One-Dimensional Hyperbolic Partial Differential Equations

(I) Initial Value Problems

Example 1: Linear Advection Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where $c \in \mathbb{R}$, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. It can be shown very easily that the exact solution for this equation with the initial condition of the form:

$$u(x, 0) = u_0(x), \qquad \forall x \in \mathbb{R}$$

is simply

$$u(x, t) = u_0(x - ct).$$

That is, the solution is obtained by shifting the initial profile by the distance ct to the right or left depending the sign of the "propagating" speed c.

As an example, take $c = 1$, and consider the Gaussian profile $u_0(x) = \exp\left(-x^2/0.02\right)$. The graph of the solution will be shown next up to time $t = 1$.