91學年度第1學期

課程編號: 221 U1310 學分: 3
科目名稱: 數值偏微分方程式一
課程網站: http://www.math.ntu.edu.tw/~shyue/myclass/npde02

Homework ♯ 5

Assign: 12/11/2002 Due: 1/5/2003

- Include your computer program(s), when turning the homework set

1. Show that the *box* scheme

$$\frac{1}{2k}\left[\left(U_j^{n+1}+U_{j+1}^{n+1}\right)-\left(U_j^n+U_{j+1}^n\right)\right]+\frac{a}{2h}\left[\left(U_{j+1}^{n+1}-U_j^{n+1}\right)+\left(U_{j+1}^n+U_j^n\right)\right]=f_j^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of $\lambda = ak/h$, where $U_j^n \approx u(x_j, t_n)$; $a > 0$.

2. Show that the group velocity for the leapfrog scheme

$$\frac{1}{2k}\left(U_j^{n+1}+U_j^{n-1}\right)+\frac{a}{2h}\left(U_{j+1}^n-U_{j-1}^n\right)=0$$

is given by

$$a_g(\omega)=a\frac{\cos(\omega h)}{\sqrt{1-\lambda^2\sin^2(\omega h)}},$$

where $\lambda = ak/h$ and U_j^n is the numerical approximation of the solution $u_t + au_x = 0$ at the time t_n and point x_j. What is the group velocity for the *box* scheme considered in Problem ♯1 ?

3. Consider the one-way wave equation

$$u_t+u_x=0, \qquad -2\le x\le 2, \qquad t>0$$

with periodic boundary condition $u(-2,t) = u(2,t)$ and the initial condition

$$u(x,0)=\begin{cases}\cos(5\pi x)\cos^2(\pi x/2) & \text{if } |x|<1\\ 0 & \text{otherwise.}\end{cases}$$

Solve the problem using both the leapfrog and Lax-Wendroff schemes. Demonstrate that the wave packet moves with the group velocity and that the high-frequency mode travels with the phase velocity.

4. Consider the inviscid Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad 0 \le x \le 1, \qquad t > 0,$$

with the periodic boundary condition $u(0,t) = u(1,t)$ and the initial condition

$$u(x,0) = \sin(2\pi x).$$

a) Solve this problem using both the upwind and Lax-Wendroff schemes up to time $t = 1.5$. Plot the computed solution as a three-dimensional surface over the (x,t) plane. In addition, compute numerically the order of accuracy of the method in both the discrete $1-$ and max-norm.

b) Repeat the same computation as done in a), but up to a slightly longer time $t = 2$. Do you observe the same error behavior and the same order of accuracy as in case a) before ?

5. Consider the linear acoustic wave equations of the form

$$\begin{aligned} p_t + \kappa(x) u_x &= 0 \\ \rho(x) u_t + p_x &= 0. \end{aligned}$$

Here p and u are the pressure disturbance and the velocity, respectively. The coefficient functions $\rho(x)$ and $\kappa(x)$ are the density and the bulk modulus of elasticity, respectively. An an example, we take $\kappa(x) = 1$ for all x and $\rho(x) = 1$ if $x < 1$, and $\rho(x) = 4$ otherwise, *i.e.*we have two different media separated by an interface located at $x = 1$. Note that across the interface, we assume there is no jump in both the pressure and velocity. The initial conditions we choose are

$$p(x,0) = u(x,0) = \exp\left[-48(x - 1/2)^2\right].$$

Now, solve this problem using the upwind method up to $t = 1$, and $x \in [0,2]$ with outflow boundary conditions at the both end. Plot the computed solutions at three different times: $t = 0.16, 0.7$, and 1.