91學年度第1學期

課程編號: 221 U1310
科目名稱: 數值偏微分方程式一
課程網站: http://www.math.ntu.edu.tw/~shyue/myclass/npde02

Homework ♯ 3

Assign: 10/30/2002 Due: 11/13/2002

- *Include your computer program(s), when turning the homework set*

This problem is concerned with the motion of an elastic beam arising from mechanics. We assume that the beam is inextensible of length 1 and thin. So we neglect shearing forces and rotatory inertia. We further want to allow it arbitrarily large movements. Thus, the most natural coordinate system to use is the angle θ as a function of arc length s and time t. We further suppose that the beam clamped at $s = 0$ and a force $\vec{F} = (F_x, F_y)$ acting at the free end $s = 1$. The beam is then described by the equation

$$x(s,t) = \int_0^s \cos\left[\theta(\sigma,t)\right]\, d\sigma, \qquad y(s,t) = \int_0^s \sin\left[\theta(\sigma,t)\right]\, d\sigma.$$

By applying the Lagrange theory and using the Hamilton principle (see [1] for example), it can be shown that the equation of motion for this problem can be described by the following integral-differential equation:

$$\begin{aligned}\int_0^1 G(s,\sigma)\left\{\cos\left[\theta(s,t)-\theta(\sigma,t)\right]\ddot{\theta}(\sigma,t)+\sin\left[\theta(s,t)-\theta(\sigma,t)\right]\left[\dot{\theta}(\sigma,t)\right]^2\right\}\, d\sigma = \\ \theta^{''}(s,t)+F_y(t)\cos\left[\theta(s,t)\right]-F_x(t)\sin\left[\theta(s,t)\right], \qquad 0 \le s \le 1,\end{aligned} \tag{1}$$

with the boundary conditions

$$\theta(0,t) = 0 \qquad \text{and} \qquad \theta^{'}(1,t) = 0, \tag{2}$$

and

$$G(s,\sigma) = 1 - \max(s,\sigma).$$

Note the dots and primes represent derivatives with respect to t and s, respectively.

Now if we discretize the integral with the help of midpoint rule, *i.e.*,

$$\int_0^1 f\left[\theta(\sigma,t)\right]\, d\sigma = \frac{1}{n}\sum_{k=1}^{n} f(\theta_k), \qquad \theta_k = \theta\left(\frac{k-1/2}{n}, t\right), \quad k = 1,2,\cdots,n.$$

and $\theta^{''}$ by a standard central difference formula, Equation (1) becomes a system of nonlinear ordinary differential equations for the unknown θ at a discrete arc length s and a continuous time t spaces,

$$\sum_{k=1}^{n}\left[a_{lk}\ddot{\theta}_k + b_{lk}\left(\dot{\theta}_k\right)^2\right] = n^4(\theta_{l-1}-2\theta_l+\theta_{l+1}) + n^2(F_y\cos\theta_l - F_x\sin\theta_l), \quad l = 1,2,\cdots,n,$$

where

$$a_{lk} = g_{lk} \cos(\theta_l - \theta_k), \qquad b_{lk} = g_{lk} \sin(\theta_l - \theta_k), \qquad g_{lk} = n + \frac{1}{2} - \max(l, k).$$

We use the following numerical boundary conditions,

$$\theta_0 = -\theta_1, \qquad \theta_{n+1} = \theta_n,$$

for the discrete representation of (2).

An an example, we choose the initial conditions

$$\theta(s, 0) = 0, \qquad \dot{\theta}(s, 0) = 0,$$

and apply the exterior forces

$$F_x = -\varphi(t), \qquad F_y = \varphi(t), \qquad \varphi(t) = \begin{cases} \alpha \sin^2 t & 0 \leq t \leq \pi \\ 0 & \pi < t, \end{cases}$$

for some constant α.

a) Devise an *efficient* algorithm to solve the problem numerically with parameters: $\alpha = 3/2$, $n = 40$, for example, and $0 \leq t \leq 5$. Plot the resulting solution in the (x, y, t) space.

b) Perform the same computations as in a), but with a slightly perturbed initial data, say at the free end $s = 1$ only. What kind of solution behavior do you observe ?

References

[1] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems.* Springer-Verlag, 2 revised edition, 1996.