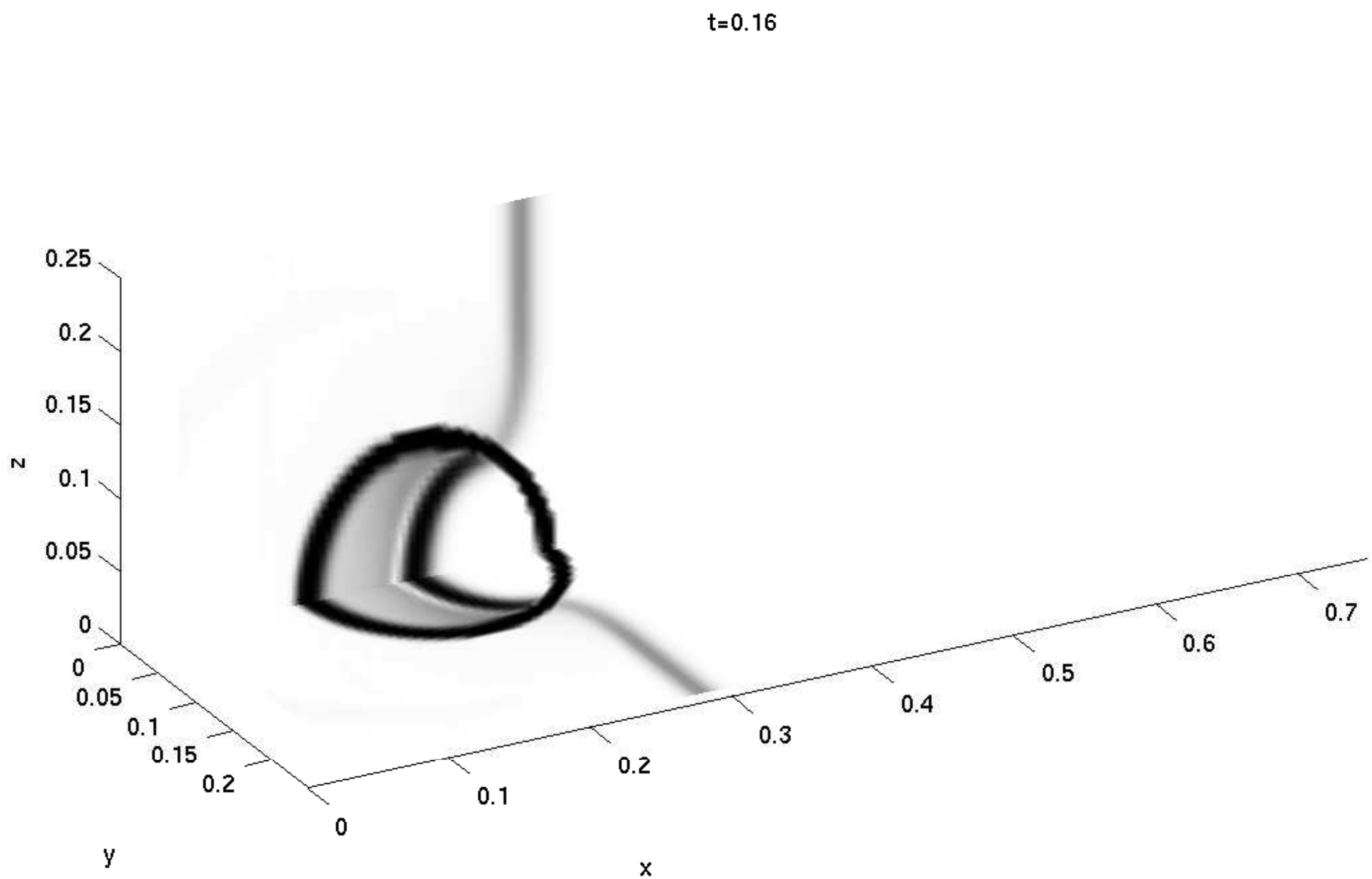
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



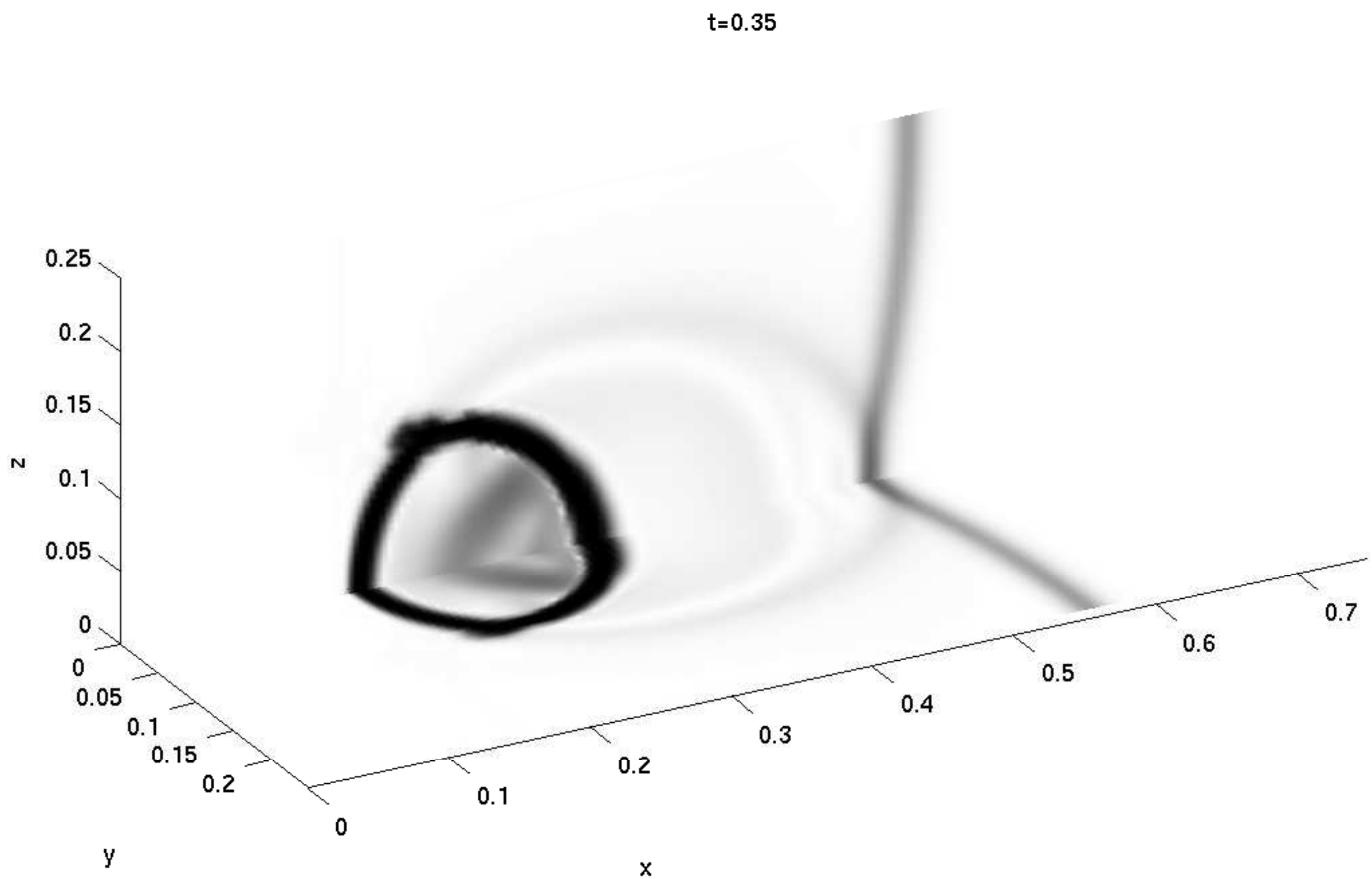
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



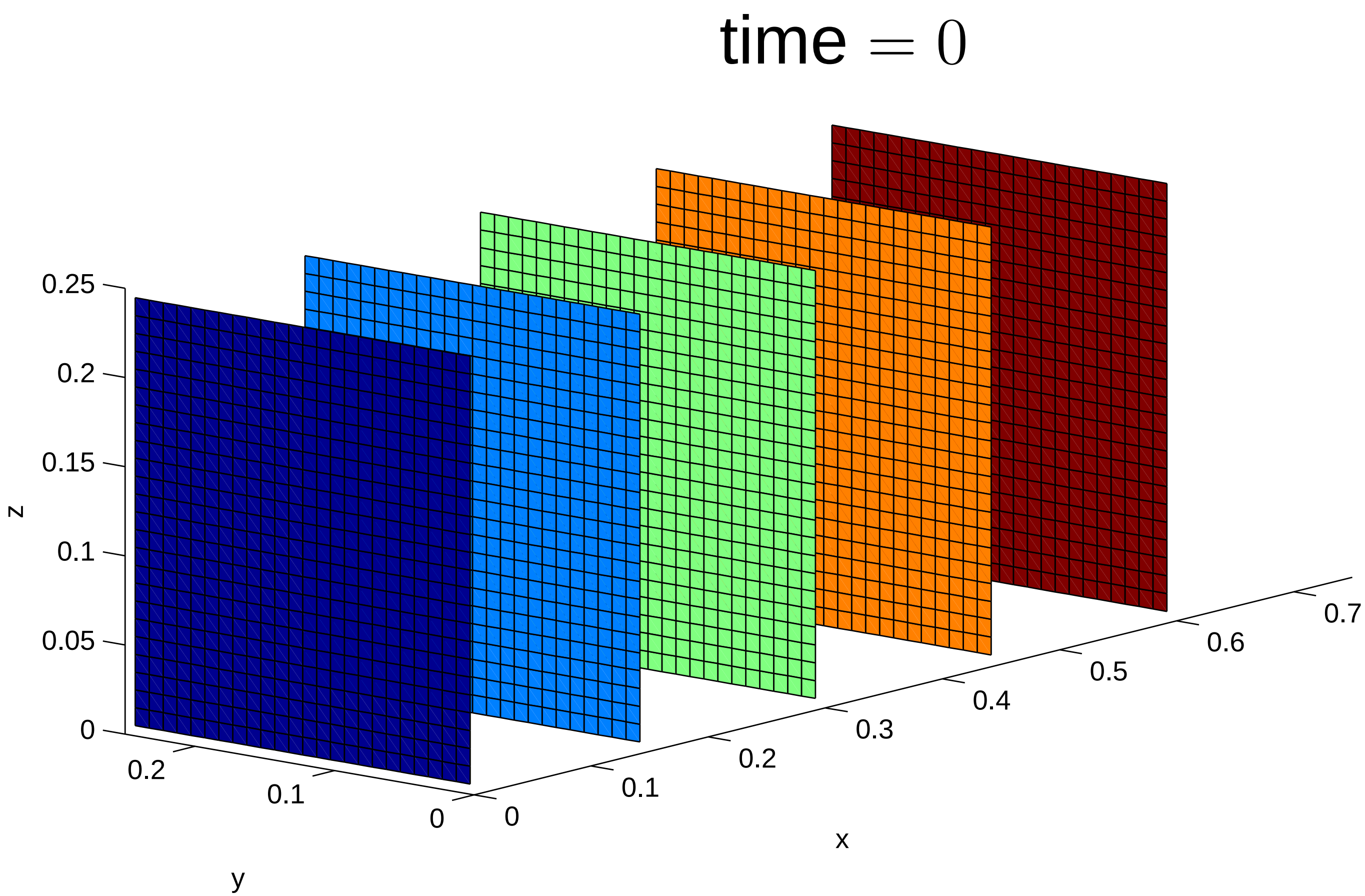
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



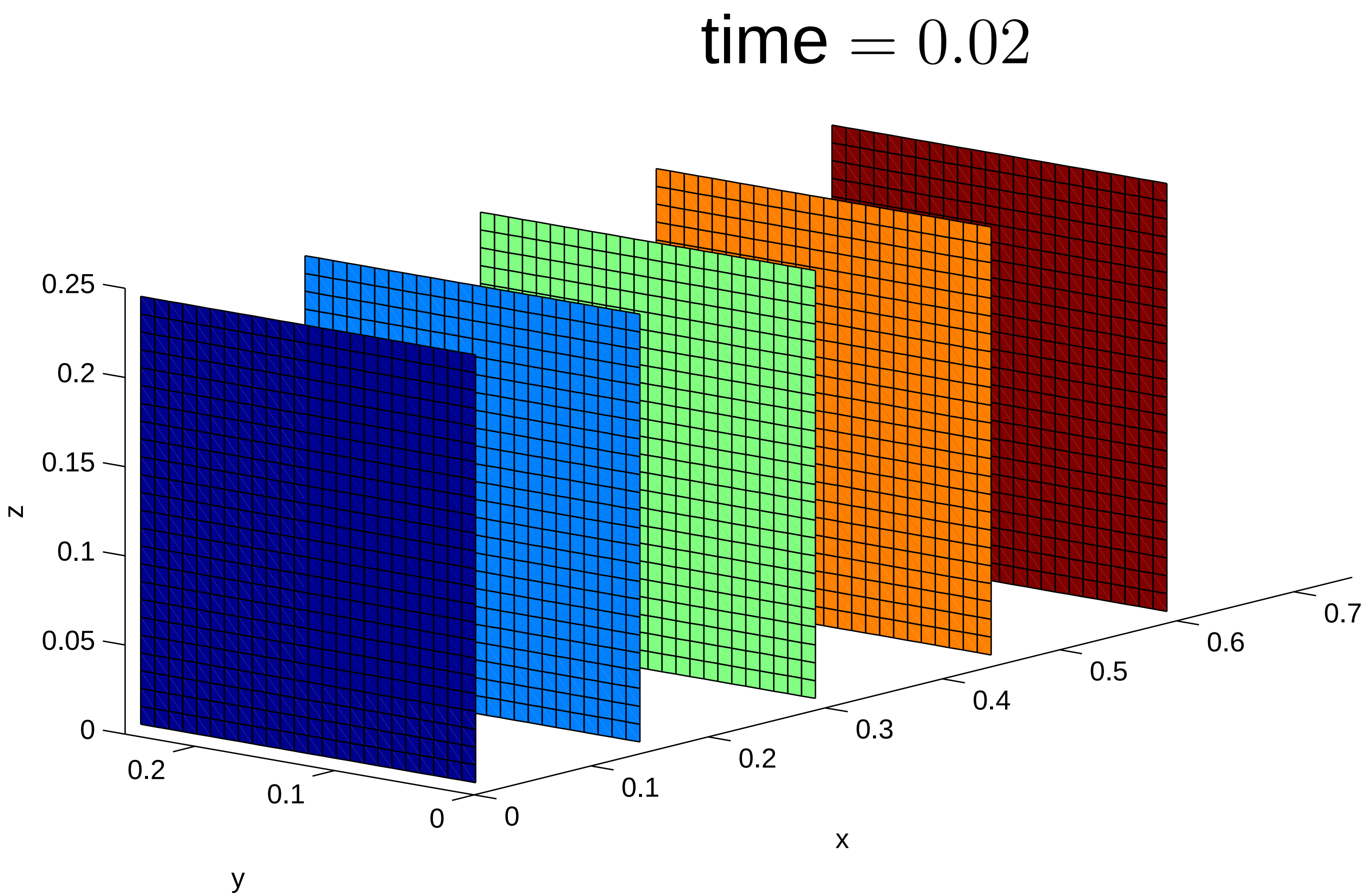
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



Shock-Bubble (R22) (Cont.)

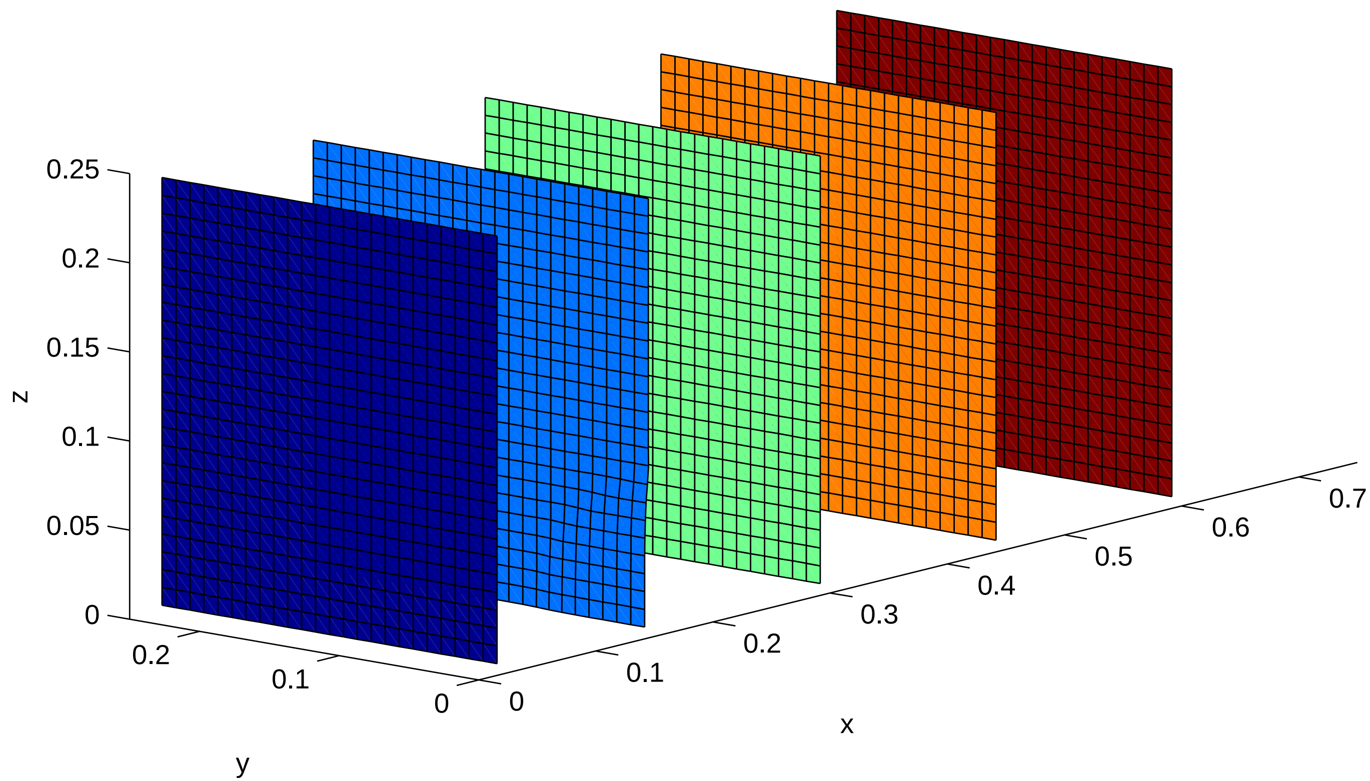
- Grid system (coarsen by factor 2) with $h_0 = 0.6$



Shock-Bubble (R22) (Cont.)

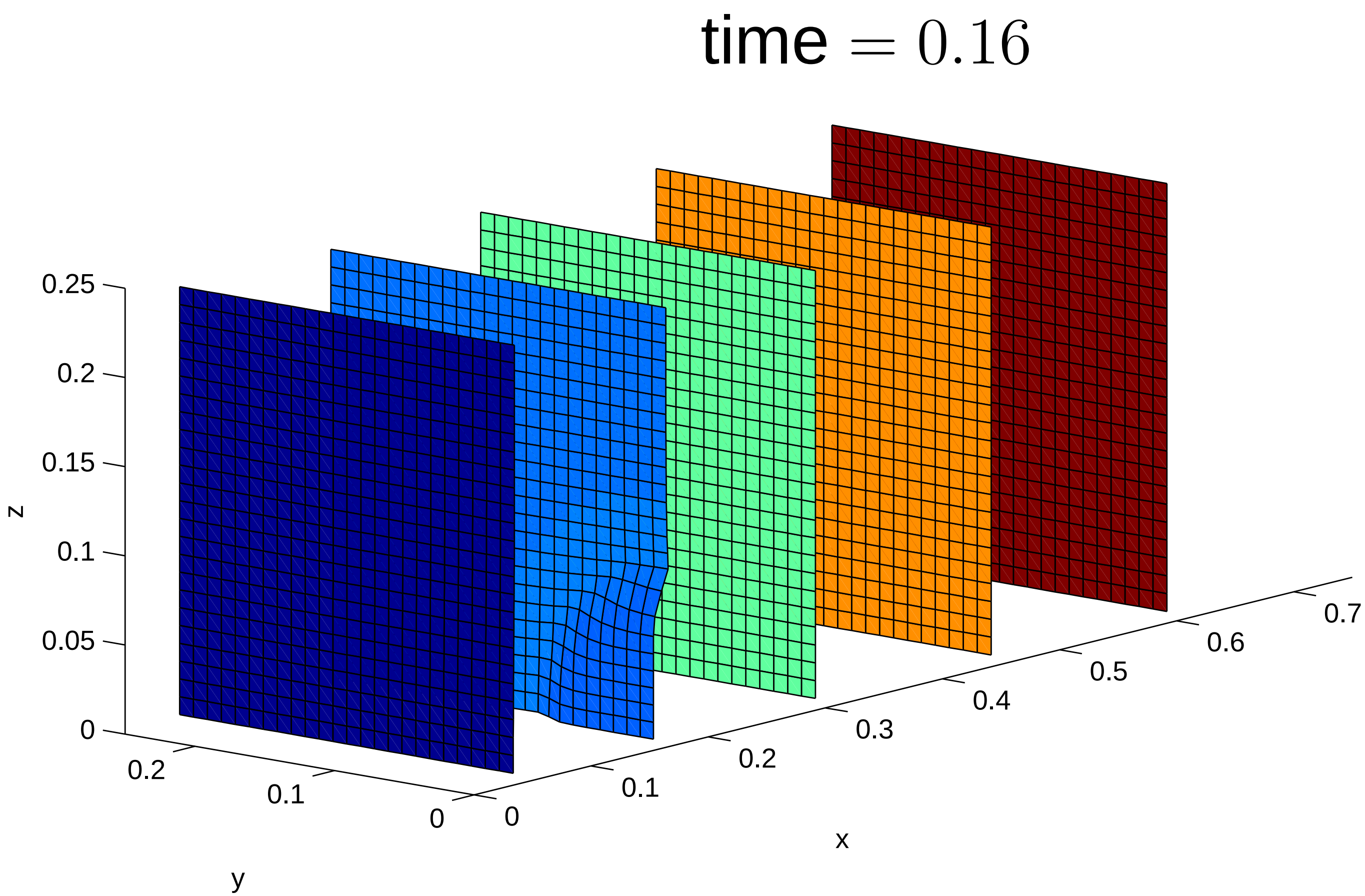
- Grid system (coarsen by factor 2) with $h_0 = 0.6$

time $= 0.08$



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$