One-Dimensional Parabolic Partial Differential Equations

Example 1: Linear Heat Equation

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2},$$

where $\varepsilon \in \mathbb{R}^+$ (typically a small number), $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. As in the previous example, it can be shown very easily that under certain condition the exact solution for this equation with the initial condition of the form:

$$u(x, 0) = u_0(x), \qquad \forall x \in \mathbb{R}$$

is simply

$$u(x, t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} \exp\left[\frac{-(x-\xi)^2}{4\varepsilon t}\right] u_0(\xi)\, d\xi.$$