One-Dimensional Hyperbolic Partial Differential Equations

(I) Initial Value Problems

Example 3: Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

where $c \in \mathbb{R}$, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. As we will learn in the course, the exact solution for the equation the initial conditions of the form:

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad \forall x \in \mathbb{R}$$

is simply

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi.$$

As an example, take $c = 1$, and consider the initial conditions:

$$f(x) = \exp\left(-x^2/0.02\right), \qquad g(x) = 0, \qquad x \in [-1,1],$$

and

$$f(x) = \exp\left(-x^2/0.02\right), \qquad g(x) = 1, \qquad x \in [-1,1],$$

For simplicity, we use the outflow boundary condition at both the left and right boundaries. The graph of the solution will be shown next up to time $t = 1$.