Solutions of nonlinear dispersive shallow water equations: analytical and numerical study

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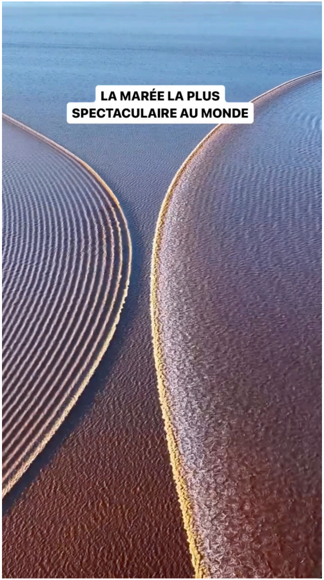
Dispersive shallow water flow: undular bore

Undular bore (Favre wave) on Severn river near Gloucester, UK

Qiantang river bore (from Dailymotion)



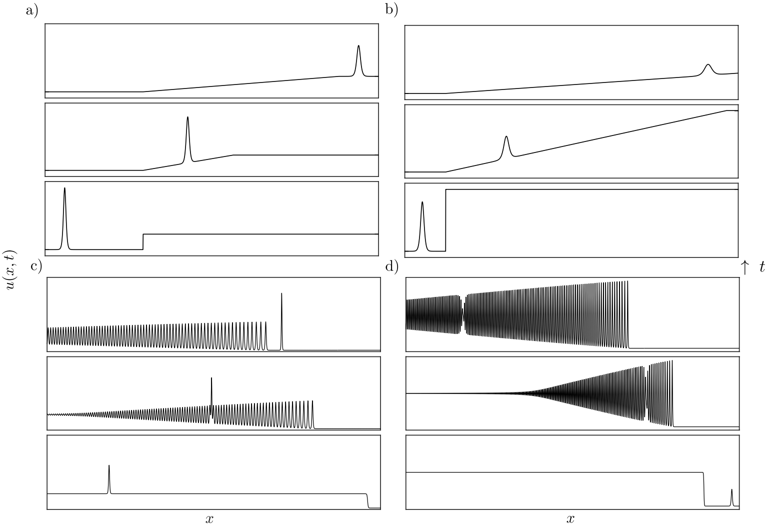
Talk outline

1. Whitham modulation theory for solitary wave-mean flow interaction for dispersive equations
   - Benjamin-Bona-Mahony (BBM) equation
   - Serre-Green-Naghdi (SGN) equations
2. Numerical algorithm for $2D$ SGN equations with bottom topography

Dispersive shallow water flow applications include:
tsunami modeling, river hydraulics, & study of geohazards like avalanches, debris flows

Joint work with S. Gavrilyuk, B. Nkonga, G. El, M. Hoefer, T. Congy and many others

Solitary wave-mean flow interaction scenarios



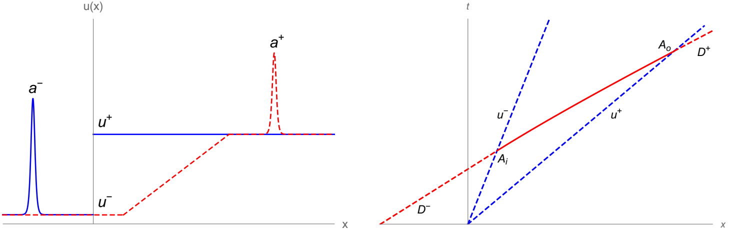
Figures referred from Ablowitz *et al.* 2023 for Korteweg-de Vries (KdV) equation

Solitary wave-mean flow: dispersive equations

For dispersive solitary wave-mean flow interaction, consider initial condition:

$$(\overline{u}, a)(x, 0) = \begin{cases} (u^-, a^-), & x < 0 \\ (u^+, a^+(?)), & x > 0 \end{cases} \qquad (1)$$

Sketch of positive rarefaction wave (RW) case: $0 < u^- < u^+$



Could one can obtain analytical formula for a^+ ? Yes, for KdV, BBM, SGN, conduit equation, ...

BBM equation

BBM equation was proposed as a unidirectional model of weakly non-linear waves in shallow water:

$$v_t + v_x + vv_x - v_{xxt} = 0$$

After change of variable $v = u - 1$ one gets

$$u_t + uu_x - u_{xxt} = 0 \tag{2}$$

BBM (2) admits only three independent conservation laws:

$$(u - u_{xx})_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{3a}$$

$$\left(\frac{u^2}{2} + \frac{u_x^2}{2}\right)_t + \left(\frac{u^3}{3} - uu_{tx}\right)_x = 0 \tag{3b}$$

$$\left(\frac{u^3}{3}\right)_t - \left(u_t^2 - u_{xt}^2 + u^2 u_{xt} - \frac{u^4}{4}\right)_x = 0 \tag{3c}$$

Periodic travelling wave solutions: BBM equation

BBM periodic travelling wave solutions $u(x,t) = u(\xi)$, $\xi = x - Dt$, satisfies

$$(u')^2 = \frac{2}{D}\left(-\frac{u^3}{6} + D\frac{u^2}{2} + c_1 u + c_2\right) = \frac{1}{3D}\mathrm{P}(u) \qquad \text{(4a)}$$

$\mathrm{P}(u) = (u - u_1)(u - u_2)(u_3 - u)$, $D > 0$

Its solution (three-parameter family) is:

$$u(\xi) = u_2 + a\,\mathrm{cn}^2\left(\frac{1}{2}\sqrt{\frac{u_3 - u_1}{u_1 + u_2 + u_3}}\xi, m\right) \qquad \text{(4b)}$$

cn is Jacobi function,

$$a = u_3 - u_2,\ m = (u_3 - u_2)/(u_3 - u_1),\ D = (u_1 + u_2 + u_3)/3$$

Define wave averaged of any function $f(u)$ as

$$\overline{f(u)} = \frac{1}{L}\int_{\xi_2}^{\xi_3} f(u)\, d\xi = \frac{2}{L}\int_{u_2}^{u_3} f(u)\frac{\sqrt{3D}}{\sqrt{\mathrm{P}(u)}}\, du$$

Wave averaged of u is

$$\overline{u} = \int_{u_2}^{u_3} \frac{u du}{\sqrt{\mathrm{P}(u)}} \Big/ \int_{u_2}^{u_3} \frac{du}{\sqrt{\mathrm{P}(u)}} = u_1 + (u_3 - u_1)\frac{\mathrm{E}(m)}{\mathrm{K}(m)}$$

Wavelength L is

$$L = \int_{\xi_2}^{\xi_3} d\xi = 2\int_{u_2}^{u_3} \frac{\sqrt{3D}}{\sqrt{\mathrm{P}(u)}}\, du = 4\sqrt{3}\sqrt{\frac{Dm}{a}}\,\mathrm{K}(m)$$

$\mathrm{K}(m)$ & $\mathrm{E}(m)$: first & second type complete elliptic integrals

Solitary wave solution obtained in $L \to \infty$, $u_1 = u_2 > 0$ is

$$u(\xi) = u_2 + a\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{u_3 - u_2}{2u_2 + u_3}}\xi\right)$$

Whitham modulation system: BBM equation

Assume BBM solution $u(\xi, X, T, \varepsilon)$; L-periodic with respect to ξ & varies slowly to X & T:

$$\xi = \frac{X - DT}{\varepsilon} = x - Dt, \quad X = \varepsilon x, \quad T = \varepsilon t,\ \varepsilon \ll 1$$

One can obtain modulation system using two equivalent methods: averaging of conservation laws & method of averaged Lagrangian (Whitham 1965,1974)

For BBM equation, from (3), we have modulation system:

$$\left[\frac{\frac{\overline{u^2}}{2} + \frac{\overline{u}}{\frac{\overline{\mathrm{P}(u)}}{6D}}}{\frac{1}{L}}\right]_T + \left[\frac{\frac{\overline{u^3}}{3} - \frac{\frac{\overline{u^2}}{2}}{\frac{\overline{\mathrm{P}(u)}}{3}}}{\frac{D}{L}}\right]_X = 0 \qquad (5)$$

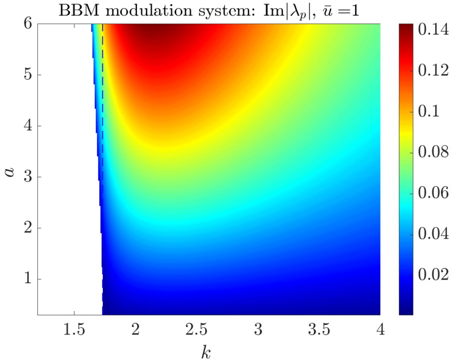
In quasi-linear form, (5) is written as

$$Aq_T + Bq_X = 0 \tag{6}$$

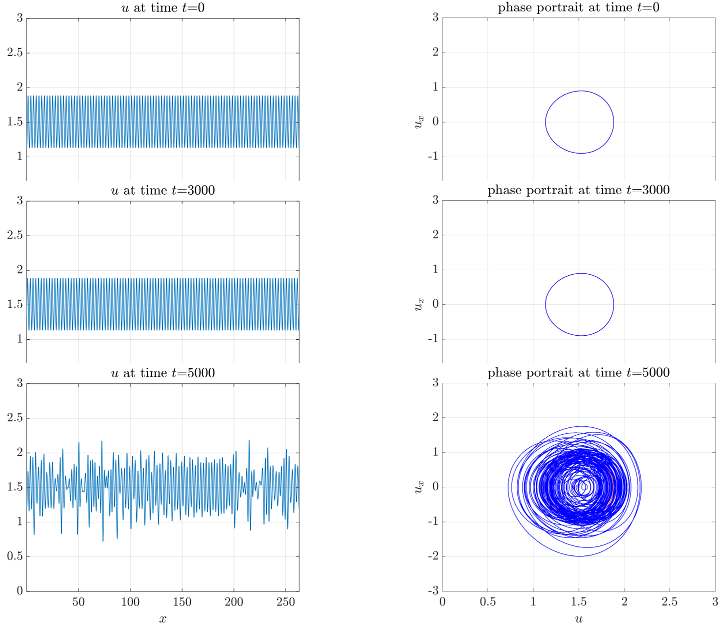
Let λ_j be eigenvalue & r_j be associated right eigenvector:

$$(B - \lambda_j A)\, r_j = 0, \quad j = 1, 2, 3$$

It is known that (6) is mixed elliptic-hyperbolic system, *i.e.*, there exists $\lambda_j \in \mathbf{C}$ (Congy, Gavrilyuk, Tso, ...).



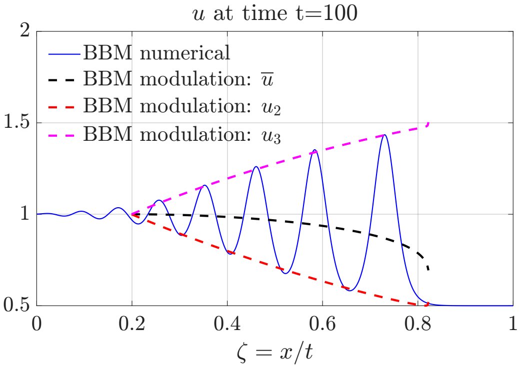
Modulational instability ($\overline{u} = 1$, $a = \frac{1}{2}$, $k \approx 2.389$)



BBM modulation system: simple wave solution

Assume solution is self-similar: $q(x,t) = q(\zeta)$, $\zeta = x/t$. If $\nabla_q \lambda_j \cdot r_j \neq 0$, one obtains $q(\zeta)$ by solving ODEs numerically:

$$\frac{dq}{d\zeta} = r_j / \left(\nabla_q \lambda_j \cdot r_j\right), \quad \zeta = \lambda_j, \quad j = 1, 2, 3$$



Amplitude modulation equation: BBM equation

Taking $L \to \infty$ to modulation system (5) (solitary limit), one obtains only equations for $\overline{u}$

$$\overline{u}_T + \left(\frac{(\overline{u})^2}{2}\right)_X = 0, \quad \left(\frac{(\overline{u})^2}{2}\right)_T + \left(\frac{(\overline{u})^3}{3}\right)_X = 0$$

from first two equations & trivial identity from third

To find equation for amplitude $a(\overline{u})$, employing method of averaged Lagrangian, we have action conservation laws:

$$\overline{u}_T + \left(\frac{\overline{u^2}}{2}\right)_X = 0 \tag{7a}$$

$$\left(\frac{\overline{u^2} - (\overline{u})^2}{2k} + \frac{\overline{\mathrm{P}(u)}}{6Dk}\right)_T + \left(D\left(\frac{\overline{u^2} - (\overline{u})^2}{2k} - \frac{\overline{\mathrm{P}(u)}}{6Dk}\right)\right)_X = 0 \tag{7b}$$

BBM modulation system: solitary limit

Now taking $L \to \infty$ to (7) we have governing equations for Riemann problem (1):

$$\overline{u}_T + \left(\frac{(\overline{u})^2}{2}\right)_X = 0 \tag{8a}$$

$$F(\overline{u}, a)_T + G(\overline{u}, a)_X = 0 \tag{8b}$$

$$F(\overline{u}, a) = \frac{a^{3/2}(2a + 5\overline{u})}{\sqrt{a + 3\overline{u}}},\ G(\overline{u}, a) = \frac{a^{3/2}(4a + 15\overline{u})\sqrt{a + 3\overline{u}}}{9}$$

Quasilinear form of (8) is with $D = \overline{u} + \dfrac{a}{3}$:

$$\overline{u}_T + \overline{u}\,\overline{u}_X = 0 \tag{9a}$$

$$a_T + Da_X + \frac{a}{3}\frac{14a^2 + 75a\overline{u} + 90\overline{u}^2}{8a^2 + 40a\overline{u} + 45\overline{u}^2}\overline{u}_X = 0 \tag{9b}$$

BBM solitary limit: self-similar solution

Assuming solution in (8) is smooth & self-similar in $\zeta = X/T$.
Let $z = a/\overline{u}$. From (8), we have separable ODE:

$$\overline{u}\frac{dz}{d\overline{u}} = -f(z), \quad f(z) = z + \frac{14z^2 + 75z + 90}{8z^2 + 40z + 45}$$

For Riemann problem (1), perform integration, we have algebraic equation to be solved for z^+:

$$\ln\left(\frac{u^+}{u^-}\right) = \Psi\left(z^-\right) - \Psi\left(z^+\right), \quad z^\pm = (a/u)^\pm$$

$$\Psi(z) = -\frac{1}{12}\sqrt{15}\ \tan^{-1}\left(\frac{15+8z}{\sqrt{15}}\right) - \frac{1}{4}\ln(3+z) + \frac{5}{8}\ln\left(15 + 15z + 4z^2\right)$$

Approximate BBM solitary limit: fitting method

Dispersion relation to BBM equation linearized on constant background $u = \overline{u}$ is

$$c_p = \frac{\omega_0}{k} = \frac{\overline{u}}{1+k^2} \quad (c_p \text{ phase velocity})$$

DSW fitting method of G. El (2005) is to assume that solitary wave motion on simple-wave background is governed by

$$\overline{u}_T + \left(\frac{\overline{u}^2}{2}\right)_X = 0 \tag{10a}$$

$$\tilde{k}_T + \widetilde{\omega_0}\left(\overline{u}, \tilde{k}\right)_X = 0 \tag{10b}$$

$$\widetilde{\omega}_0\left(\overline{u}, \tilde{k}\right) = -i\omega_0\left(\overline{u}, i\tilde{k}\right) = \frac{\overline{u}\tilde{k}}{1-\tilde{k}^2}, \quad D = \frac{\widetilde{\omega_0}}{\tilde{k}} = \overline{u} + \frac{a}{3}$$

$\widetilde{\omega}_0$ conjugate dispersion relation & $\tilde{k}$ conjugate wave number

It follows from (10) that one obtains ODE

$$\frac{d\tilde{k}}{d\overline{u}} = \frac{(\widetilde{\omega}_0)_{\overline{u}}}{\overline{u} - (\widetilde{\omega}_0)_{\tilde{k}}} = \frac{1 - \tilde{k}^2}{\overline{u}(-3\tilde{k} + \tilde{k}^3)}$$

For Riemann problem (1), perform integration, we have algebraic equation to be solved for $\tilde{k}^+$:

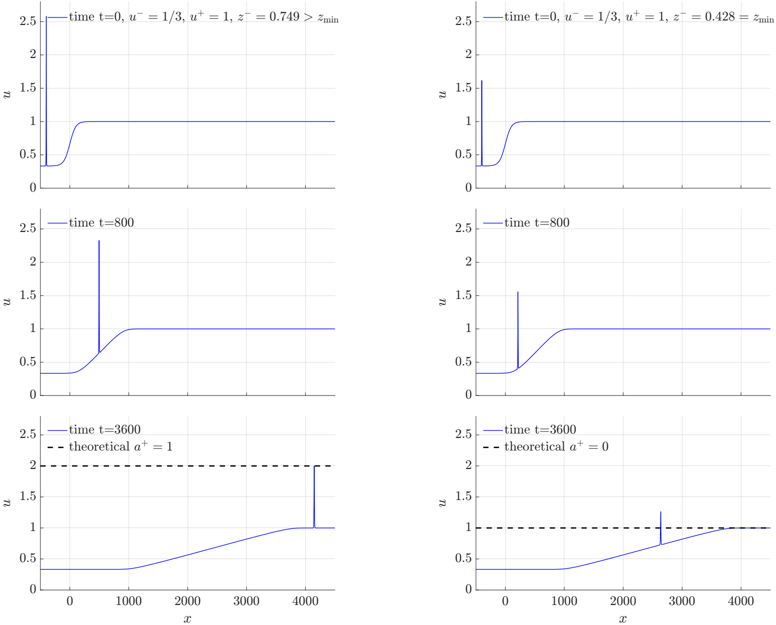
$$\ln\left(\frac{u^+}{u^-}\right) = \Psi\left(\tilde{k}^+\right) - \Psi\left(\tilde{k}^-\right), \quad \tilde{k}^- = 0$$

$$\Psi\left(\tilde{k}\right) = -\frac{\tilde{k}^2}{2} + \ln\left(\tilde{k}^2 - 1\right)$$

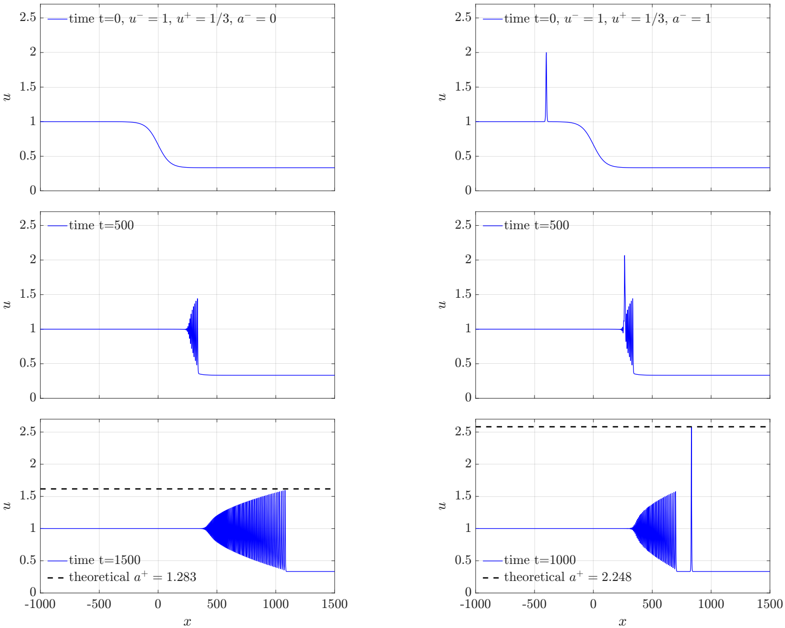
Solitary wave amplitude a^+ is:

$$a^+ = 3\left(\frac{\widetilde{\omega}_0\left(u^+, \tilde{k}^+\right)}{\tilde{k}^+} - u^+\right) = 3\frac{u^+(\tilde{k}^+)^2}{1 - (\tilde{k}^+)^2}$$

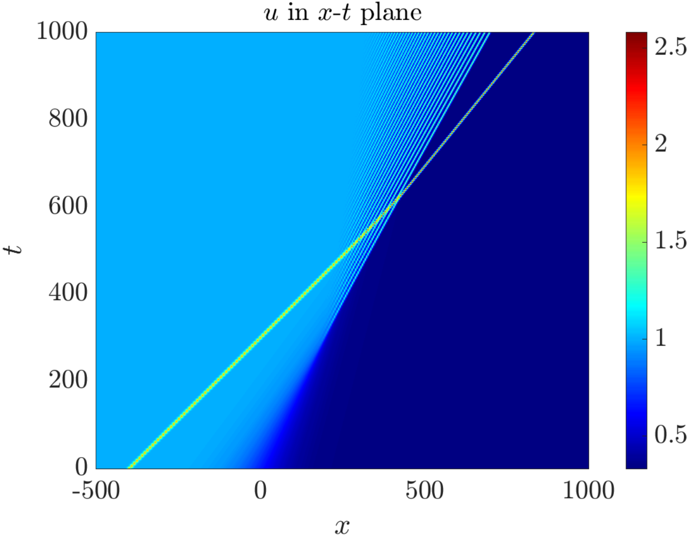
Solitary wave over RW: BBM equation



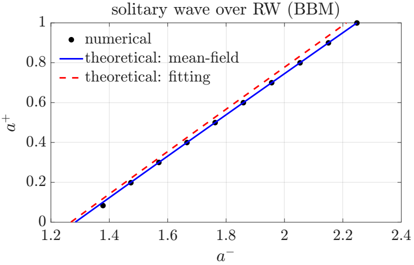
Solitary wave over DSW: BBM equation

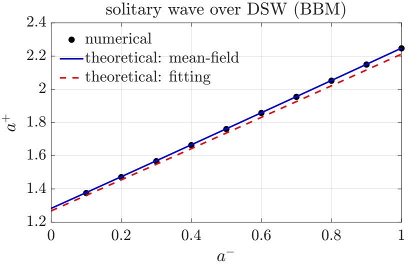


Solitary wave over DSW: BBM equation

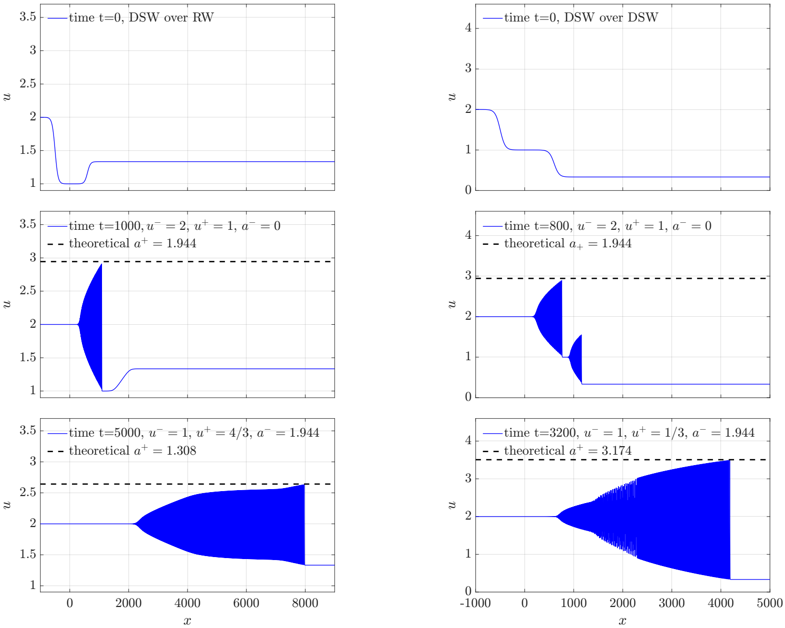


Transmission: analytical & numerical comparision





DSW over RW/DSW: BBM equation



Numerical approximation: BBM equation

BBM equation is

$$u_t + \left(\frac{1}{2}u^2\right)_x - u_{xxt} = 0$$

- u-based elliptic operator inversion

$$K_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad \text{(hyperbolic step)}$$
$$-u_{xx} + u = K \quad \text{(elliptic step)}$$

- ϖ-based elliptic operator inversion

$$u_t + \left(\frac{1}{2}u^2 + \varpi\right)_x = 0 \quad \text{(hyperbolic step)}$$
$$-\varpi_{xx} + \varpi = \left(\frac{u^2}{2}\right)_{xx} \quad \text{(elliptic step)}$$

SGN equations

SGN equations over a flat bottom approximating free-surface Euler equations in long wave limit are

$$h_t + (hu)_x = 0 \tag{11a}$$

$$u_t + uu_x + gh_x = \frac{1}{h}\left(\frac{h^3}{3}\left(u_{xt} + uu_{xx} - u_x^2\right)\right)_x \tag{11b}$$

h is total depth, u is depth-averaged horizontal velocity, and g is acceleration due to gravity (set $g = 1$ in what follows)

SGN equations admit energy conservation law:

$$\left(\frac{1}{2}h\left(h + u^2 + \frac{1}{3}h^2u_x^2\right)\right)_t + \left(hu\left(h + \frac{1}{2}u^2 + \frac{1}{2}h^2u_x^2 - \frac{1}{3}h^2\left(u_{xt} + uu_{xx}\right)\right)\right)_x = 0 \tag{11c}$$

Periodic travelling wave solutions: SGN equations

Periodic travelling wave solutions $h(x,t) = h(\xi)$, $u(x,t) = u(\xi)$, $\xi = x - Dt$ to SGN equations (11) follows

$$\left(h'\right)^2 = \frac{3}{h_1 h_2 h_3}(h-h_1)(h-h_2)(h_3-h) \tag{12a}$$

$$u = D - \sigma \frac{\sqrt{h_1 h_2 h_3}}{h} \tag{12b}$$

We have periodic travelling wave solution (4-parameter family):

$$h(\xi) = h_2 + a\,\mathrm{cn}^2\left(\frac{1}{2}\sqrt{\frac{3(h_3-h_1)}{h_1 h_2 h_3}}\xi, m\right) \tag{13}$$

$a = h_3 - h_2$, $m = \dfrac{h_3 - h_2}{h_3 - h_1}$, $\sigma = +1(\text{fast})$, $\sigma = -1(\text{slow})$

Averages of depth $\overline{h}$ & velocity $\overline{u}$ can be written as

$$\overline{h} = h_1 + (h_3 - h_1)\frac{\mathrm{E}(m)}{\mathrm{K}(m)}, \; \overline{u} = D - \sigma\sqrt{h_1 h_2 h_3}\,\frac{\Pi\left(1 - \frac{h_2}{h_3}, m\right)}{h_3 \mathrm{K}(m)}$$

When $h_2 \to h_1$ $(m \to 1)$, solitary wave limit of (13) on background $h = \overline{h}, u = \overline{u}$ is

$$h(x,t) = \overline{h} + a\,\mathrm{sech}^2\left(\frac{\sqrt{3a}}{\overline{h}\sqrt{\overline{h}+a}}(x - Dt)\right) \quad (14a)$$

$$u(x,t) = D - \sigma\frac{\overline{h}\sqrt{\overline{h}+a}}{h(x,t)}, \quad D = \overline{u} + \sigma\sqrt{\overline{h}+a} \quad (14b)$$

When $h_2 \to h_3$ $(m \to 0)$, harmonic limit of (13) yields small-amplitude linear wave characterised by dispersion relation

$$\omega = kD = \omega_0(k, \overline{h}, \overline{u}) \equiv k\overline{u} + \sigma k\sqrt{\frac{\overline{h}}{1 + \overline{h}^2 k^2/3}} \quad (15)$$

Whitham modulation system: SGN equation

Assume SGN periodic solution $h(\xi, X, T, \varepsilon)$, $u(\xi, X, T, \varepsilon)$ with respect to ξ & slowly varying to X & T. Applying Whitham averaging procedure to three SGN conservation laws (11) & augmenting them by wave conservation equation, one obtains SGN modulation system (El *et al.* 2006):

$$\overline{h}_T + (\overline{hu})_X = 0$$

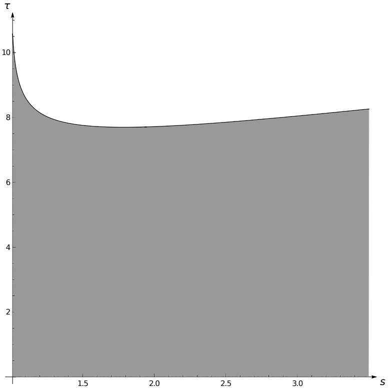
$$(\overline{hu})_T + \left(\overline{hu^2} + \frac{1}{2}\overline{h^2} - \frac{1}{3}\overline{h^3\left((u-D)u'' - (u')^2\right)}\right)_X = 0$$

$$\left(\frac{1}{2}\overline{h\left(h + u^2 + \frac{1}{3}h^2(u')^2\right)}\right)_T + \left(\overline{hu\left(h + \frac{1}{2}u^2 + \frac{1}{2}h^2(u')^2 - \frac{1}{3}h^2(u-D)u''\right)}\right)_X = 0$$

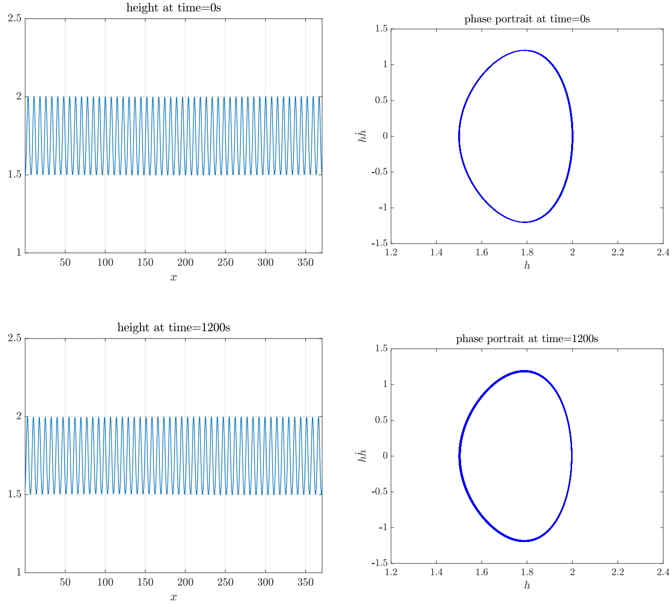
$$k_T + (kD)_X = 0$$

SGN modulational system: hyperbolicity

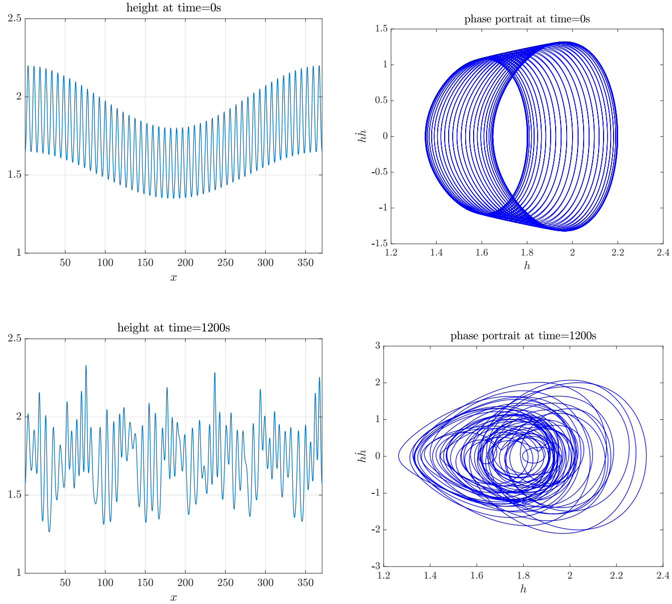
Region $s = h_2 > 1,\ \tau = h_3 - h_2 > 0$ is divided by smooth curve into two sub-regions: grey & white, corresponds to eigenvalues sign change. In both regions, all eigenvalues are real & distinct; system is genuinely nonlinear & strictly hyperbolic (Tkachenko *et al.* 2020)



Modulational instability: small amplitude $a = 10^{-3}$



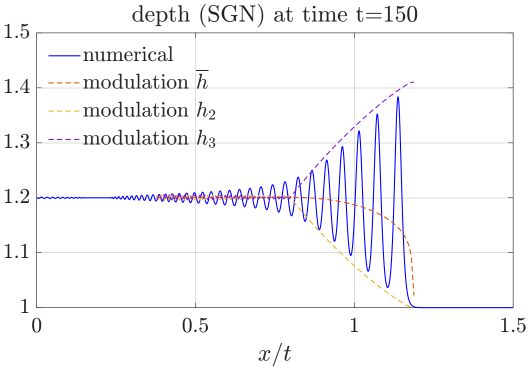
Modulational instability: large amplitude $a = 10^{-1}$



SGN modulation system: simple wave solution

Assume solution is self-similar: $q(x,t) = q(\zeta)$, $\zeta = x/t$. If $\nabla_q \lambda_j \cdot r_j \neq 0$, one obtains $q(\zeta)$ by solving ODEs numerically:

$$\frac{dq}{d\zeta} = r_j / \left(\nabla_q \lambda_j \cdot r_j\right), \quad \zeta = \lambda_j, \quad j = 1, 2, 3$$



SGN modulation system: solitary limit

Without going into details, we write down exact solitary limit for SGN modulation system as follows (Congy *et al.* 2025):

$$\overline{h}_T + \left(\overline{h}\overline{u}\right)_X = 0 \tag{16a}$$

$$\overline{u}_T + \overline{u}\,\overline{u}_X + \overline{h}_X = 0 \tag{16b}$$

$$\begin{aligned} z_T + \left(\overline{u} + \sigma\sqrt{\overline{h}(1+z^2)}\right) z_X + \\ \sigma\left(\frac{3\left(z^2+1\right)^{3/2}}{2z}\,\frac{z\sqrt{z^2+1}-\sinh^{-1}(z)}{2z\sqrt{z^2+1}-\sinh^{-1}(z)}\,\frac{1}{\sqrt{\overline{h}}}\right)\overline{h}_X + \\ \left(-\frac{\sqrt{z^2+1}}{2z}\,\frac{3z+2z^3-3\sqrt{z^2+1}\sinh^{-1}(z)}{2z\sqrt{z^2+1}-\sinh^{-1}(z)}\right)\overline{u}_X = 0 \end{aligned} \tag{16c}$$

Here $z^2 = a/\overline{h}$ & $\sigma = \pm 1$

SGN solitary limit: self-similar solution

Riemann invariants for (16a), (16b) are $r_\pm = \overline{u} \pm 2\sqrt{\overline{h}}$,

$$(r_\pm)_T + V_\pm (r_\pm)_X = 0, \quad V_\pm = \overline{u} \pm \sqrt{\overline{h}}$$

Let $z^2 = a/\overline{h}$. Assume solution in (16) is self-similar in $\zeta = X/T = V_+ = r_- + 3\sqrt{\overline{h}}$ (2-wave) with $r_- = \text{const}$. From (16), we have

$$\begin{aligned} &\overline{h}_T + \left(r_- + 3\sqrt{\overline{h}}\right)\overline{h}_X = 0 \\ &z_T + \left(r_- + 2\sqrt{\overline{h}} + \sigma\sqrt{\overline{h}(1+z^2)}\right) z_X + \sigma\frac{g_\sigma(z)}{\sqrt{\overline{h}}}\overline{h}_X = 0 \end{aligned} \tag{17}$$

$$g_\sigma(z) = \frac{3\left(z^2+1\right)^{3/2}}{2z}\frac{z\sqrt{z^2+1} - \sinh^{-1}(z)}{2z\sqrt{z^2+1} - \sinh^{-1}(z)} - \sigma\frac{1+z^2}{2z}\frac{(3z+2z^3)(1+z^2)^{-1/2} - 3\sinh^{-1}(z)}{2z\sqrt{1+z^2} - \sinh^{-1}(z)}$$

$\overline{h}$ is one Riemann invariant of (17). Employing simple-wave ansatz $z = z(\overline{h})$ in (17), we have ODE

$$\frac{dz}{d\overline{h}} + \frac{g_\sigma(z)}{\overline{h}\left(\sqrt{1+z^2}-\sigma\right)} = 0 \tag{18a}$$

Perform integration, we have

$$\ln \overline{h} = \Psi(z), \quad \Psi(z) = \int^z -\frac{\sqrt{1+s^2}-\sigma}{g_\sigma(s)}\, ds \tag{18b}$$

As in DSW fitting method for BBM equation, using approximate solitary limit equation (10b), one obtains

$$\frac{d\widetilde{k}}{d\overline{h}} = \frac{\widetilde{\omega}_{\overline{h}}}{V_+(\overline{h}) - \widetilde{\omega}_{\widetilde{k}}} = \frac{\widetilde{k}\left(\left(1+\frac{\overline{h}^2\widetilde{k}^2}{3}\right) + 2\left(1-\frac{\overline{h}^2\widetilde{k}^2}{3}\right)^{3/2} + 1\right)}{2\overline{h}\left(\left(1-\frac{\overline{h}^2\widetilde{k}^2}{3}\right)^{3/2} - 1\right)} \tag{19}$$

Solitary wave-mean flow: SGN equations

To solve SGN solitary wave-mean flow interaction problem with initial condition:

$$\left(\overline{h}, \overline{u}, a\right)(x,0) = \begin{cases} \left(h^-, u^-, a^-\right), & x < 0, \\ \left(h^+, u^+, a^+(?)\right), & x > 0 \end{cases}$$

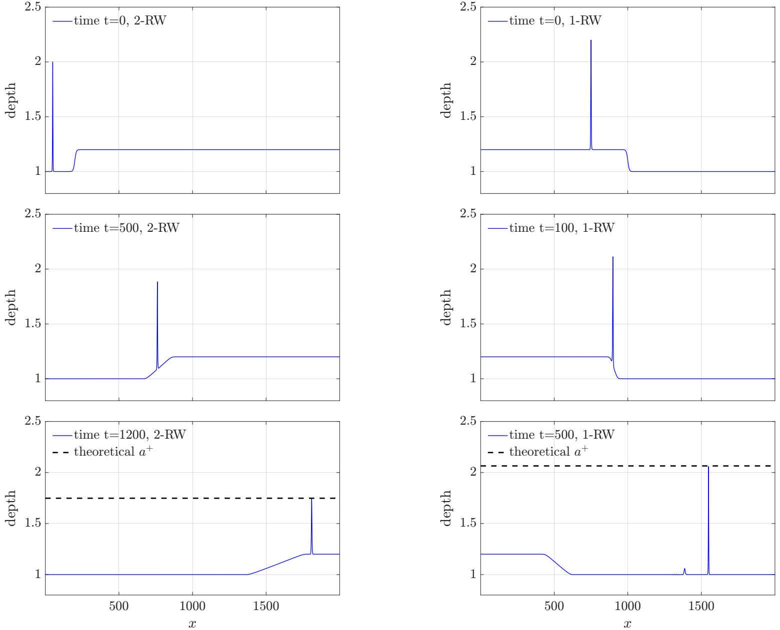
one may use (18) or (19) for governing equation

For 2-wave case,, condition for solitary wave trapping is $z^+ = 0$. One need to take $z^- > z^-_{\min}$ which is a unique root of

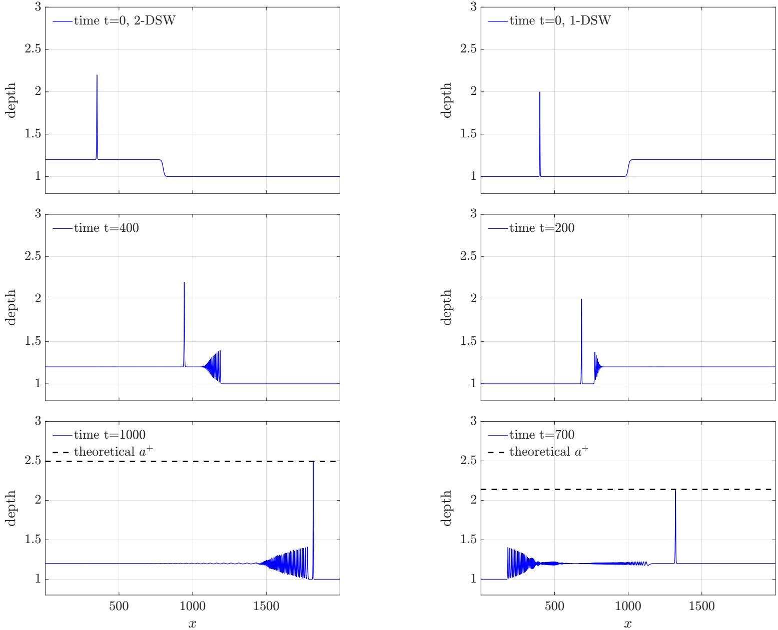
$$\Psi(z^-_{\min}) - \Psi(0) - \ln\left(\frac{h^+}{h^-}\right) = 0$$

to have solitary wave transmission through initial step function

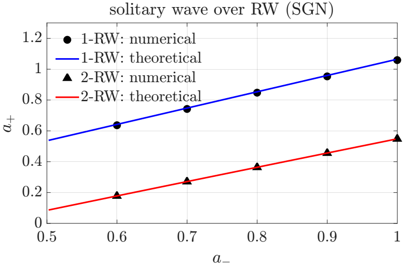
Solitary wave over RW: SGN equations

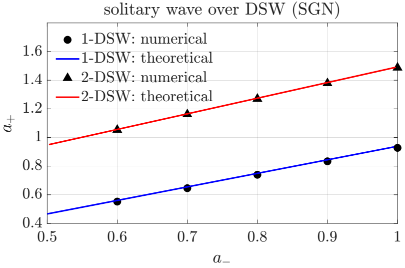


Solitary wave over DSW: SGN equations

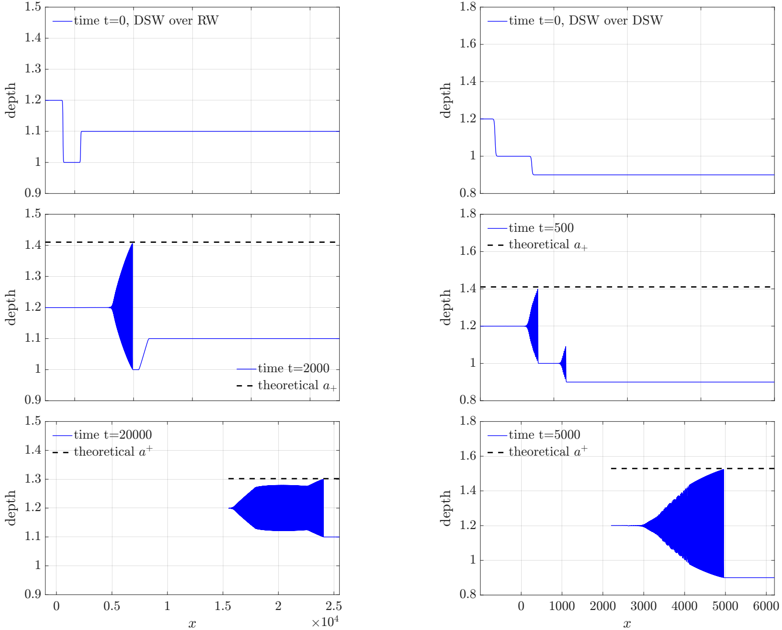


Transmission: analytical & numerical comparison





DSW over RW/DSW: SGN equations

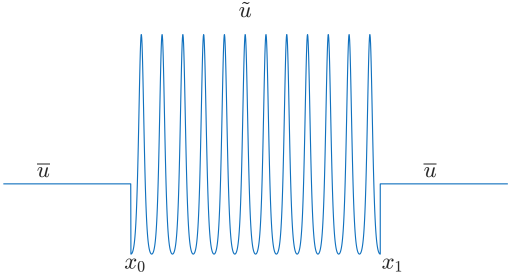


Generalized Riemann problem: dispersive equations

Generalized Riemann problem (GRP) for dispersive equations is Cauchy problem with initial condition:

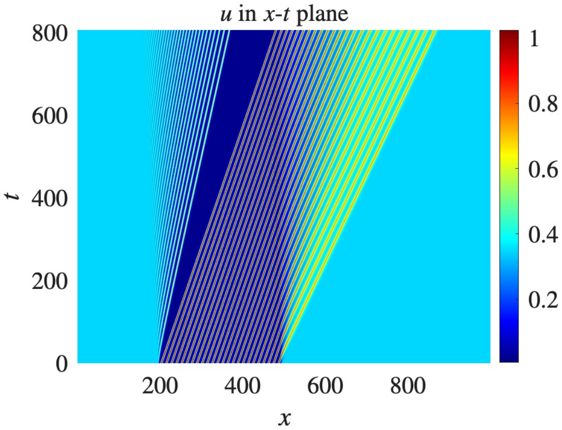
$$u(x,0) = \begin{cases} \tilde{u}(x), & x \in (x_0, x_1)\,, \\ \overline{u}, & x \in \,]x_0, x_1[\,. \end{cases}$$

$\tilde{u}(x)$ is wave train of periodic travelling wave solution & $\overline{u}$ is wave average of $\tilde{u}$

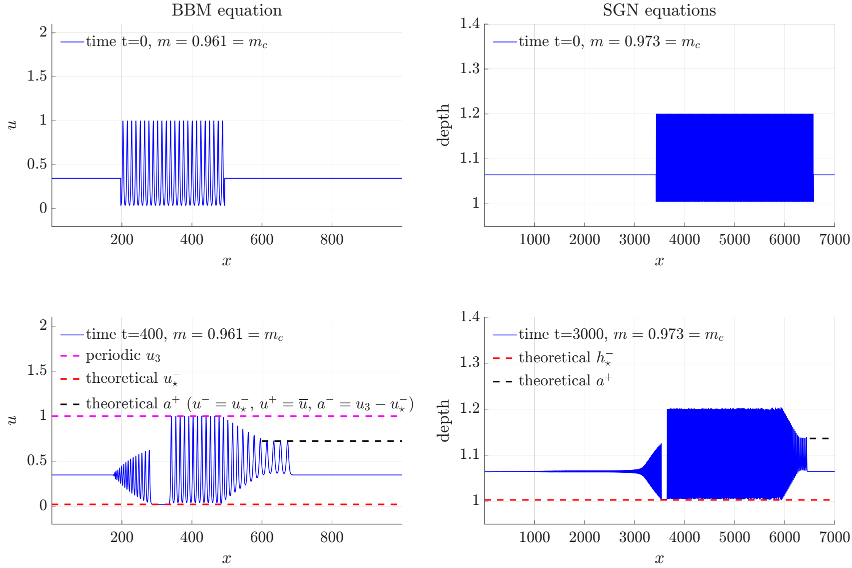


GRP solution structure: dispersive equations

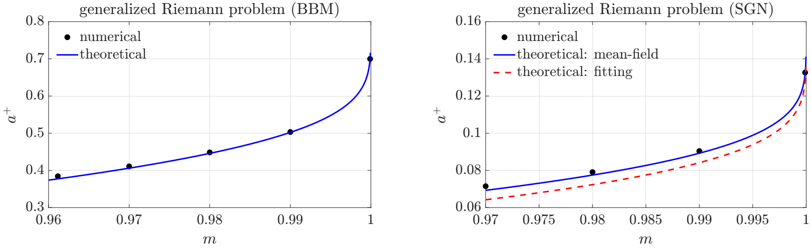
On the right, a right-facing DSW is formed followed by a left-facing RW. A constant state, denoted by u_*^-, is formed on the left of the initial periodic wave train



GRP solutions: dispersive equations



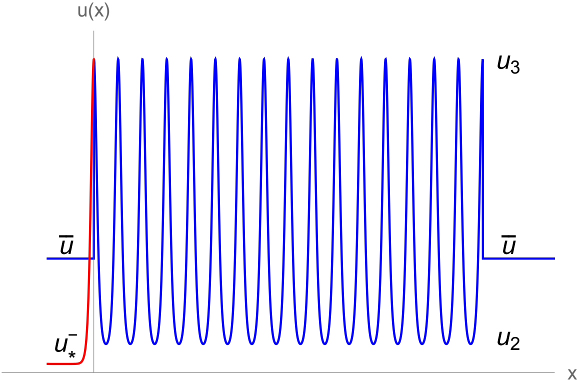
Transmission: analytical & numerical comparison



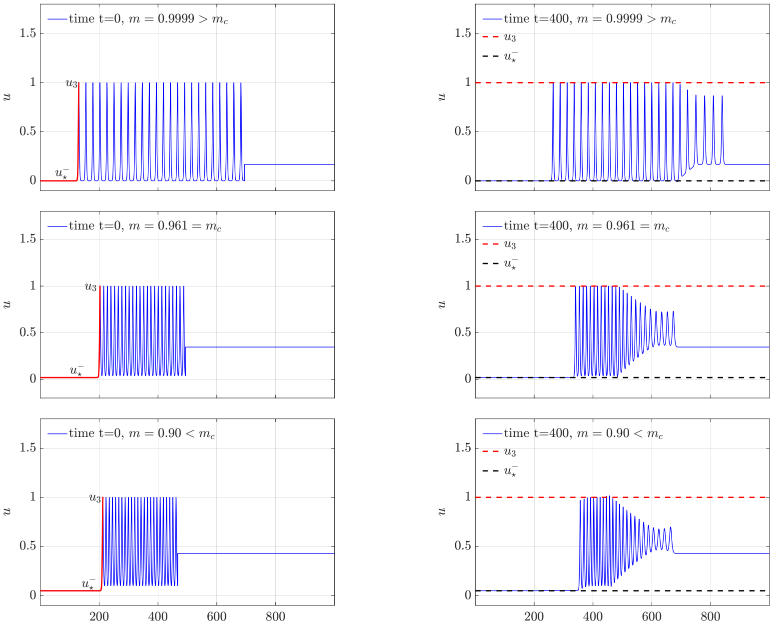
- $u_\star^-$ & $h_\star^-$ are solutions of generalized Rankine-Hugoniot conditions for BBM & SGN modulation equations, respectively
- Recall $m = \dfrac{u_3 - u_2}{u_3 - u_1}$ or $m = \dfrac{h_3 - h_2}{h_3 - h_1}$. Critical value $m = m_c$ satisfies $D = \overline{\lambda}$ (phase speed = dispersonless characteristic velocity)

Stable shock-like travelling front

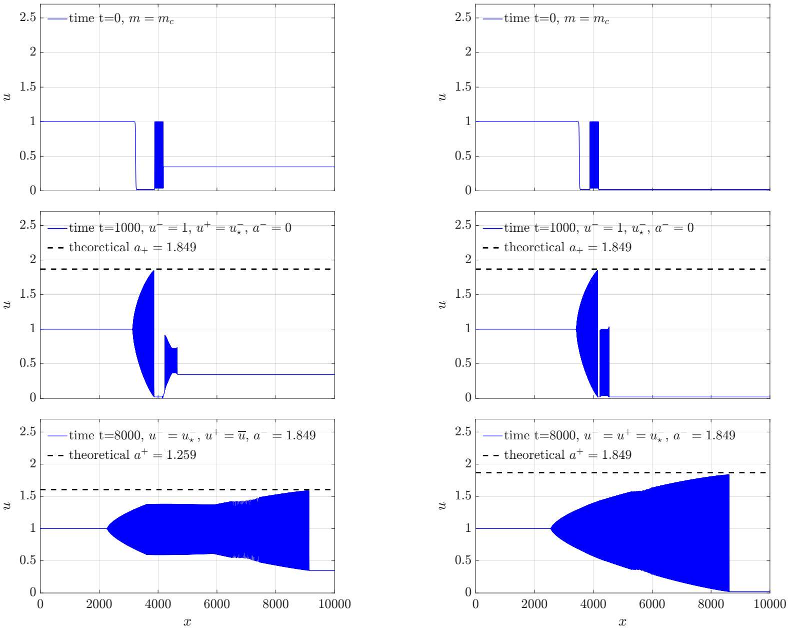
If, initially, instead of $\overline{u}$, we put $u_\star^-$ on left connected with wave train by half-solitary wave (red curve), would left boundary of wave train remain invariable in time ?



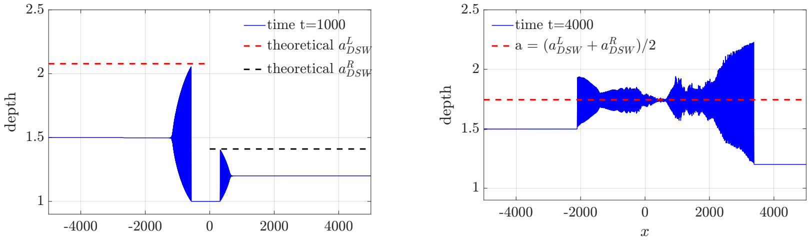
Stable shock-like travelling front: BBM equation



DSW over multi-hump: BBM equation

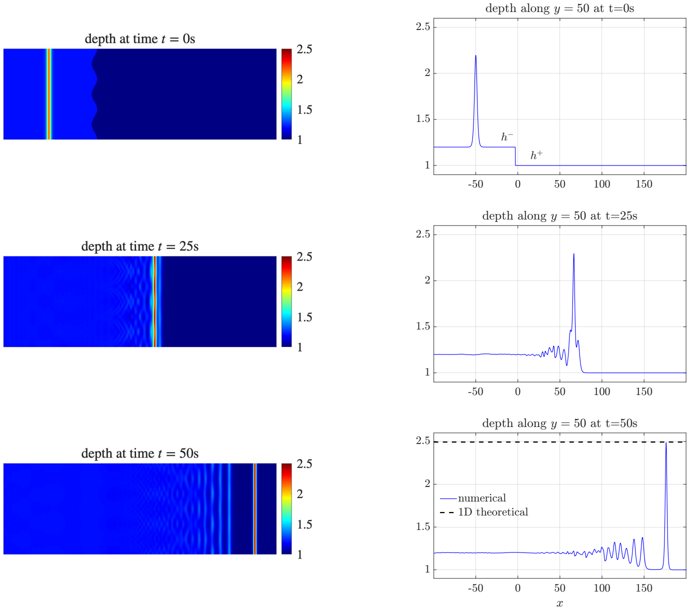


Headon DSW–DSW interaction: SGN equations

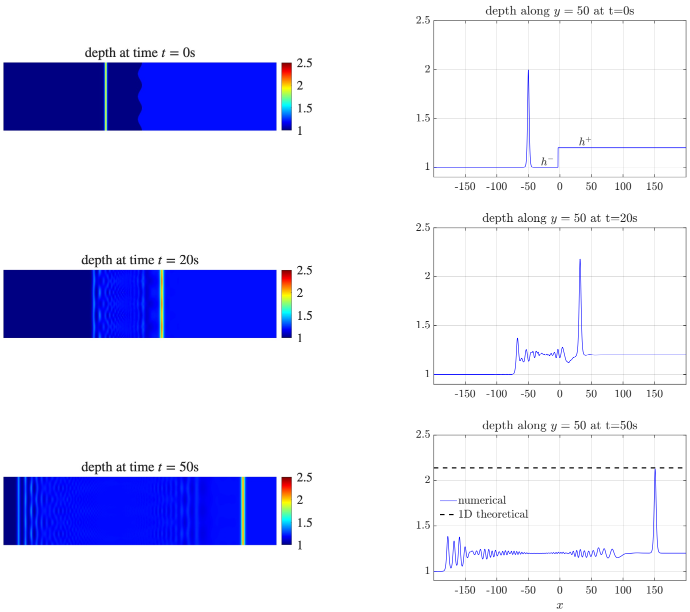


- Solitary limit construction does not apply to head-on DSW interaction, due to unidirectional nature of wave generation in our Cauchy problems
- Solitary wave limit solution works only outside self-similar fan, but not within fan

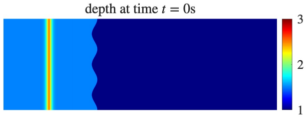
Solitary wave over perturbed 2-DSW: $h^+/h^- = 1.2$

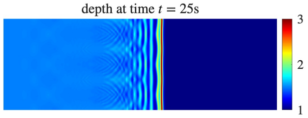


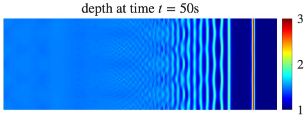
Solitary wave over perturbed 1-DSW: $h^+/h^- = 1.2$

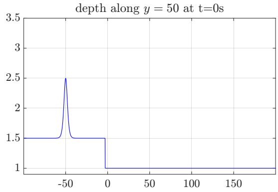


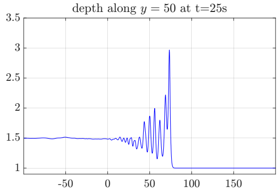
Solitary wave over perturbed 2-DSW: $h^+/h^- = 1.5$

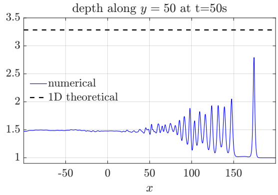




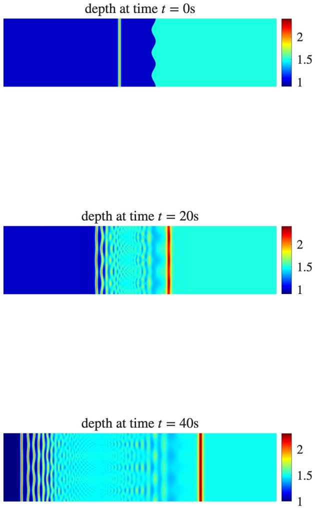


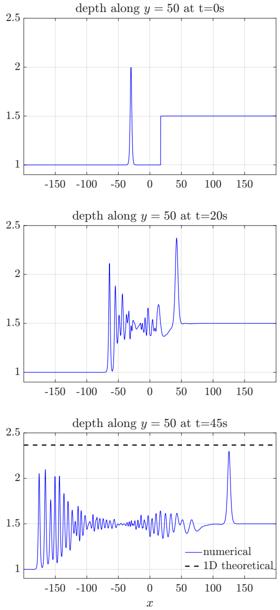






Solitary wave over perturbed 1-DSW: $h^+/h^- = 1.5$





SGN equations: alternative form

Recall SGN momentum equation (11b) reads

$$u_t + uu_x + gh_x = \frac{1}{h}\left(\frac{h^3}{3}\left(u_{xt} + uu_{xx} - u_x^2\right)\right)_x$$

Equivalent form of (11b) is

$$(hu)_t + \left(hu^2 + p\right)_x = 0, \quad p = \frac{1}{2}gh^2 + \frac{1}{3}h^2\ddot{h}$$

p is integrated fluid pressure divided by constant density ρ,
$\dot{h} = h_t + uh_x$, $\ddot{h} = \dot{h}_t + u\dot{h}_x$

Bernoulli conservation law for $\mathcal{K}$ representing tangent component of fluid velocity at free surface (Gavrilyuk *et al.* 2015) is:

$$\mathcal{K}_t + \left(\mathcal{K}u + gh - \frac{u^2}{2} - \frac{1}{2}h_x^2u^2\right)_x = 0, \quad \mathcal{K} = u - \frac{1}{3h}\left(h^3u_x\right)_x$$

Numerical approximation: SGN equation

1. u-based elliptic operator inversion
   - Hyperbolic step for $(h, h\mathcal{K})$

$$h_t + (hu)_x = 0, \quad (h\mathcal{K})_t + \left(h\mathcal{K}u + \frac{1}{2}gh^2\right) = \left(\frac{2}{3}h^3\left(u_x\right)^2\right)_x$$

   - Elliptic step for u

$$u - \frac{1}{3h}\left(h^3 u_x\right)_x = \mathcal{K}$$

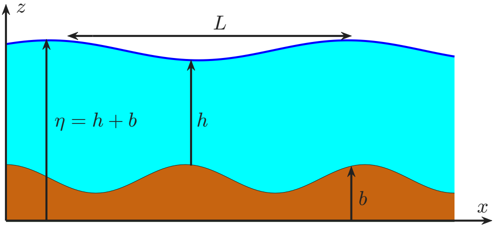
2. ϖ-based elliptic operator inversion
   - Hyperbolic step for (h, hu)

$$h_t + (hu)_x = 0, \quad (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -\varpi_x$$

   - Elliptic step for ϖ $(p = \varpi + \frac{1}{2}gh^2)$

$$-\frac{h^3}{3}\left(\frac{\varpi_x}{h}\right)_x + \varpi = \frac{2}{3}h^3 u_x^2 + \frac{h^3}{3}gh_{xx}$$

SGN equations over topography



SGN equations are fully nonlinear Boussinesq-type model (cf. weakly nonlinear models: Nwogu and Madsen & Sørensen). With bottom topography, it can be written as:

$$h_t + \operatorname{div}(h\mathbf{u}) = 0 \tag{20a}$$

$$(h\mathbf{u})_t + \operatorname{div}\left(h\mathbf{u} \otimes \mathbf{u}\right) + \nabla p = -p\,\Big|_{z=b} \nabla b \tag{20b}$$

$$p = \frac{gh^2}{2} + \frac{h^2}{3}\left(\ddot{h} + \frac{3}{2}\ddot{b}\right), \qquad p\,\Big|_{z=b} = gh + h\left(\ddot{b} + \frac{1}{2}\ddot{h}\right)$$

Enhanced SGN equations with topography

Improved formulation of SGN equations (cf. Bonneton *et al.* 2011, Tissier *et al.* 2012, Berger & LeVeque 2024) is:

$$h_t + (hu)_x + (hv)_y = 0 \tag{21a}$$

$$(hu)_t + \left(hu^2\right)_x + (huv)_y + gh\eta_x = h\left(\frac{g}{\alpha}\eta_x - \psi_1\right) \tag{21b}$$

$$(hv)_t + (huv)_x + \left(hv^2\right)_y + gh\eta_y = h\left(\frac{g}{\alpha}\eta_y - \psi_2\right) \tag{21c}$$

Vector $\boldsymbol{\psi} = [\psi_1, \psi_2]$ satisfies elliptic equation of form

$$(I + \alpha\mathcal{T})\,\boldsymbol{\psi} = \boldsymbol{Z}, \quad \mathcal{T} = \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{bmatrix}, \quad \boldsymbol{Z} = [Z_1, Z_2]$$

$$\mathcal{T}_{11} = -\frac{h^2}{3}\partial_x^2 - hh_x\partial_x + \frac{h}{2}b_{xx} + b_x\eta_x$$

$$\mathcal{T}_{12} = -\frac{h^2}{3}\partial_x\partial_y + \frac{h}{2}b_y\partial_x - h\left(h_x + \frac{1}{2}b_x\right)\partial_y + \frac{h}{2}b_{xy} + b_y\eta_x$$

$$\mathcal{T}_{21} = -\frac{h^2}{3}\partial_x\partial_y - h\left(h_y + \frac{1}{2}b_y\right)\partial_x + \frac{h}{2}b_x\partial_y + \frac{h}{2}b_{xy} + b_y\eta_y$$

$$\mathcal{T}_{22} = -\frac{h^2}{3}\partial_y^2 - hh_y\partial_y + \frac{h}{2}b_{yy} + b_y\eta_y$$

$$Z_1 = \frac{g}{\alpha}\eta_x + 2h\left(\frac{h}{3}\phi_x + \phi\left(h_x + \frac{1}{2}b_x\right)\right) + \frac{h}{2}w_x + w\eta_x$$

$$Z_2 = \frac{g}{\alpha}\eta_y + 2h\left(\frac{h}{3}\phi_y + \phi\left(h_y + \frac{1}{2}b_y\right)\right) + \frac{h}{2}w_y + w\eta_y$$

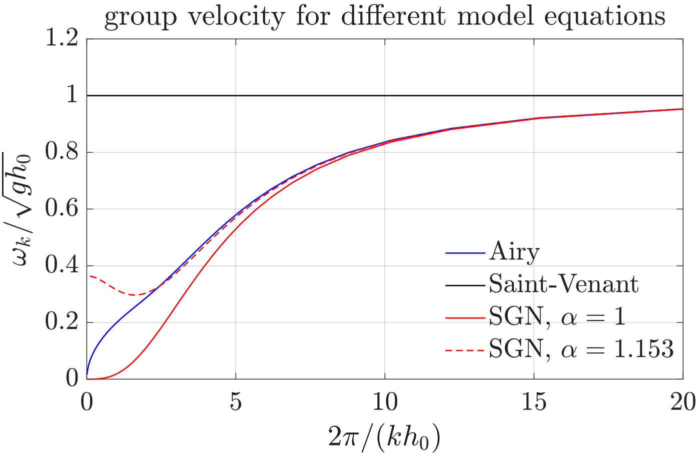
$$\phi = v_x u_y - u_x v_y + (u_x + v_y)^2\,, \quad w = u^2 b_{xx} + 2uvb_{xy} + v^2 b_{yy}$$

To solve (21) numerically, one approach is fractional-step method:

- Elliptic step for dispersive source term $\boldsymbol{\psi}$
- Hyperbolic step for h, hu, & hv

Detailed numerical procedures would vary depending on employed discretization methods

Shallow water models: group velocity comparison



Numerical SGN model: pressure-based formulation

To simplify elliptic-step computation in 2D SGN, Gavrilyuk & Shyue (2023) derived scalar elliptic equation for averaged pressure p defined in (20) as

$$
\begin{aligned}
&-\frac{h^3}{3}\nabla\cdot\left(\frac{\nabla p}{h}\right)-\frac{h^3}{2}\nabla\cdot\left(\frac{p\nabla b}{h^2}\right)+p= \\
&\quad \frac{2h^3}{3}\left[(\nabla\cdot\mathbf{u})^2-\det\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\right]+\frac{1}{2}gh^2+\frac{1}{2}h^2\ddot{b}-\frac{h^3}{3}\nabla\cdot\Upsilon
\end{aligned}
$$

with

$$
\Upsilon=\begin{bmatrix}-\left(g+\ddot{b}\right)b_x/4\\ -\left(g+\ddot{b}\right)b_y/4\end{bmatrix}
$$

Numerical SGN model: ϖ-based formulation

Let $\varpi = p - \frac{1}{2}gh^2 = \frac{h^2}{3}\left(\ddot{h} + \frac{3}{2}\ddot{b}\right)$ be non-hydrostatic part of averaged pressure p

Alternative elliptic equation written in ϖ is:

$$-\frac{h^3}{3}\left[\left(\frac{\varpi_x}{h}\right)_x + \left(\frac{\varpi_y}{h}\right)_y\right] - \frac{h^3}{2}\left[\left(\frac{b_x}{h^2}\varpi\right)_x + \left(\frac{b_y}{h^2}\varpi\right)_y\right] + \varpi =$$
$$\frac{2h^3}{3}\left[(U_x + V_y)^2 - (U_x V_y - U_y V_x)\right] + \frac{h^2}{2}\ddot{b} +$$
$$\frac{h^3}{3}\left[\left(g\,(h+b)_x + \frac{1}{4}\ddot{b}\,b_x\right)_x + \left(g\,(h+b)_y + \frac{1}{4}\ddot{b}\,b_y\right)_y\right]$$

Separating ϖ from p has advantage in hyperbolic-step approximation. Here $\mathbf{u} = (U, V)^T$ & $\mathbf{x} = (x, y)^T$

Numerical method for ϖ-based SGN model

We employ hyperbolic-elliptic splitting in solving SGN model

- Hyperbolic step: well-balanced scheme

$$\boldsymbol{q}_t + \operatorname{div}\boldsymbol{\mathcal{F}}(\boldsymbol{q}) + \boldsymbol{\zeta}(\boldsymbol{q}, \nabla b) = \boldsymbol{\psi}\left(\boldsymbol{q}, \nabla \boldsymbol{q}, \nabla b, \nabla^2 b, \varpi, \nabla\varpi\right)$$

$$\boldsymbol{q} = \begin{bmatrix} h \\ hU \\ hV \\ b \end{bmatrix}, \quad \boldsymbol{\mathcal{F}} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \end{bmatrix} = \begin{bmatrix} hU & hV \\ hU^2 + \frac{1}{2}gh^2 & hUV \\ hUV & hV^2 + \frac{1}{2}gh^2 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\zeta} = \begin{bmatrix} 0 \\ -ghb_x \\ -ghb_y \\ 0 \end{bmatrix}, \qquad \boldsymbol{\psi} = \begin{bmatrix} 0 \\ -\frac{3\varpi}{2h} b_x - \frac{h}{4}\ddot{b} b_x - \varpi_x \\ -\frac{3\varpi}{2h} b_y - \frac{h}{4}\ddot{b} b_y - \varpi_y \\ 0 \end{bmatrix}$$

- Elliptic step

$$-\frac{h^3}{3}\nabla\cdot\left(\frac{1}{h}\nabla\varpi\right) - \frac{h^3}{2}\nabla\cdot\left(\frac{\nabla b}{h^2}\varpi\right) + \varpi = \varphi\left(\boldsymbol{q}, \nabla\boldsymbol{q}, \cdots\right)$$

Derivation of SGN averaged pressure equation

SGN equations (20) in 2D are:

$$h_t + (hU)_x + (hV)_y = 0$$
$$(hU)_t + \left(hU^2 + p\right)_x + (hUV)_y = -\left[gh + h\left(\ddot{b} + \frac{1}{2}\ddot{h}\right)\right] b_x$$
$$(hV)_t + (hUV)_x + \left(hV^2 + p\right)_y = -\left[gh + h\left(\ddot{b} + \frac{1}{2}\ddot{h}\right)\right] b_y$$

With fluid pressure p defined by $p = \dfrac{gh^2}{2} + h^2\left(\dfrac{1}{2}\ddot{b} + \dfrac{1}{3}\ddot{h}\right)$,

rewrite momentum equations as:

$$\dot{U} + \frac{p_x}{h} = -\frac{1}{4}\left(g + \ddot{b} + \frac{6p}{h^2}\right) b_x$$
$$\dot{V} + \frac{p_y}{h} = -\frac{1}{4}\left(g + \ddot{b} + \frac{6p}{h^2}\right) b_y$$

Taking divergence of above system, we find

$$-\nabla \cdot \left(\frac{\nabla p}{h}\right) + \nabla \cdot \Psi = \nabla \cdot (\dot{\mathbf{u}})\,, \quad \Psi = \begin{bmatrix} -\left(g + \ddot{b} + 6p/h^2\right) b_x/4 \\ -\left(g + \ddot{b} + 6p/h^2\right) b_y/4 \end{bmatrix}$$

Using vector calculus, we have

$$\nabla \cdot (\dot{\mathbf{u}}) = \overbrace{(\nabla \cdot \mathbf{u})}^{\cdot} + (\nabla \cdot \mathbf{u})^2 - 2 \det \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right), \qquad \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \in \mathbf{R}^{2\times 2}$$

Thus one obtains:

$$-\nabla \cdot \left(\frac{\nabla p}{h}\right) + \nabla \cdot \Psi = \overbrace{(\nabla \cdot \mathbf{u})}^{\cdot} + (\nabla \cdot \mathbf{u})^2 - 2 \det \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)$$

which with $\nabla \cdot \mathbf{u} = -\dot{h}/h$ gives

$$-\nabla \cdot \left(\frac{\nabla p}{h}\right) + \nabla \cdot \Psi = -\frac{\ddot{h}}{h} + 2\frac{\dot{h}^2}{h^2} - 2 \det \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)$$

This leads to

$$\frac{1}{3}h^2\ddot{h} = \frac{2}{3}h^3\left(\frac{\dot{h}^2}{h^2} - \text{det}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\right) + \frac{1}{3}h^3\nabla\cdot\left(\frac{\nabla p}{h}\right) - \frac{1}{3}h^3\nabla\cdot\Psi$$
$$= p - \frac{1}{2}gh^2 - \frac{1}{2}h^2\ddot{b}$$

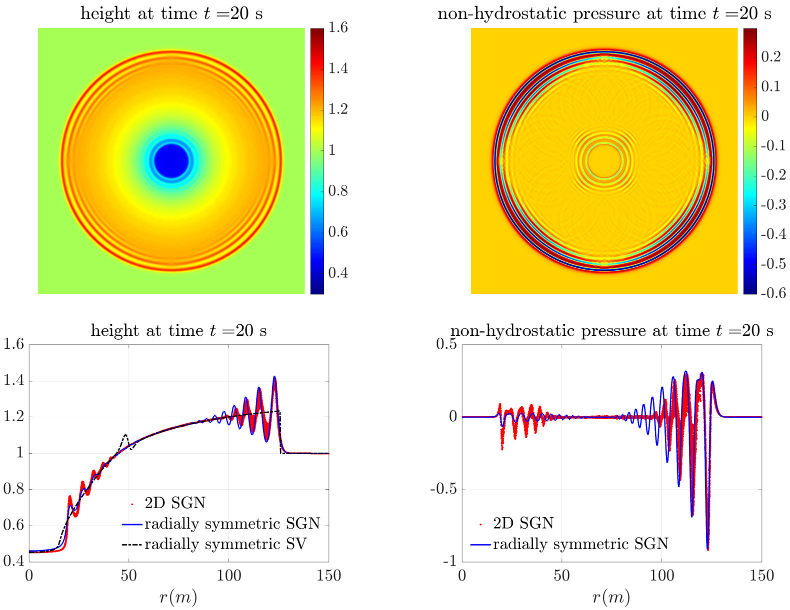
Thus we arrive at

$$-\frac{h^3}{3}\nabla\cdot\left(\frac{\nabla p}{h}\right) - \frac{h^3}{2}\nabla\cdot\left(\frac{p\nabla b}{h^2}\right) + p =$$
$$\frac{2h^3}{3}\left[(\nabla\cdot\mathbf{u})^2 - \text{det}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\right] + \frac{1}{2}gh^2 + \frac{1}{2}h^2\ddot{b} - \frac{h^3}{3}\nabla\cdot\Upsilon$$

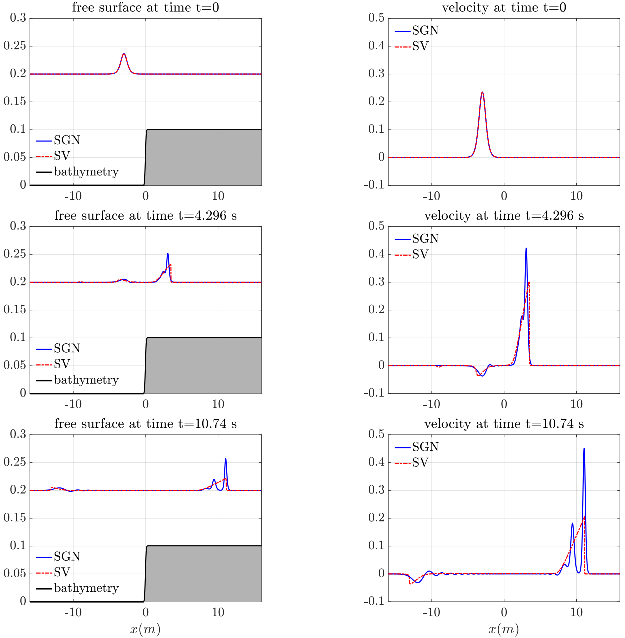
with

$$\Upsilon = \begin{bmatrix} -\left(g+\ddot{b}\right)b_x/4 \\ -\left(g+\ddot{b}\right)b_y/4 \end{bmatrix}$$

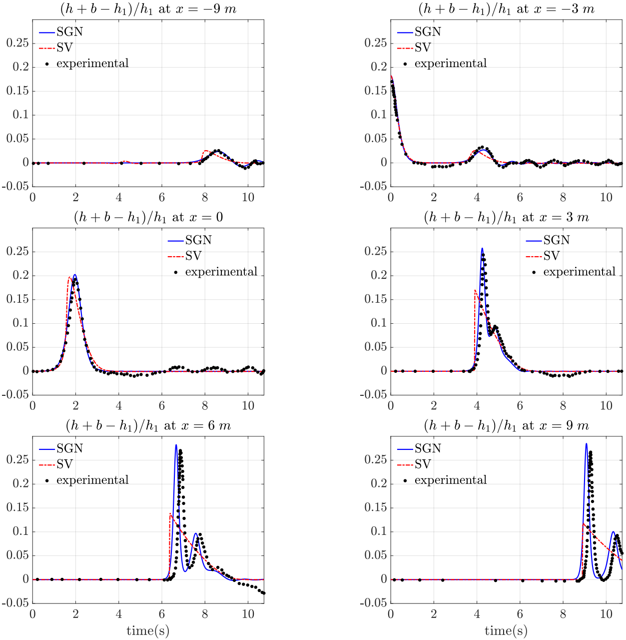
Radially symmetric problems: SGN equations



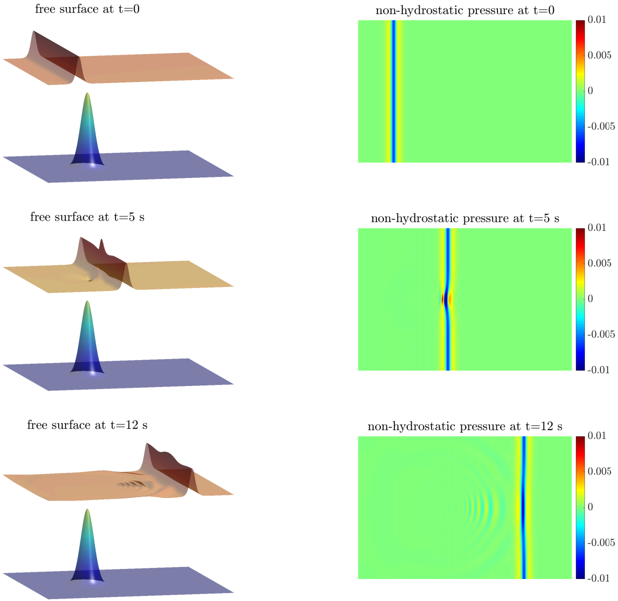
Solitary wave over bottom step: SGN equations



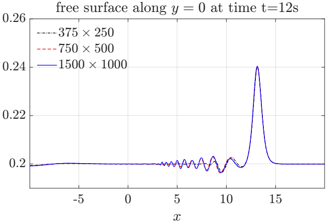
Solitary wave over bottom step: time history

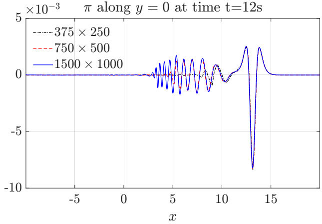


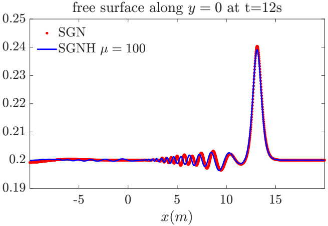
Solitary wave over Gaussian hump

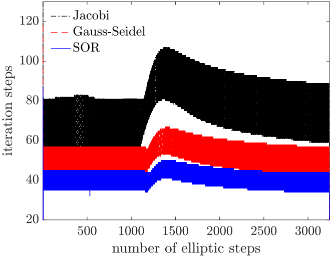


Solitary wave over Gaussian hump: Convergence









Hyperbolic SGN (SGNH) model over topography

To avoid elliptic operator inversion Favrie & Gavrilyuk (2016) considered hyperbolic approximation of SGN equations:

$$
\begin{aligned}
& h_t + \operatorname{div}(h\mathbf{u}) = 0 \\
& (h\mathbf{u})_t + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u}) + \nabla p_\mu = -\left[gh + \frac{\mu}{2}\left(\frac{\eta}{h} - 1\right)\right]\nabla b \\
& (h\eta)_t + \operatorname{div}(h\eta\mathbf{u}) = hw \\
& (hw)_t + \operatorname{div}(hw\mathbf{u}) = -\mu\left(\frac{\eta}{h} - 1\right)
\end{aligned}
$$

$\mu \in \mathbf{R}_+$ relaxation parameter & p_μ relaxed pressure

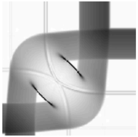
$$p_\mu = \frac{1}{2}gh^2 - \frac{\mu}{3}\left(\frac{\eta}{h} - 1\right)\eta$$

Model is hyperbolic with real eigenvalues & complete set of eigenvectors

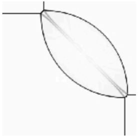
Duchene (2019) gave rigorous mathematical justification of hyperbolic model ($b = 0$ case) to SGN as $\mu \to \infty$

Compressible gas dynamics: 2D Riemann problems

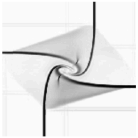

Configuration 1


Configuration 2


Configuration 3


Configuration 4

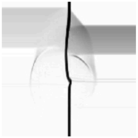

Configuration 5

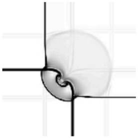

Configuration 6

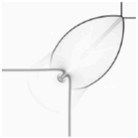

Configuration 7

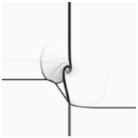

Configuration 8


Configuration 9


Configuration 10

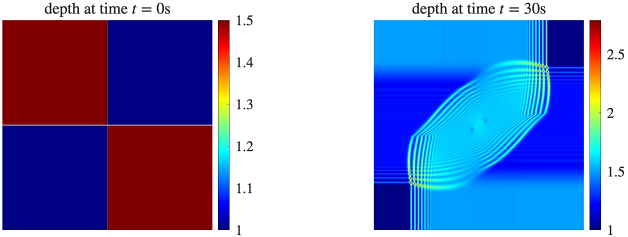




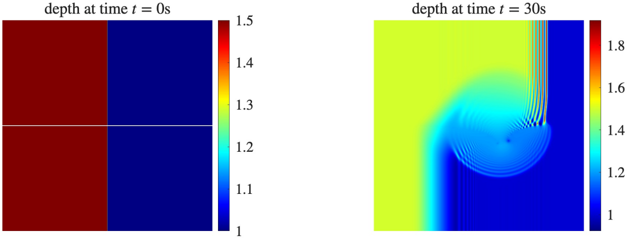


SGN equations: 2D Riemann problems

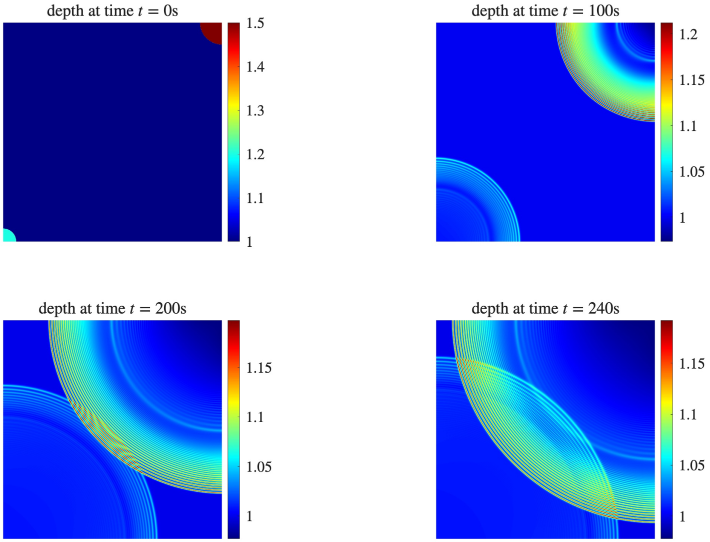
2 DSW–1 DSW case



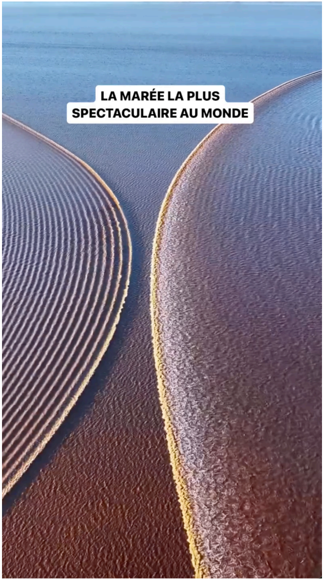
2 DSW–1 RW case



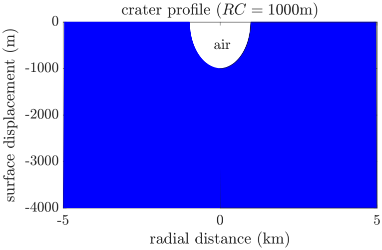
Headon DSW–DSW interaction

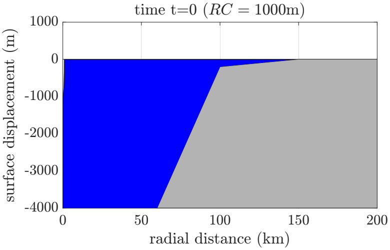


Qiantang river bore (from Dailymotion)



Impact induced tsunami wave over reef



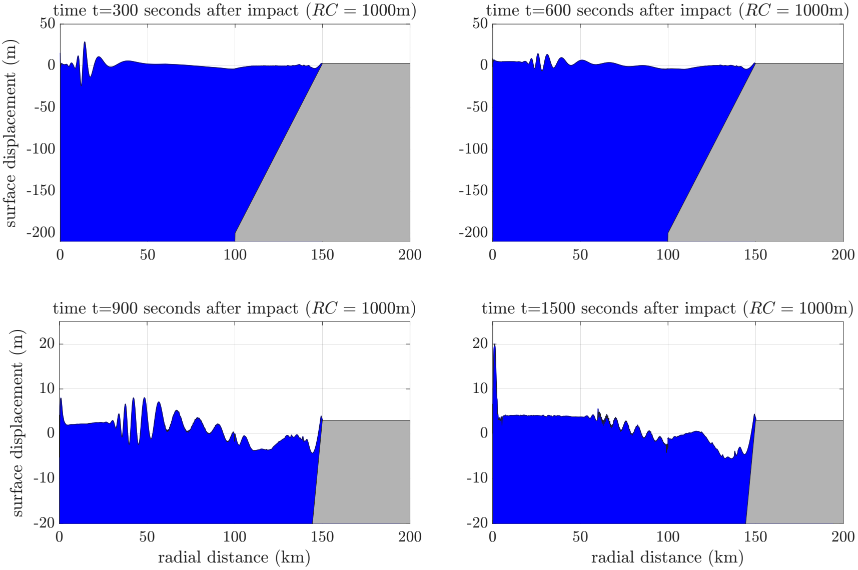


Numerical modelling approach:

(i) Initial wave generation
Direct numerical impact simulation

(ii) Wave propagation & run-up
Depth-average Boussinesq simulation

Impact induced waves: direct numerical simulation



- 5-equation transport model

$$
\begin{aligned}
&(\alpha_1\rho_1)_t + (\alpha_1\rho_1 u)_r + (\alpha_1\rho_1 v)_z = -\frac{1}{r}\alpha_1\rho_1 u \\
&(\alpha_2\rho_2)_t + (\alpha_2\rho_2 u)_r + (\alpha_2\rho_2 v)_z = -\frac{1}{r}\alpha_2\rho_2 u \\
&(\rho u)_t + \left(\rho u^2 + p\right)_r + (\rho u v)_z = -\frac{1}{r}\rho u^2 \\
&(\rho v)_t + (\rho u v)_r + \left(\rho v^2 + p\right)_z = -\frac{1}{r}\rho u v - \rho g \\
&(\rho E)_t + (\rho H u)_r + (\rho H v)_z = -\frac{1}{r}\rho H u - \rho v g \\
&\alpha_{1,t} + u\alpha_{1,r} + v\alpha_{1,z} = 0
\end{aligned}
$$

- Equation of state: stiffened gas

$$
\rho e(p,\alpha) = \alpha\left(\frac{p+\gamma_1 p_{\infty,1}}{\gamma_1 - 1}\right) + (1-\alpha)\left(\frac{p+\gamma_2 p_{\infty,2}}{\gamma_2 - 1}\right)
$$

- $\rho = \alpha_1\rho_1 + \alpha_2\rho_2, \quad \alpha_1 + \alpha_2 = 1$

Hyperbolic-step solver for SGN equations: remark

1. Solitary wave problem

	Godunov		MUSCL		WENO 3	
N	$E^1(h)$	order	$E^1(h)$	order	$E^1(h)$	order
1200	2.595e+02		4.894e+00		2.088e−01	
2400	1.470e+02	0.82	1.210e+00	2.02	5.237e−02	2.00
4800	7.834e+01	0.91	3.005e−01	2.01	1.310e−02	2.00
9600	4.044e+01	0.95	7.487e−02	2.01	3.273e−03	2.04

2. Periodic wave problem

	Godunov		MUSCL		WENO 3	
N	$E^1(h)$	order	$E^1(h)$	order	$E^1(h)$	order
300	1.346e−01		5.250e−03		3.521e−03	
600	7.749e−02	0.83	1.094e−03	2.37	4.563e−04	3.09
1200	4.100e−02	0.92	2.482e−04	2.15	5.927e−05	2.96
2400	2.112e−02	0.96	6.072e−05	2.03	7.923e−06	2.90

Adaptive reconstruction scheme

Let q_L^ι & q_R^ι be interpolated states at left & right cell edges of numerical reconstruction scheme $Q_j^\iota(x)$ for $\iota = A$, B

Boundary variation diminising (BVD) selection algorithm consists of two steps (Deng *et al.* 2018):

1. Compute values of minimum total boundary variation (mTBV) for schemes ι for $\iota = A$ & B

$$\begin{aligned}\mathrm{mTBV}_\mathrm{j}^\iota = \min\Big(&\left|q_{L,j-1/2}^A - q_{R,j-1/2}^\iota\right| + \left|q_{L,j+1/2}^\iota - q_{R,j+1/2}^A\right|,\\ &\left|q_{L,j-1/2}^A - q_{R,j-1/2}^\iota\right| + \left|q_{L,j+1/2}^\iota - q_{R,j+1/2}^B\right|,\\ &\left|q_{L,j-1/2}^B - q_{R,j-1/2}^\iota\right| + \left|q_{L,j+1/2}^\iota - q_{R,j+1/2}^A\right|,\\ &\left|q_{L,j-1/2}^B - q_{R,j-1/2}^\iota\right| + \left|q_{L,j+1/2}^\iota - q_{R,j+1/2}^B\right|\Big)\end{aligned}$$

2. Compare values of mTBV^A & mTBV^B

$$Q_j(x) = \begin{cases} Q_j^A(x), & \text{if } \mathrm{mTBV}^\mathrm{A} < \mathrm{mTBV}^\mathrm{B},\\ Q_j^B(x), & \text{otherwise}\end{cases}$$

Thank you