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Hyperbolicity study of the modulation equations for the Benjamin–Bona–Mahony equation

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Abstract

A complete study of the modulation equations for the Benjamin–Bona–Mahony equation is performed. In particular, the boundary between the hyperbolic and elliptic regions of the modulation equations is found. When the wave amplitude is small, this boundary is approximately defined by $k = \sqrt{3}$, where k is the wave number. This particular value corresponds to the inflection point of the linear dispersion relation for the BBM equation. Numerical results are presented showing the appearance of the Benjamin–Feir instability when the periodic solutions are inside the ellipticity region.

KEYWORDS

dispersive equations, modulational instability

JEL CLASSIFICATION

35L65, 35Q35

1 | INTRODUCTION

The Benjamin–Bona–Mahony (BBM) equation is an unidirectional model of weakly nonlinear waves in shallow water[1]:

$$v_t + v_x + v v_x - v_{txx} = 0.$$

It involves one dependent variable $v(t, x)$ and two independent variables t (time) and x (space coordinate). The last term v_{txx} is responsible for the nonlocal nature of the BBM equation. It also appears as an asymptotic model of finite amplitude waves propagating in one-dimensional rods made of a compressible neo-Hookean material.[2] After the change of variables $v = u - 1$, one gets the equation

$$u_t + uu_x - u_{txx} = 0. \tag{1}$$

The dispersion relation to (1) linearized on a constant solution $u = u_0$ is

$$c_p = \frac{u_0}{1 + k^2}, \tag{2}$$

where c_p is the phase velocity, and k is the wave number. The phase and group velocities are uniformly bounded for any waves numbers.

P. Olver[3] justified that (1) admits only three independent conservation laws

$$(u - u_{xx})_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{3a}$$

$$\left(\frac{u^2}{2} + \frac{u_x^2}{2}\right)_t + \left(\frac{u^3}{3} - uu_{tx}\right)_x = 0, \tag{3b}$$

$$\left(\frac{u^3}{3}\right)_t - \left(u_t^2 - u_{xt}^2 + u^2 u_{xt} - \frac{u^4}{4}\right)_x = 0, \tag{3c}$$

and proposed a Hamiltonian formulation of the BBM equation.[4] In particular, the Lagrangian for the BBM equation is

$$\mathcal{L} = -\frac{\varphi_t \varphi_x}{2} + \frac{\varphi_t \varphi_{xxx}}{2} - \frac{\varphi_x^3}{6}, \quad u = \varphi_x. \tag{4}$$

The conservation law (3a) is the Euler–Lagrange equation for (4). The conservation laws (3b) and (3c) are the consequence of the Noether's theorem that correspond to the invariance of the Lagrangian under space and time translations.

Much work has been done for the BBM equation, both mathematical (e.g., Ref. 5 and references therein) and numerical (e.g., Ref. 6 and references therein). Here, we study the hyperbolicity of the corresponding Whitham's modulation equations. The study of the hyperbolic character or not of the Whitham system is important at least for two reasons. From the physical point of view, the ellipticity of the modulation equations is "responsable" for the Benjamin–Feir instability, well known in the theory of water waves.[7] From the mathematical point of view, the modulational stability (hyperbolicity of the modulational equations) is necessary for the spectral stability of periodic traveling waves.[8] The study of the Benjamin–Feir instability for the BBM equation in the case of small amplitudes can be found in Ref. 9. The singular soliton limit ($k \to 0$) was studied analytically and numerically in Ref. 10. In this limit, the governing equations for the wave amplitude and its wave mean are hyperbolic. Such a singular limit is obtained by using the action integral for the modulation equations for (3) following the papers Refs. 11, 12. It was mentioned in Ref. 13 that the modulation equations for the BBM equation could be elliptic, but the whole study of the modulation equations was not performed. Here, we provide the analysis of the hyperbolicity in the whole three-dimensional domain of wave parameters. The analytical results are illustrated by the numerical ones.

2 | WHITHAM MODULATION EQUATIONS FOR THE BBM SYSTEM

The periodic traveling wave solutions of the BBM equation (1) in the form

$$u = u(\xi), \quad \xi = x - Dt$$

satisfy the equation

$$-Du' + uu' + Du''' = 0. \tag{5a}$$

Here, "prime" means the derivative with respect to ξ. In what follows, we will study only the solutions with positive D. The case with negative D is recovered by the change of variables $u \to -u$, and $D \to -D$. Performing two integrations, we have

$$(u')^2 = \frac{2}{D}\left(-\frac{u^3}{6} + D\frac{u^2}{2} + c_1 u + c_2\right), \tag{5b}$$

where c_1 and c_2 are constants. Equation (5b) has real periodic solutions if and only if the right-hand side of (5b), which is a cubic polynomial, has three different real roots u_i, $i = 1, 2, 3$, $u_1 < u_2 < u_3$. For this, the discriminant of this polynomial should be positive. We introduce $P(u) = (u - u_1)(u - u_2)(u_3 - u)$, and write (5b) as:

$$(u')^2 = \frac{1}{3D}P(u) = \frac{1}{3D}(u - u_1)(u - u_2)(u_3 - u). \tag{5c}$$

The relations between these new constants and D, c_1, and c_2 are

$$D = \frac{1}{3}(u_1 + u_2 + u_3), \; c_1 = -\frac{1}{6}(u_1u_2 + u_1u_3 + u_2u_3), \; c_2 = \frac{1}{6}u_1u_2u_3.$$

From (5c), the three-parameter family of periodic solutions defined up to a shift in space, takes the form:

$$u(\xi) = u_2 + a\,\mathrm{cn}^2(\eta, m), \tag{6}$$

where

$$m = \frac{u_3 - u_2}{u_3 - u_1}, \quad a = u_3 - u_2, \quad \eta = \frac{\xi + \xi_0}{2\sqrt{3D}}\sqrt{\frac{a}{m}}, \quad \xi_0 = \text{const.}$$

Here, $\mathrm{cn}(\eta, m) = \cos(\varphi(\eta, m))$, where φ is defined implicitly from

$$\eta = \int_0^{\varphi(\eta,m)} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}.$$

The wavelength L is given as

$$L = \int_{\xi_2}^{\xi_3} d\xi = 2\int_{u_2}^{u_3} \frac{\sqrt{3D}}{\sqrt{P(u)}}\,du = 4\sqrt{\frac{3Dm}{a}}\,K(m), \tag{7}$$

where the interval $[\xi_2, \xi_3]$ has the length L, and

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

is the complete elliptic integral of the first kind.[14] In particular, the solitary wave solution obtained in the limit $L \to \infty$ and for the values $u_1 = u_2 > 0,\ a = u_3 - u_2$ is in the form

$$u(\xi) = u_2 + \frac{a}{\cosh^2(\eta)}, \quad \eta = \frac{\xi + \xi_0}{2\sqrt{1 + (3u_2/a)}}, \quad D = u_2 + \frac{a}{3}, \quad \xi_0 = \text{const.}$$

We will define the wave averaged of any function $f(u)$ as

$$\overline{f(u)} = \frac{1}{L}\int_{\xi_2}^{\xi_3} f(u)\, d\xi = \frac{2}{L}\int_{u_2}^{u_3} f(u)\frac{\sqrt{3D}}{\sqrt{P(u)}}\, du.$$

In particular, the wave averaged of u (denoted below by $\overline{u}$) is given by:

$$\begin{aligned} \overline{u} &= \int_{u_2}^{u_3} \frac{udu}{\sqrt{P(u)}} \Big/ \int_{u_2}^{u_3} \frac{du}{\sqrt{P(u)}} \\ &= u_1 + (u_3 - u_1)Q(m) = u_2 + \frac{a}{m}(Q(m) + m - 1), \quad Q(m) = \frac{E(m)}{K(m)}. \end{aligned} \tag{8}$$

Here, $E(m)$ is the complete elliptic integral of the second kind[14]:

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\, d\theta.$$

In Whitham's modulation theory, we are interested in the three-parameter solution (6), which is periodic with respect to ξ and whose parameters u_1, u_2, and u_3 vary slowly with respect to T and X. The solution period L is thus also a slowly varying function. Commutating the averaging with respect to ξ, and time and space derivatives, we obtain the BBM modulation system in conservation form[10,13]:

$$\boldsymbol{F}(\boldsymbol{q})_T + \boldsymbol{G}(\boldsymbol{q})_X = 0, \tag{9}$$

with the state vector $\boldsymbol{F}$ and the flux $\boldsymbol{G}$ given as

$$\boldsymbol{F}(\boldsymbol{q}) = \begin{bmatrix} \bar{u} \\ \overline{u^2}/2 + \overline{u'^2}/2 \\ k \end{bmatrix} \quad \text{and} \quad \boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \overline{u^2}/2 \\ \overline{u^3}/3 - D\overline{u'^2} \\ Dk \end{bmatrix}. \tag{10}$$

Here, $\boldsymbol{q}$ is a state vector, for example, $\boldsymbol{q} = (u_1, u_2, u_3)^T$ and $k = 2\pi/L$ is the wave number. The last equation for k is the phase conservation law. It is equivalent to the averaged equation (3c). To simplify the notations, we will use in the following for the independent variables again the notations (x, t) instead of (X, T).

The choice of variables $\boldsymbol{q}$ in which the hyperbolicity of Whitham's system of modulation equations is studied is not essential if the Jacobian matrices corresponding to the change of variables are nondegenerate. In other words, if the modulation equations are hyperbolic in one set variables, they are hyperbolic in another set of variables provided the coordinate change between them is invertible. However, a specific choice of $\boldsymbol{q}$ can simplify the study of the modulation equations. We will use several such change of variables. For example, by their better physical interpretation, the choice $\boldsymbol{q} = (\bar{u}, a, m)^T$ or $\boldsymbol{q} = (D, a, m)^T$ is sometimes more suitable than that of (u_1, u_2, u_3). The corresponding Jacobian matrices are nondegenerate[10]:

$$\det\left(\frac{\partial(u_1, u_2, u_3)}{\partial(\overline{u}, a, m)}\right) = \frac{a}{m^2} \neq 0$$

and

$$\det\left(\frac{\partial(D, a, m)}{\partial(\overline{u}, a, m)}\right) = \frac{\partial D}{\partial \overline{u}} = 1.$$

In the limit of small amplitudes, the choice $\boldsymbol{q} = (\overline{u}, a, k)^T$ is also appropriate. The nondegeneracy of the corresponding Jacobian matrix is proven in the Appendix.

3 | HYPERBOLICITY STUDY IN THE VARIABLES $\boldsymbol{q} = (\boldsymbol{D}, \boldsymbol{a}, \boldsymbol{m})^T$

For further analysis, we will write the governing equations in quasilinear form by using the variables $\boldsymbol{q} = (D, a, m)^T$. One can find the following expressions for the averaged quantities[10]:

$$\begin{aligned}
\overline{u} &= D + af_1(m), \\
\overline{u^2} &= D^2 + 2af_1(m)D + a^2 f_2(m), \\
\overline{u^3} &= D^3 + 3af_1(m)D^2 + 3a^2 f_2(m)D + a^3 f_3(m), \\
\overline{P} &= a^3 \tilde{P}(m), \\
L &= 4\sqrt{3}\sqrt{\frac{Dm}{a}}K(m),
\end{aligned}$$

with

$$\begin{aligned}
f_1(m) &= \frac{1}{m}\left(Q - \frac{2-m}{3}\right), \\
f_2(m) &= \frac{1 - m + m^2}{9m^2}, \\
f_3(m) &= \frac{27(1 - m + m^2)Q + 5m^3 - 21m^2 + 33m - 22}{135m^3}, \\
\tilde{P}(m) &= \frac{2(1 - m + m^2)Q - m^2 + 3m - 2}{15m^3}, \\
Q(m) &= \frac{E(m)}{K(m)}.
\end{aligned}$$

The corresponding quasilinear system is:

$$\boldsymbol{F}(\boldsymbol{q})_t + \boldsymbol{G}(\boldsymbol{q})_x = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}}\boldsymbol{q}_t + \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{q}}\boldsymbol{q}_x = \boldsymbol{A}\boldsymbol{q}_t + \boldsymbol{B}\boldsymbol{q}_x = 0.$$

Here,

$$\mathbf{F} = \begin{bmatrix} D + af_1 \\ 3D^2 + 6af_1D + 3a^2f_2 + \frac{a^3\tilde{P}}{D} \\ \sqrt{\frac{a}{Dm}}\frac{1}{K} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \frac{D^2}{2} + af_1D + \frac{a^2f_2}{2} \\ 2D^3 + 6af_1D^2 + 6a^2f_2D + 2a^3f_3 - 2a^3\tilde{P} \\ \sqrt{\frac{aD}{m}}\frac{1}{K} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 1 & f_1 & af_1' \\ 6D + 6af_1 - \frac{a^3\tilde{P}}{D^2} & 6f_1D + 6af_2 + 3\frac{a^2\tilde{P}}{D} & 6af_1'D + 3a^2f_2' + \frac{a^3\tilde{P}'}{D} \\ -\frac{1}{2}\sqrt{\frac{a}{D^3m}}\frac{1}{K} & \frac{1}{2}\sqrt{\frac{1}{aDm}}\frac{1}{K} & -\frac{1}{2}\sqrt{\frac{a}{Dm^3}}\frac{1}{K}\frac{Q(m)}{1-m} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} D + af_1 & f_1D + af_2 & af_1'D + \frac{a^2f_2'}{2} \\ 6D^2 + 12af_1D + 6a^2f_2 & B_{22} & B_{23} \\ \frac{1}{2}\sqrt{\frac{a}{Dm}}\frac{1}{K} & \frac{1}{2}\sqrt{\frac{D}{am}}\frac{1}{K} & -\frac{1}{2}\sqrt{\frac{aD}{m^3}}\frac{1}{K}\frac{Q(m)}{1-m} \end{bmatrix},$$

$$B_{22} = 6f_1D^2 + 12af_2D + 6a^2f_3 - 6a^2\tilde{P},$$

$$B_{23} = 6af_1'D^2 + 6a^2f_2'D + 2a^3f_3' - 2a^3\tilde{P}',$$

with $(\cdot)' \equiv \frac{\partial(\cdot)}{\partial m}$. The system would be hyperbolic if the three roots $\lambda_1, \lambda_2, \lambda_3$ of the third-degree polynomial

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0$$

are real. After some simplification, and letting $\alpha = a/D$, we can transform the above equation to

$$g(\beta, m, \alpha) = \det(\tilde{\mathbf{B}} - \beta\tilde{\mathbf{A}}) = 0, \quad \beta = \frac{\lambda/D - 1}{\alpha},$$

with

$$\tilde{\mathbf{A}} = \begin{bmatrix} m^2 & m\left(Q - \frac{2-m}{3}\right) & \frac{m}{2}\left(\frac{1-m}{3} - Q^2\right) \\ 2m^2(3Q - (2-m)) & \tilde{A}_{22} & \tilde{A}_{23} \\ -\alpha & 1 & -Q \end{bmatrix},$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} m\left(Q - \frac{2-m}{3}\right) & \frac{1-m+m^2}{9} & \frac{-(2-m)(1-m)}{18} \\ \tilde{B}_{21} & \frac{(1-2m)(2+m-m^2)}{9} & \frac{(m^2+2m-2)(1-m)}{9} \\ 2 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}_{22} = \frac{2m(1-m+m^2)}{3} + \frac{2\alpha}{15}[2(1-m+m^2)Q-(1-m)(2-m)],$$

$$\tilde{A}_{23} = \frac{-m(2-m)(1-m)}{3} +$$

$$\frac{\alpha}{15}\left[5(1-m)^2 - 3(2-m)(1-m)Q + (1-m+m^2)Q^2\right],$$

$$\tilde{B}_{21} = \frac{2m(1-m+m^2)}{3} - \frac{\alpha}{15}[2(1-m+m^2)Q-(1-m)(2-m)].$$

Notice that, since D and α are real numbers, β_i would be real if and only if λ is real. The coefficient of β^3 is $-\det\tilde{\mathbf{A}}$, which is not 0. Therefore, the above equation $g(\beta, m, \alpha)$ is a third-degree polynomial.

Notice that its coefficients are determined only by two parameters: α and m. We have already supposed that $D > 0$, which implies that the domain to study the corresponding third-degree polynomial is: $\alpha = a/D > 0$ and $0 < m < 1$.

Using Wolfram Mathematica, we can plot in the (m, α)-plane the resultant of the polynomials g and its derivative g_β with respect to β, $R(m, \alpha) = \mathrm{res}(g, g_\beta) = 0$, which is shown by red line in Figure 1A.

Below the red line, three roots are real and the system of modulation equations is hyperbolic. It has been numerically verified that on the red line two coinciding eigenvalues $\beta_1 = \beta_2$ are negative, the third one, β_3, is positive. In terms of the eigenvalues λ, it means that on this curve, one has $\lambda_1 = \lambda_2 < D$ and $\lambda_3 > D$. In the domain above the curve the system is elliptic: two eigenvalues become complex. A special case, $u_1 = 0$ is shown in Figure 1A in blue line. Below this curve, the roots u_i, $u_1 < u_2 < u_3$ of the polynomial $P(u)$ (5c) are positive. In the (m, α)-plane the curve $u_1 = 0$ is given by the relation $\alpha = \frac{3m}{2-m}$. In particular, for $0 \le u_1 < u_2 < u_3$, the modulation equations are hyperbolic.

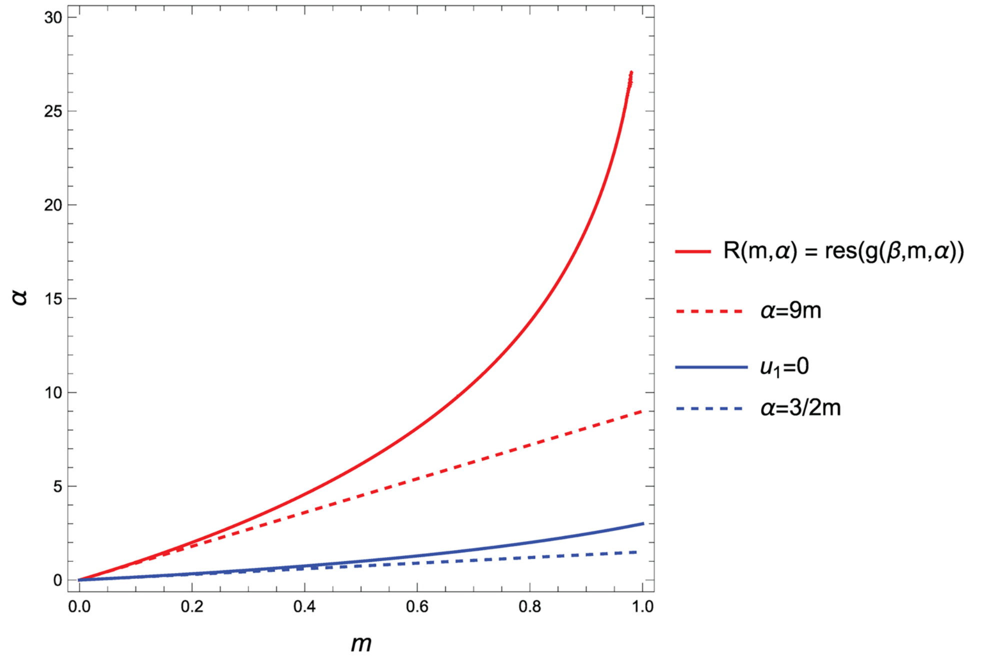
We study now the behavior of the curve $R(m, \alpha) = 0$ separating the hyperbolic and elliptic region. For small m, we parameterize this curve as $\alpha = \alpha(m)$, with $\alpha = \gamma_1 m + \gamma_2 m^2 + \cdots$, with unknown coefficients γ_i, $i = 1, 2, \ldots$. Replacing this asymptotic expression into $R(m, \alpha(m)) = 0$, one can find $\gamma_1 = 9$ and $\gamma_2 = \frac{9}{2}$. Thus, for small m, one has $\alpha \approx 9m + \frac{9}{2}m^2$. Analogously, in the soliton limit, when $m \to 1$, one can show, by using symbolic computations, that $\alpha \to 3(5 + 2\sqrt{10})$. Now, we map the (m, α)-plane to the (m, k)-plane where k is the wave number. By definition,

$$k = \frac{2\pi}{L} = \frac{\pi}{2\sqrt{3}}\sqrt{\frac{\alpha}{m}}\frac{1}{K(m)}.$$

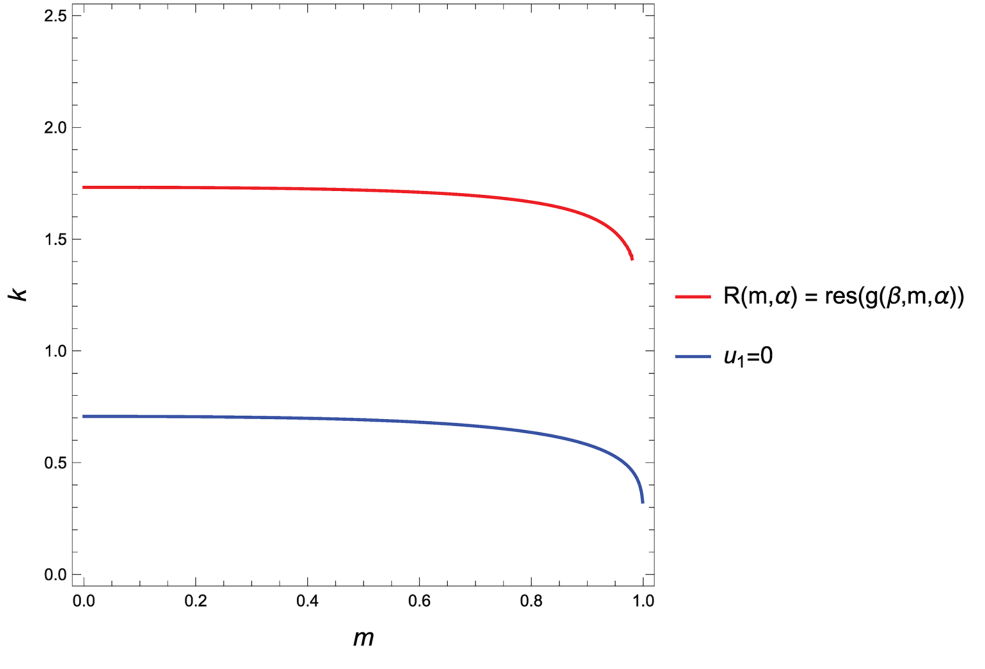
When $m \to 0$, one has $\alpha \approx 9m + \frac{9}{2}m^2$, and $K(m) \to \frac{\pi}{2}$. Therefore, $k \to \sqrt{3}$. Furthermore, one can check the corresponding slope is vanishing: $\lim_{m\to 0} \frac{dk}{dm} = 0$. When $m \to 1$, $\alpha \to 3(5 + 2\sqrt{10})$, and $K(m) \to \infty$, so $k \to 0$.

The qualitative behavior of the resultant in the (m, k)-plane is shown in continuous red line in Figure 1B. It is interesting to note that in the ellipticity region, one always has $k > \sqrt{3}$. This specific value of $k = \sqrt{3}$ corresponds to the inflection point in the linear dispersion relation (2) (see Figure 2). Indeed, Equation (2) can be written as

$$\omega = \frac{u_0 k}{1 + k^2}. \tag{11}$$



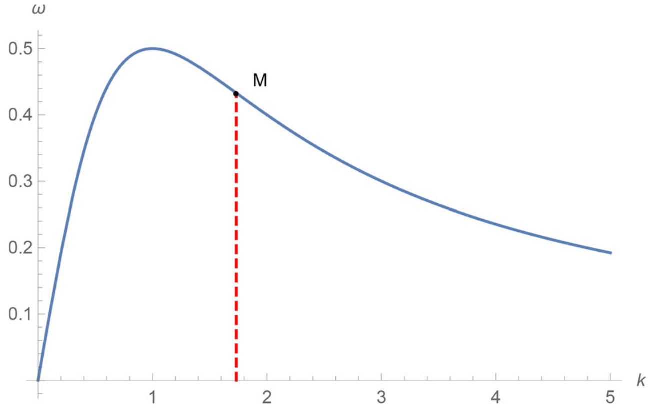
(A)



(B)

FIGURE 1 The hyperbolicity region in the (m, α)-plane (A) and (m, k)-plane (B). The red continuous curve is the boundary between elliptic (above the red curve) and hyperbolic (below the red curve) regions. The blue curve corresponds to $u_1 = 0$ (below the blue curve we have the inequality $0 < u_1 < u_2 < u_3$). The dashed blue and red straight lines in (A) are tangent lines to the corresponding continuous curves at $m = 0$.

FIGURE 2 The point M of the dispersion curve having the abscissa $\sqrt{3}$ is the inflection point of the dispersion relation (11).



The second derivative $\omega''(k)$ is

$$\omega''(k) = 2\frac{u_0 k(k^2-3)}{(1+k^2)^3}.$$

Thus, $k = \sqrt{3}$ corresponds to the inflection point of the dispersion relation $\omega = \omega(k)$. The curve in the $(m-k)$-plane corresponding to $u_1 = 0$ is also shown in blue.

4 | NUMERICAL STUDY OF EIGENFIELDS IN $(\overline{\boldsymbol{u}}, \boldsymbol{a}, \boldsymbol{m})$ VARIABLES

Let λ_j be the eigenvalue, $\boldsymbol{\ell}_j$ and $\boldsymbol{r}_j$ be the left and right eigenvectors of the quasi-linear system of the BBM modulation system satisfying $\boldsymbol{\ell}_j(\boldsymbol{B}-\lambda_j\boldsymbol{A}) = 0$ and $(\boldsymbol{B}-\lambda_j\boldsymbol{A})\boldsymbol{r}_j = 0$, respectively. Assume that the solution is self-similar in the variable $\zeta = x/t$, that is, $\boldsymbol{q}(x,t) = \boldsymbol{q}(\zeta)$ is the simple wave solution, the modulation system is reduced to a system of ordinary differential equations (ODEs):

$$(\boldsymbol{B}-\boldsymbol{A}\zeta)\frac{d\boldsymbol{q}}{d\zeta} = 0. \tag{12}$$

If $\nabla_{\boldsymbol{q}}\lambda_j \cdot \boldsymbol{r}_j \neq 0$, the simple wave solution of (12) corresponding to the eigenfield j is determined by a set of ODEs in the form[15]:

$$\frac{d\boldsymbol{q}}{d\zeta} = \boldsymbol{r}_j/\left(\nabla_{\boldsymbol{q}}\lambda_j \cdot \boldsymbol{r}_j\right), \quad \zeta = \lambda_j, \quad j = 1,2,3. \tag{13}$$

Unlike the case of the Korteweg-de Vries (KdV) equation where the modulation system can be written in Riemann invariant form, and where we can therefore find the closed-form solutions of the corresponding ODEs, for the BBM equation, we can only find the eigenvectors and eigenvalues and solve the ODEs (13) numerically.

In Figure 3, we present numerical results for the eigenvalues λ_j and the quantities $\nabla_{\boldsymbol{q}}\lambda_j \cdot \boldsymbol{r}_j$, $j = 1,2,3$, as a function of $m = (u_3-u_2)/(u_3-u_1)$ for $u_1 = 0$ and $u_3 = 1$. As it has been proved in Section 3 (see Figure 1A,B), in this case (since $u_1 = 0$) the equations are hyperbolic. We observe the occurrence of double characteristic roots only in the limits $m \to 0$ and $m \to 1$ as it is usually the case of integrable (cf. Ref. 11) and nonintegrable (cf. Ref. 16) systems.

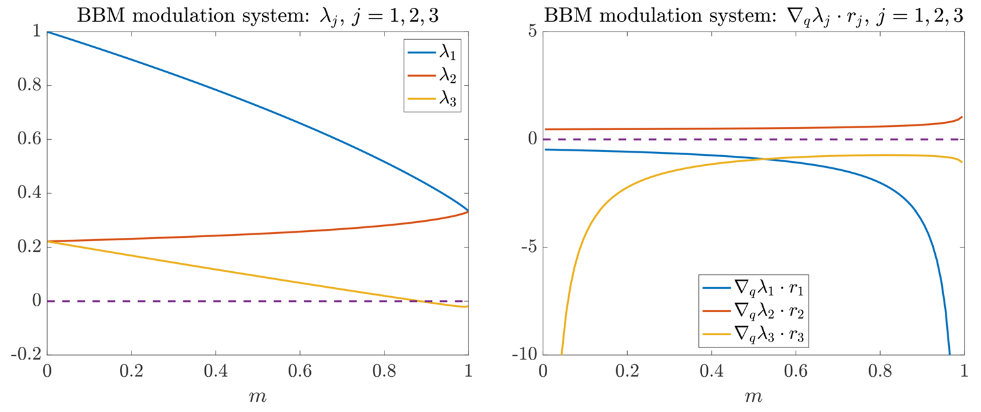


FIGURE 3 Numerical results for the eigenvalue problem of the Benjamin–Bona–Mahony (BBM) modulation system in $(\overline{u}, a, m)$ variables. On the left column, the eigenvalues λ_1, λ_2, and λ_3 are shown as a function of $m = (u_3 - u_2)/(u_3 - u_1)$ with $u_1 = 0$ and $u_3 = 1$. As has been mentioned in Section 3 (see Figure 3), in this case (since $u_1 = 0$), the equations are hyperbolic. On the right column, the quantities $\nabla_{\boldsymbol{q}} \lambda_1 \cdot \boldsymbol{r}_1$, $\nabla_{\boldsymbol{q}} \lambda_2 \cdot \boldsymbol{r}_2$, and $\nabla_q \lambda_3 \cdot \boldsymbol{r}_3$ are shown.

In addition, we observe the genuine nonlinearity of all the characteristic fields with this set of parameter values ($\nabla_{\boldsymbol{q}} \lambda_i \cdot \boldsymbol{r}_i \neq 0$). Obviously, the property of the genuine nonlinearity is not satisfied if the wave parameters are in the vicinity of the hyperbolic–elliptic boundary.

To give an example for the undular bore solution of the BBM equation (1) and the simple wave solution of the BBM modulation system (9), we consider an initial-boundary value problem studied in Ref. 17 for the magma equation. For the undular bore part of the problem, the boundary conditions we take on the left at $x = 0$ is $u_L = 1$ at all time, and on the right at $x = 100$ is the nonreflecting boundary. Initially, in the interior of the domain where $x \in (0, 100]$, we take $u = 1/2$. For the simple wave solution problem, the solution is an expansion fan along the λ_2 characteristic field. On the left of this expansion fan, we have the acoustic wave limit $m_L \to 0$, $\bar{u}_L \to 1$, and on the right of the fan, we have the soliton limit $m_R \to 1$ and $\bar{u}_R \to 1/2$; the parameter on the amplitude a_L and a_R in the state vector $q = (\bar{u}, a, m)$ of the modulation system can be determined numerically once the phase velocities are set on the left and on the right, respectively. Figure 4 shows the numerical results, at time $t = 100$, observing qualitative agreement of the results on the wave envelopes and the average state between the BBM bore solution and the modulation simple wave solution. Numerical results obtained using the hyperbolized BBM model (BBMH)[18] are included also where $\hat{c} = c = 10^3$ is used in the computation (see Ref. 18 for details). Here, $\hat{c}$ and c are two parameters of the hyperbolized model. The BBMH model is a very accurate hyperbolic approximation of the exact BBM equation and is obtained as the Euler–Lagrange equations for a two-parameter "extended" Lagrangian. The advantage of such an hyperbolic approximation is a possibility to use a full arsenal of numerical methods developed for hyperbolic systems of conservations laws. Such a method of "extended" Lagrangian was efficiently used for dispersive models appearing both in classical and quantum fluids (e.g., Refs. 18–24). map

5 | STOKES EXPANSION FOR THE BBM MODULATION SYSTEM

If the flow state of interests is in the weakly nonlinear regime, we can obtain an approximate periodic solution via the Stokes expansion[25]:

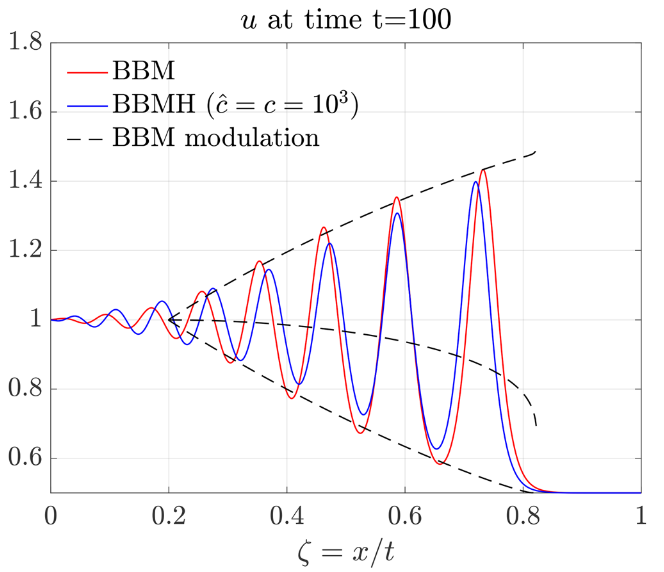


FIGURE 4 Numerical results for the initial-boundary value problem of the Benjamin–Bona–Mahony (BBM) equation and the BBM modulation system. In the graph, the solid red line is the solution obtained using the BBM equation (1), the blue solid line is the solution obtained using the hyperbolized BBM equation,[18] and the black dashed line is the solution using the BBM modulation system (13).

$$u(\xi) = \bar{u} + \varepsilon\tilde{u}_1(\xi) + \varepsilon^2\tilde{u}_2(\xi) + \varepsilon^3\tilde{u}_3(\xi) + \cdots,$$

$$D(k) = \frac{\bar{u}}{1+k^2} + \varepsilon D_1(k) + \varepsilon^2 D_2(k) + \cdots$$

for $0 < \varepsilon \ll 1$. It follows from (8) that the condition to be imposed on each $\tilde{u}_j$ is

$$\int_{\xi_2}^{\xi_3} \tilde{u}_j(\xi)\,d\xi = 0, \quad j = 1, 2, \ldots.$$

Inserting this ansatz into (5a), and collecting terms of the equal powers of ε, we find the solution up to $O(\varepsilon^2)$:

$$u(\xi) = \bar{u} + \varepsilon\cos(k\xi) + \varepsilon^2\left(\frac{1+k^2}{12\bar{u}k^2}\right)\cos(2k\xi) + O(\varepsilon^3),$$

$$D(k) = \frac{\bar{u}}{1+k^2} + \frac{\varepsilon^2}{24\bar{u}k^2} + O(\varepsilon^3).$$

Now if we assume the wave amplitude $a = 2\varepsilon$, from the above equations, we arrive at the approximate periodic solution to the BBM equation:

$$u(\xi) = \bar{u} + \frac{a}{2}\cos(k\xi) + a^2\left(\frac{1+k^2}{48\bar{u}k^2}\right)\cos(2k\xi) + O(a^3), \tag{14a}$$

$$D(k) = \frac{\bar{u}}{1+k^2} + \frac{a^2}{96\bar{u}k^2} + O(a^3). \tag{14b}$$

By definition, this series is an asymptotic one, if with a given ξ_0 we have $|u_2| \ll |u_1|$ for $\xi \to \xi_0$, which means

$$a \ll \frac{24\bar{u}k^2}{1+k^2}.$$

We continue by inserting (14) into (9) that gives the approximate modulation system up to $O(a^2)$:

$$\bar{u}_t + \left(\frac{1}{2}\bar{u}^2 + \frac{a^2}{16}\right)_x = 0, \tag{15a}$$

$$\left(\frac{1}{2}\bar{u}^2 + \frac{a^2}{16}(1+k^2)\right)_t + \left(\frac{1}{3}\bar{u}^3 + \frac{a^2}{8}\frac{\bar{u}}{1+k^2}\right)_x = 0, \tag{15b}$$

$$k_t + \left(\frac{k\bar{u}}{1+k^2} + \frac{a^2}{96k\bar{u}}\right)_x = 0. \tag{15c}$$

Assume that the state vector $\boldsymbol{q}$ is given by $\boldsymbol{q} = (\overline{u}, a, k)^T$. In a matrix form, the approximate model (15) can be written as

$$\boldsymbol{A}\boldsymbol{q}_t + \boldsymbol{B}\boldsymbol{q}_x = 0,$$

where the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (by abuse of notations, we use the same symbols for the approximate matrices) are defined by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{u} & \frac{a(1+k^2)}{8} & \frac{a^2k}{8} \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \bar{u} & \frac{a}{8} & 0 \\ \bar{u}^2 + \frac{a^2}{8(1+k^2)} & \frac{a\bar{u}}{4(1+k^2)} & \frac{-a^2k\bar{u}}{4(1+k^2)^2} \\ \frac{k}{1+k^2} - \frac{a^2}{96k\bar{u}^2} & \frac{a}{48k\bar{u}} & \frac{(1-k^2)\bar{u}}{(1+k^2)^2} - \frac{a^2}{96k^2\bar{u}} \end{bmatrix},$$

respectively.

Employing the standard perturbation procedures (cf. Ref. 26), one obtains the eigenvalues of $\tilde{\mathcal{A}} = \boldsymbol{A}^{-1}\boldsymbol{B}$ computed from the equation $\det(\boldsymbol{B} - \lambda\boldsymbol{A}) = 0$ in the form

$$\lambda_{\tilde{\mathcal{A}},1} = \bar{u} + O(a^2),$$

$$\lambda_{\tilde{\mathcal{A}},2} = \frac{(1-k^2)\bar{u}}{(1+k^2)^2} + \frac{a}{12}\sqrt{\frac{3(3+5k^2)(3-k^2)}{(3+k^2)(1+k^2)^3}} + O(a^2),$$

$$\lambda_{\tilde{\mathcal{A}},3} = \frac{(1-k^2)\bar{u}}{(1+k^2)^2} - \frac{a}{12}\sqrt{\frac{3(3+5k^2)(3-k^2)}{(3+k^2)(1+k^2)^3}} + O(a^2).$$

Again, we recover the critical value $k = \sqrt{3}$ below, which the modulation equations for waves of small amplitude are hyperbolic.

Figure 5 shows a comparison plot of the approximate periodic solution to the analytic solution, where the parameters we used are $\bar{u} = 1$, $a = 1/5$, and $k = 1$, observing good agreement of results.

6 | STABILITY OF THE BBM PERIODIC SOLUTIONS

Here, we present numerical results on the stability or instability of periodic solutions depending whether the parameters of the periodic solution belong to the hyperbolic or elliptic region of the modulation equations. The numerical results are consistent with the analytical estimations of the boundary between hyperbolic and elliptic regions of the modulation equations.

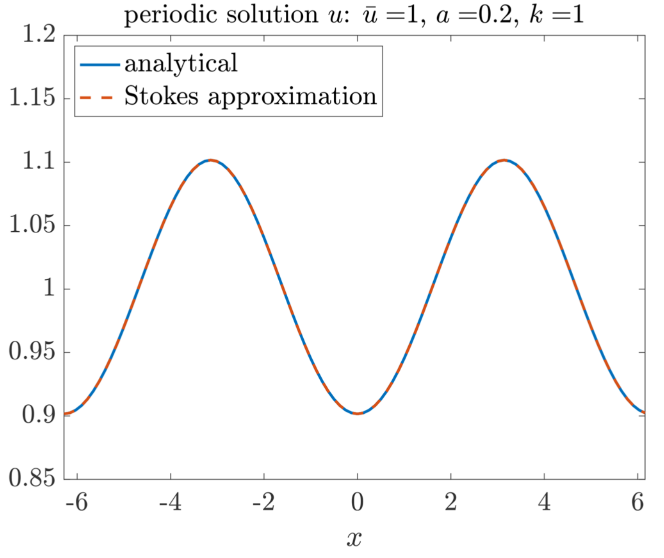


FIGURE 5 Numerical comparison of the periodic Stokes wave approximation to the analytic Benjamin–Bona–Mahony (BBM) solution; parameters $\bar{u} = 1$, $a = 1/5$, and $k = 1$ were used in the computation.

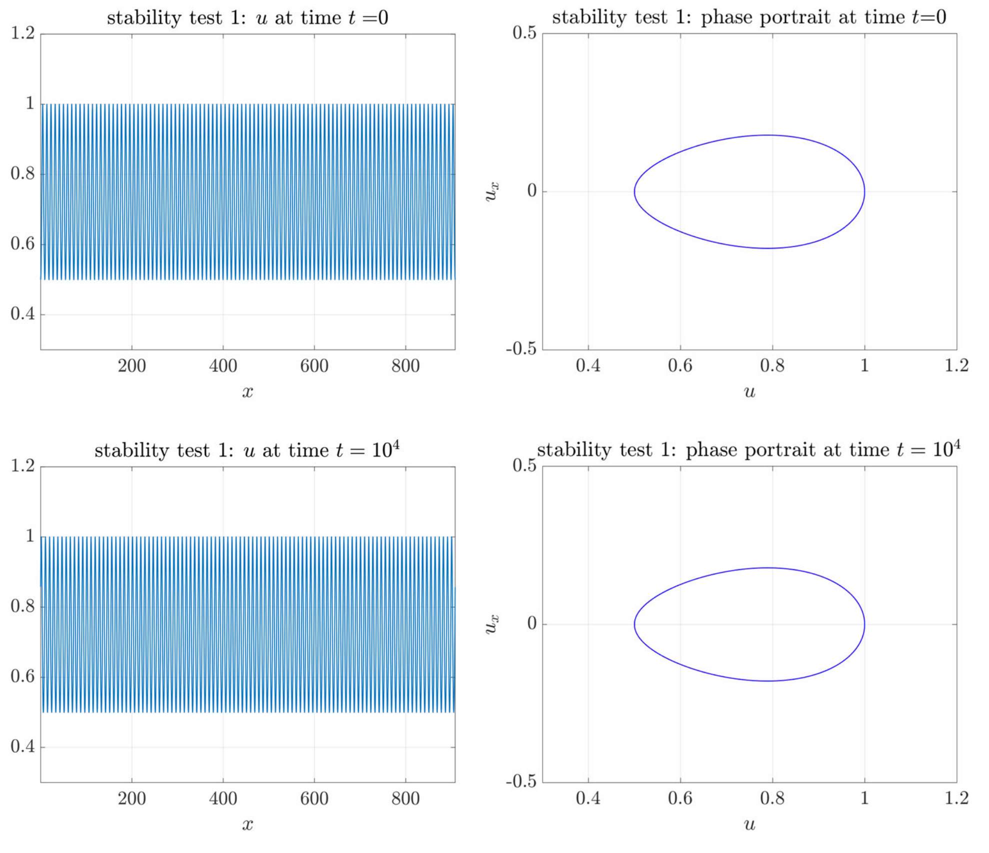


FIGURE 6 Numerical results for the modulational stability of Benjamin–Bona–Mahony (BBM) periodic solutions with the parameters $\bar{u} \approx 0.728473$, $a = 0.5$, and $k \approx 0.691747$ belonging to the hyperbolic region of the exact modulation equations (9)–(10). The BBM solution is shown on the left column, and the phase portrait of the solution is shown on the right column.

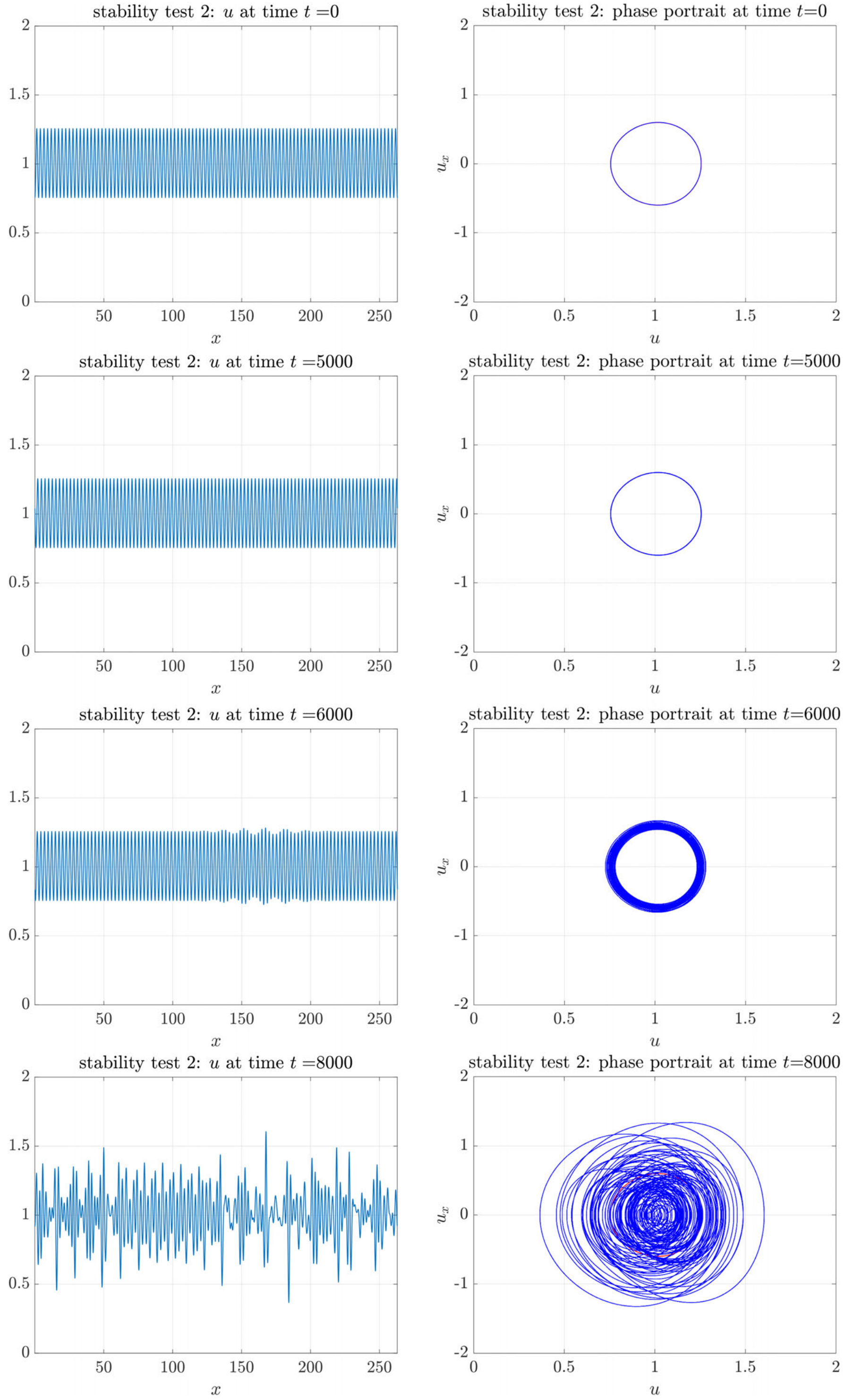


FIGURE 7 Numerical results for the modulational stability of Benjamin–Bona–Mahony (BBM) periodic solutions with the parameters $\bar{u} = 1$, $a = 0.5$, and $k = 2.3895$ belonging to the elliptic region of the exact modulation equations (9)–(10). The graphs are displayed in the same manner as in Figure 6.

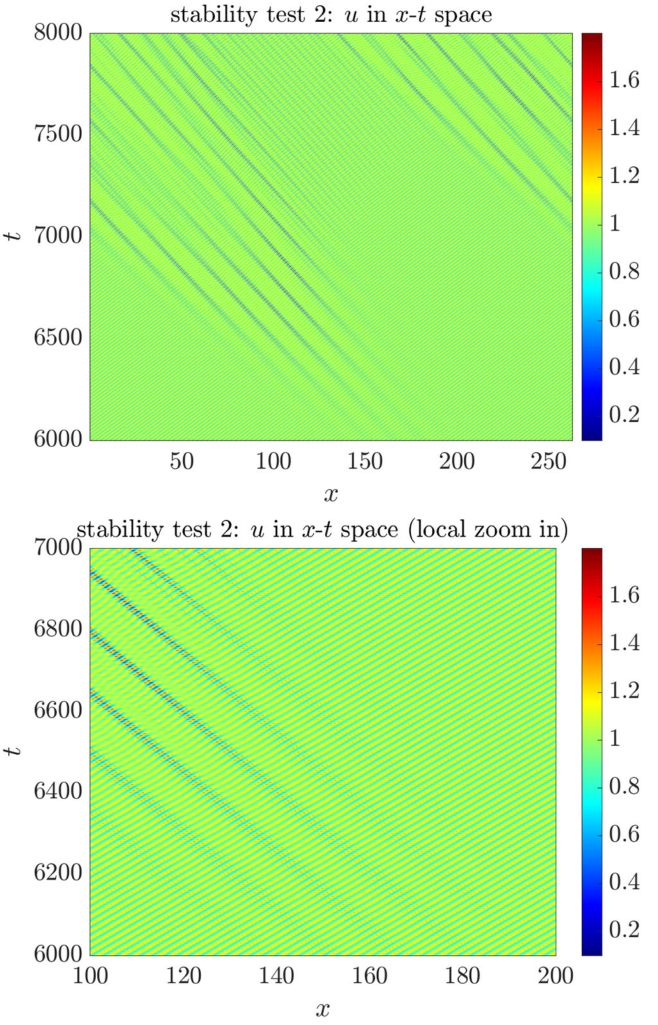


FIGURE 8 The pseudo-color plot of the numerical solution in the (x, t)-plane for the test shown in Figure 7. On the top the solutions in the time interval $t \in [6000, 8000]$ are shown, and on the bottom the zoom-in solutions for $(x, t) \in [100, 200] \times [6000, 7000]$ are shown.

The perturbations are introduced only by the numerical method solving directly the BBM equation (1). The numerical method we used is described in Refs. 10, 18.

We first choose the parameters $\bar{u}$, a, and m for the solution of a single periodic wave over a wave length L. We then setup the problem initially by forming a wave train that consists of N aforementioned single periodic waves (6). In the tests shown below, we take $N = 100$.

In the first example, we take the parameters of periodic wave belonging to the hyperbolicity region: $\bar{u} \approx 0.728473$, $a = 0.5$, and $k \approx 0.691747$. The corresponding value of m can be estimated as $m \approx 0.5$. It gives the real eigenvalues $\lambda_1 \approx 0.0929218$, $\lambda_2 \approx 0.250354$, and $\lambda_3 \approx 0.724104$. The numerical solution shown in Figure 6 is stable up to time $t = 10^4$.

In the second example, we take the parameters of periodic wave belonging to the ellipticity region: $\bar{u} = 1$, $a = 0.5$, and $k \approx 2.3895$. The corresponding value of m can be estimated as $m \approx 0.177317$. It gives two complex conjugate and one real eigenvalue: $\lambda_1 \approx -0.104009 - i\,0.0129987$, $\lambda_2 \approx -0.104009 + i\,0.0129987$, and $\lambda_3 \approx 0.997296$. The numerical solution shown in Figure 7 is stable up to time $t \approx 5 \times 10^3$, becomes unstable at time $t \approx 6 \times 10^3$, and proceeds afterwards. It is clearly seen from the pseudo-color plot of the solution in the x-t plane shown in Figure 8 for $t \in [6000, 8000]$ and the zoom-in for $(x, t) \in [100, 200] \times [6000, 7000]$.

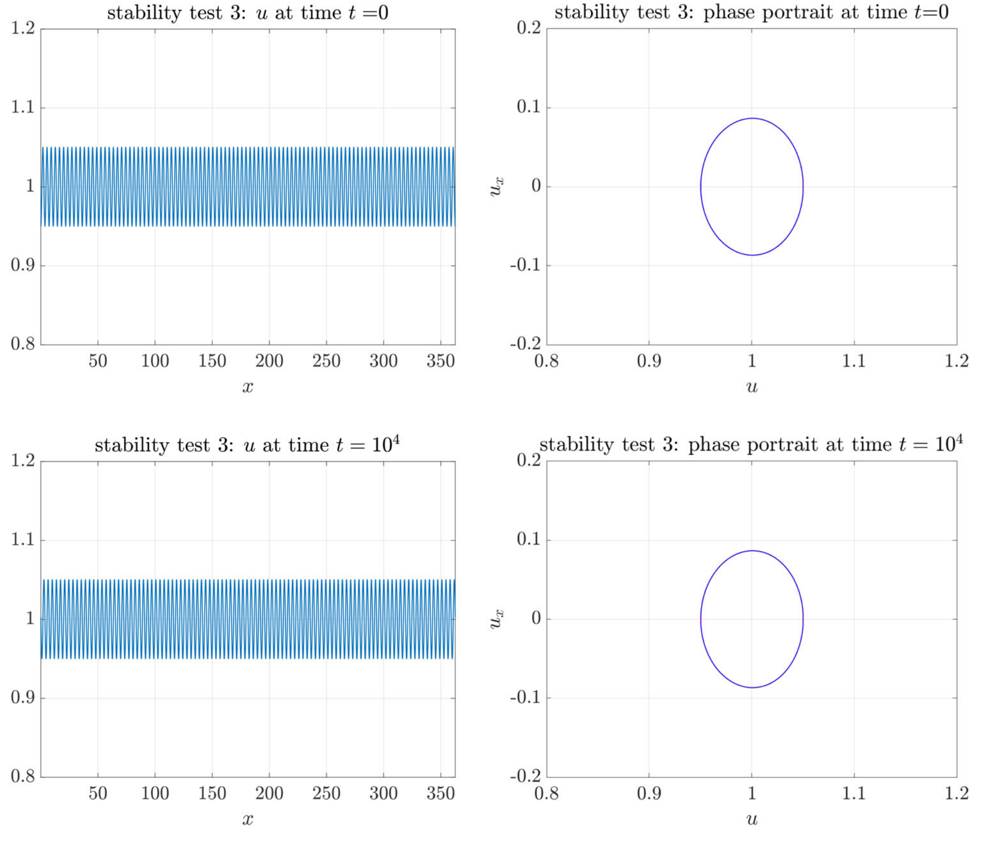


FIGURE 9 Numerical results for the modulational stability of Benjamin–Bona–Mahony (BBM) periodic solutions with the parameters $\bar{u} = 1$, $a = 0.1$, and $k = 1.73242$ belonging to the boundary between elliptic and hyperbolic region of the exact modulation equations (9)–(10). The graphs are displayed in the same manner as in Figure 6.

Finally, in the third example, we take the parameters of periodic wave at the boundary between hyperbolic and elliptic regions: $\bar{u} = 1$, $a = 0.1$, and $k \approx 1.73242$. The corresponding value of m can be estimated as $m \approx 0.04346$. It gives the multiple eigenvalues $\lambda_1 = \lambda_2 \approx -0.124983$ and $\lambda_3 \approx 0.999861$. The numerical solution shown in Figure 9 is stable up to time $t = 10^4$.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX: THE NONDEGENERACY OF THE JACOBIAN MATRIX IN $(\overline{\boldsymbol{u}}, \boldsymbol{a}, \boldsymbol{k})$ VARIABLES

In the limit of small amplitudes, the choice $\boldsymbol{q} = (\overline{u}, a, k)^T$ is also appropriate. In this case the Jacobian matrix is:

$$\det\left(\frac{\partial(u_1, u_2, u_3)}{\partial(\overline{u}, a, k)}\right) = \det\left(\frac{\partial(u_1, u_2, u_3)}{\partial(\overline{u}, a, m)}\right)\det\left(\frac{\partial(\overline{u}, a, m)}{\partial(\overline{u}, a, k)}\right)$$

$$= \det\left(\frac{\partial(u_1, u_2, u_3)}{\partial(\overline{u}, a, m)}\right)\frac{\partial m}{\partial k} = \det\left(\frac{\partial(u_1, u_2, u_3)}{\partial(\overline{u}, a, m)}\right)\Big/\frac{\partial k}{\partial m},$$

where first m is considered as a function of $\overline{u}, a, k$, and then k is considered as a function of $\overline{u}, a, m$. One has

$$2\pi\Big/\frac{\partial k}{\partial m} = -L^2\Big/\frac{\partial L}{\partial m}.$$

Using the formula

$$\frac{Dm}{a} = \frac{\overline{u}m}{a} + \frac{2-m}{3} - \frac{E(m)}{K(m)}$$

and formula (7), we can compute $\partial L/\partial m$. One has (for fixed $\overline{u}$ and a):

$$\frac{1}{4\sqrt{3}}\frac{\partial L}{\partial m} = \frac{\partial}{\partial m}\left(K(m)\sqrt{\frac{\bar{u}\,m}{a} + \frac{2-m}{3} - \frac{E(m)}{K(m)}}\right)$$

$$= K'(m)\sqrt{\frac{\bar{u}\,m}{a} + \frac{2-m}{3} - \frac{E(m)}{K(m)}} + K(m)\frac{\frac{\bar{u}}{a} - \frac{1}{3} - \left(\frac{E(m)}{K(m)}\right)'}{2\sqrt{\frac{\bar{u}\,m}{a} + \frac{2-m}{3} - \frac{E(m)}{K(m)}}}$$

$$= K'(m)\sqrt{\frac{D\,m}{a}} + K(m)\frac{\frac{\bar{u}}{a} - \frac{1}{3} - \left(\frac{E(m)}{K(m)}\right)'}{2\sqrt{\frac{D\,m}{a}}}$$

$$= \frac{1}{2\sqrt{\frac{D\,m}{a}}}\left(2K'(m)\frac{D\,m}{a} + \frac{K(m)}{m}\left(\frac{\bar{u}m}{a} - \frac{m}{3} - m\left(\frac{E(m)}{K(m)}\right)'\right)\right)$$

$$= \frac{1}{2\sqrt{\frac{D\,m}{a}}}\left(2K'(m)\frac{D\,m}{a} + \frac{K(m)}{m}\left(\frac{D\,m}{a} - \frac{2}{3} + \frac{E(m)}{K(m)} - m\left(\frac{E(m)}{K(m)}\right)'\right)\right)$$

$$= \frac{1}{2\sqrt{\frac{D\,m}{a}}}\left(\frac{D\,m}{a}\left(2K'(m)+\frac{K(m)}{m}\right)+\frac{K(m)}{m}\left(\frac{E(m)}{K(m)}-\frac{2}{3}-m\left(\frac{E(m)}{K(m)}\right)'\right)\right)$$

$$= \frac{1}{2\sqrt{\frac{D\,m}{a}}}\left(\frac{D}{a}\frac{E(m)}{(1-m)}+\frac{K(m)}{m}\left(\frac{E^2(m)}{K^2(m)}\frac{1}{2(1-m)}-\frac{1}{6}\right)\right).$$

Here, "prime" means the derivative with respect to m. Since

$$\frac{E^2(m)}{K^2(m)} > 1-m, \quad m\in(0,1),$$

one has

$$\frac{E^2(m)}{K^2(m)}\frac{1}{2(1-m)}-\frac{1}{6} > \frac{1}{3}, \quad m\in(0,1).$$

Hence, the corresponding Jacobian also does not change sign.