One-Dimensional Hyperbolic Partial Differential Equations

(I) Initial Value Problems

Example 2: Inviscid Burger's Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. As in the previous example, it can be shown very easily that under certain condition the exact solution for this equation with the initial condition of the form:

$$u(x,0) = u_0(x), \qquad \forall x \in \mathbb{R}$$

is simply

$$u(x,t) = u_0(x - ut).$$

Note that this is an implicit function in u for any given value in the space and time. Using chain rule, we find that the partial derivatives of the solution u_x and u_t would take the form

$$\frac{\partial u}{\partial x} = \frac{u_0^{'}}{1 + u_0^{'}t}, \qquad \frac{\partial u}{\partial t} = \frac{-uu_0^{'}}{1 + u_0^{'}t},$$

which means the solution will blow up at finite time, when $1 + u_0^{'}t = 0$, even the initial condition is smooth $u_0^{'} \in C^1(\mathbb{R})$ but nonincreasing $u_0^{'} < 0$ for all x in the domain.

As an example, take $c = 1$, and consider the smooth profile $u_0(x) = \sin(2\pi x)$. In this case, the solution will blow up approximately at

$$t \approx \frac{-1}{u_0^{'}(x_0)} = \frac{-1}{2\pi \cos(2\pi x_0)} = 0.159154,$$

where x_0 corresponds to the values at which $u_0^{'}(x)$ has a minimum, see the graph of the solution next. This type of behavior is typical for a nonlinear partial differential equation.