Example 2: Burger's Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2},$$

where $\varepsilon \in \mathbb{R}^+$ (typically a small number), $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. It can be shown that the exact solution for this equation with the initial condition of the form:

$$u(x,0) = u_0(x), \qquad \forall x \in \mathbb{R}$$

is

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} e^{-G/2\varepsilon}}{\int_{-\infty}^{\infty} e^{-G/2\varepsilon}},$$

where

$$G(\xi; x, t) = \frac{(x-\xi)^2}{2t} + \int_0^{\xi} u_0(\eta)\, d\eta,$$

see [1] for the details.

References

[1] G. B. Whitham. *Linear and Nonlinear Waves.* John Wiley & Sons, Inc., New York, 1974.