

The Secretary Problem: Two-Player Extensions and Going Back

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Abstract

The *Secretary Problem* is a thoroughly-studied optimal-stopping problem in which a person must try to select the best applicant from a given number of them. The problem is that he must interview them, must decide whether to accept the applicant just after the interview, and cannot return to previous applicants. The $1/e$ optimal strategy for this is well known, and many extensions of it have been studied. Here we will study some more generalizations:

Firstly, a fairly simple generalization is presented that allows the person to return to previous applicants, with a fixed probability of success. Through this problem we will return to the classic problem and demonstrate the "sum the odds to one and stop" rule, and from that the $1/e$ optimal strategy.

Next, the main course is two generalizations of the problem involving two players each. In the first one, one player is more powerful than the other, and when both players want a certain secretary the first player will receive her. We will find the optimal strategy and probability of success for this, then go on to a scenario where the weaker player can choose between helping the stronger player and receiving the privilege of a coin flip, or not helping and remaining weak. On the way, we will derive a formula for determining the probability of a set number of candidates (secretaries who are better than previous ones) occurring in a number of applicants.

摘要: 秘書問題是一個被廣泛討論的問題, 探討在 n 個候選人中如何找出最佳秘書人選. 面試官必須親自一對一面談來應徵的人, 並在面試後

馬上決定是否錄取, 而且不能反悔或者回頭去錄取先前面試的人. 最優策略的成功機率為 $1/e$, 其方式為: 以前 n/e 個候選人中的最佳者為基準, 錄取後來遇到的第一位超過這個基準的人選. 本文推廣此問題:

首先, 假設我們容許面試官回頭選擇之前的面試者, 且有一定的成功機率, 我們發現原先的解, 即 $1/e$ 原則, 仍是最優的.

接下來還有兩種情況. 第一個情況, 有兩個面試官, 其中之一比較有權力, 當兩人同時想錄取某一位面試者時, 權力較大的那位才能聘用那位秘書. 我們發現這個情況的策略及成功機率跟原秘書問題是一樣的; 另外, 弱者也有兩種方式可以應對, 第一是幫助強者且得到用硬幣正反面決定誰是擁有者的反駁機會, 第二是保持低弱的姿態什麼都不做. 在這過程中, 我們將導出一個公式來算出在若干面試者中, 取出這幾位候選者 (比前一位秘書好).

1 Introduction

The *Secretary Problem* is an optimal-stopping problem where we are asked to determine the optimal strategy for a person looking for the best secretary, the classic form of which is roughly as follows:

A person is looking for a secretary. He knows that there are exactly n applicants, from which he must choose precisely one secretary. He chooses by interviewing the applicants one by one. After each interview he can rank the applicant, but only relative to the other applicants. He cannot think of the applicant as better or worse than average or expected, because there is no such thing as an average or expected value. Immediately after he has finished interviewing an applicant, he must decide to accept or reject the applicant. He can accept only one applicant out of the n applicants, so once he accepts his game is over and he wins iff his applicant is the best applicant out of the n applicants. Once he has rejected an applicant he cannot return to accept him again, if he ever wishes to do so. (This restriction is partially removed in our first generalization.) The goal is to maximize the probability of getting the best applicant out of the n applicants.

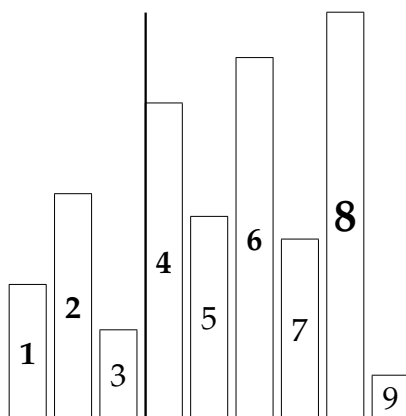
This problem has already been thoroughly studied, along with many extensions and generalizations. In this paper we will be studying some two-player extensions. Various two-player extensions have already been studied, and the author thinks the simple two-player subordinate/dominant

problem has probably already been studied, but is not yet sure because he cannot find any of the articles, although there are many similar ones. See [1] for an example of a two-player extension and a more complete discussion of previous works on two-player secretary problems. See [2] for a general discussion of many types of extensions of the secretary problem.

2 Preliminaries

First, some definitions. An *applicant* is one of the secretaries to be interviewed. We generally denote the number of applicants (which is already known) by n . A *candidate* is an applicant that is better than all the applicants before him. There may and probably will be multiple candidates; if all the applicants are ordered from worst to best, all the applicants will be candidates. The first applicant is always a candidate. Furthermore, the best applicant (which we will, for simplicity, denote #1) is always the last candidate. This is easy to see. No candidates can come after #1, because, by definition, nobody can be better than #1. And, #1 is always a candidate because he is better than all the people before him; in fact, better than all other applicants.

All the optimal strategies involved in this problem will be of this form: let the person reject the first d applicants, and after that, accept the first candidate. We will call the time after the d th person has been rejected and before the next person is interviewed the *rejection point*. As a number, the rejection point is the position of the last person to be unconditionally rejected. More complex strategies, for example requiring the player's randomization, or analyzing an opponent's possibly suboptimal strategy, will not be considered.



As examples to demonstrate our results and logic, we will use diagrams like this one to denote the ordering of the applicants, where each applicant is a rectangle and their quality is indicated by their height. The candidates have their numbers in bold; note that they form a strictly increasing sequence. #1 has a big number. The bold line is the rejection point.

The optimal strategy for the classic problem is as follows. Reject the first d applicants, and accept the first applicant who is better than all of the first d applicants. d is a function of n . As an example, here is an example of the probability $P_n(d)$ of succeeding for all d with $n = 10$.

| d | $P_n(d)$ | Out of 25200 |
|----------|---------------|--------------|
| 0 | 10% | 2520 |
| 1 | 28.29% | 7129 |
| 2 | 36.58% | 9218 |
| 3 | 39.87% | 10047 |
| 4 | 39.83% | 10036 |
| 5 | 37.28% | 9395 |
| 6 | 37.28% | 8250 |
| 7 | 32.74% | 6685 |
| 8 | 26.53% | 4760 |
| 9 | 18.89% | 2520 |
| 10 | 0% | 0 |

Obviously the optimal strategy in this case is to reject the first 3 applicants. To show how we found these values, we first derive a simple

formula for the probability of obtaining the best applicant. Let us consider the probabilities. First, if #1 is among the first d applicants to be interviewed, he will be rejected no matter what, and the employer loses. If #1 is the $(d + 1)$ th applicant, then the employer will always accept him, because he must be better than the first d applicants. More generally, if #1 is the $(d + x)$ th applicant, then we will accept him only if there are no candidates in the previous $(x - 1)$ applicants. In other words, we get #1 if and only if the best applicant among all the previous people was unconditionally rejected, because he would be the last candidate until #1.

As a result, when #1 is the $(d + x)$ th applicant, we get him if the best applicant was in the first d applicants, which occurs with probability $\frac{d}{d+x-1}$. The total probability is therefore

$$\frac{1}{n} \left(\frac{d}{d} + \frac{d}{d+1} + \cdots + \frac{d}{n-1} \right) \quad (1)$$

A good solution and introduction I found can be found online at

<http://www.math.uah.edu/stat/urn/Secretary.xhtml>

Also, this almost-obvious result about the natural log (actually, any log) will be needed.

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln n - \ln(n-1) &= \lim_{n \rightarrow \infty} \ln \frac{n}{n-1} \\ &= \ln 1 \\ &= 0. \end{aligned}$$

At last, we can begin our first (easy) generalization:

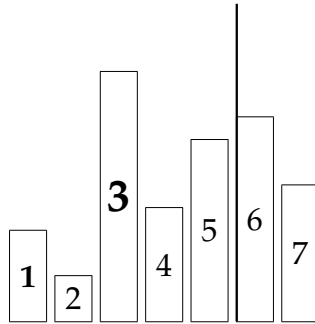
3 Going Back to Rejected Candidates

In the original problem, the interviewer was not allowed to return to previous applicants; once he rejected an applicant, the applicant had no hope of getting chosen. Let us consider a less harsh (and possibly more realistic) version, where the employer can go back to a rejected person but cannot guarantee that he will accept this time; he accepts with a fixed probability. This generalization is not hard compared to the two-player problems, so we use it as a warm-up.

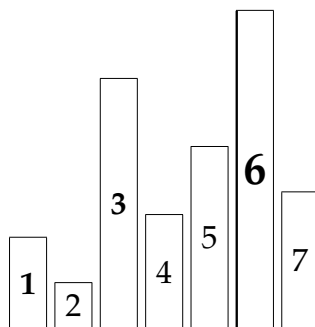
We assume that every person has a fixed probability of $P_A \geq 0$ of accepting when offered the job a second time, and that any person who rejected the second offer cannot be asked again. In this case we have a little lemma.

Lemma 1. *The only time the interviewer should go back is after all the applicants have been interviewed.*

Before we prove it, here are some examples to make the lemma clear. Suppose the interviewer has interviewed the first 5 applicants already and finds that the 3rd applicant is the best so far, and wishes to go back to ask him to accept the job. There are basically two scenarios:



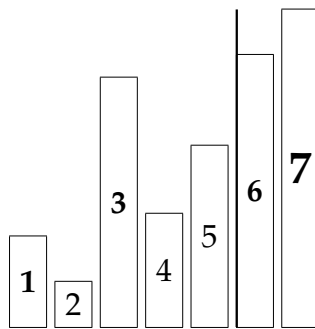
In the scenario above, the interviewer would gain nothing (but lose nothing either) if he interviews to the end. He still would want to go back to the 3rd applicant, and would succeed with the same probability.



In the scenario above, the interviewer would meet the 6th applicant and accept him. Using the original strategy he would get to meet the 6th applicant only if he was rejected by the 3rd applicant. Since $P_A \geq 0$, there is always some chance he would get accepted by the 3rd applicant and fail

in this case. So, at least in this case, interviewing all the applicants is better than interviewing only part of them before going back.

Note, however, that there is still one other case we did not treat. This is when there are two or more candidates have not yet been interviewed. Here, if the interviewer interviewed until the end without accepting, he would win; but if he interviewed until the first candidate found, he would lose. Still, if he had decided to go back to the 3rd applicant he would lose no matter what as well.



Of course, we still must prove the lemma.

Proof. Consider when the interviewer decides to go back after interviewing $k < n$ applicants. There are two different possibilities:

1. #1 is in those rejected k applicants.
2. #1 has not yet been interviewed.

Arbitrarily we can let the probability of case 1 be P_1 . What is the probability of success? In case 1, the interviewer wins if #1 accepts the second offer of the job, with probability P_A . In case 2, we cannot deduce the probability of winning, so let us just assume it is $P_{2,win}$.

As a result, the total probability of winning is $P_1 \cdot P_A + (1 - P_1) \cdot P_{2,win}$, or

$$P_A + (1 - P_1) \cdot (P_{2,win} - P_A) \tag{2}$$

Compare this to the result when the interviewer goes back after interviewing all the applicants. In this case $P_1 = 1$, so the probability of winning is simply P_A . Also consider the strategy when the interviewer just keeps interviewing. When $P_{2,win} < P_A$, going back after all the applicants is better; when $P_{2,win} > P_A$, not going back at all is better; when

$P_{2,win} = P_A$, all three strategies work equally well. All this is obvious from plugging the values into (2). Thus we discover that there is always one strategy that works as good as or better than going back before interviewing all the applicants. \square

With this lemma, we can easily discover the optimal strategy. Let the rejection point be d . Then, in the original problem, we have three possible outcomes:

1. The interviewer rejects #1 and thus reaches the end of the applicants without selecting anybody. This happens with probability $\frac{d}{n}$.
2. The interviewer successfully selects #1. The probability is given by (1).
3. The interviewer selects the wrong applicant. The exact probability of this can be easily computed by taking the probabilities of the above two cases from 1, but we will not need it.

In which of these outcomes will the interviewer succeed? In case 1, the interviewer will decide to go back; he already knows who the best applicant is, so he succeeds in case 1 with probability P_A . In case 2, the interviewer has already succeeded. The total probability is thus

$$P_G(d) = \frac{d}{n} \cdot P_A + \frac{1}{n} \left(\frac{d}{d} + \frac{d}{d+1} + \frac{d}{d+2} + \cdots + \frac{d}{n-1} \right) \quad (3)$$

Just by looking at (3), it isn't easy to figure out what the optimal value for d is. We can do so, however, by finding the difference between successive terms of P_G (that is, the difference in the probability when we add 1 to d):

$$\begin{aligned}
\Delta P_G(d) &= P_G(d+1) - P_G(d) \\
&= \frac{d+1}{n} \cdot P_A - \frac{d}{n} \cdot P_A + \\
&\quad \frac{1}{n} \left(\frac{d+1}{d+1} + \frac{d+1}{d+2} + \cdots + \frac{d+1}{n-1} \right) - \\
&\quad \frac{1}{n} \left(\frac{d}{d} + \frac{d}{d+1} + \frac{d}{d+2} + \cdots + \frac{d}{n-1} \right) \\
&= \frac{P_A}{n} + \frac{1}{n} \left(-\frac{d}{d} + \left(\frac{d+1}{d+1} - \frac{d}{d+1} \right) + \cdots + \left(\frac{d+1}{n-1} - \frac{d}{n-1} \right) \right) \\
&= \frac{P_A}{n} + \frac{1}{n} \left(\frac{1}{d+1} + \frac{1}{d+2} + \cdots + \frac{1}{n-1} - 1 \right) \tag{4} \\
P_A &= n\Delta P_G(d) + \left(1 - \frac{1}{d+1} - \frac{1}{d+2} - \cdots - \frac{1}{n-1} \right) \tag{5}
\end{aligned}$$

Clearly, $P_G(d)$ has a local maximum when $\Delta P_G(d)$ changes from positive to negative. Since $\Delta P_G(d)$ is strictly decreasing (when d increases, a term of $1/d$ disappears and decreases the overall value), the single place where the sign change occurs is the absolute maximum.

Though we now can use (5) to find what values for d maximize the probability of success, it is still cumbersome. For each value of d , we can determine what value P_A will let $\Delta P_G(d)$ be zero. That point is when the strategies to reject d and $d+1$ applicants are equally likely to succeed. When P_A is greater, rejecting $d+1$ applicants becomes better; when less, d is better. We construct a table for $n = 10$ for these values:

| d | $P_n(d)$ | /25000 | Critical P_A | /25000 |
|-----|----------|--------|----------------|------------|
| 0 | 10% | 2520 | | (Negative) |
| 1 | 28.29% | 7129 | | (Negative) |
| 2 | 36.58% | 9218 | | (Negative) |
| 3 | 39.87% | 10047 | | (Negative) |
| 4 | 39.83% | 10036 | 0.44% | 11 |
| 5 | 37.28% | 9395 | 25.44% | 641 |
| 6 | 32.74% | 8250 | 45.44% | 1145 |
| 7 | 26.53% | 6685 | 62.10% | 1565 |
| 8 | 18.89% | 4760 | 76.39% | 1925 |
| 9 | 10% | 2520 | 88.89% | 2240 |
| 10 | 0%/10% | 0 | 100% | 2520 |

So, for arbitrary P_A in the case of $n = 10$, we can use the table as follows: find the row with the Critical P_A , less than the given P_A , that is greatest; that row contains the d that is optimal. For example, if we wanted to find the optimal strategy when the probability of getting accepted the second time is 50%, we would use $d = 6$, because its P_A is 45.44%, which is the greatest value that is still less than 50%.

Now, we consider what occurs when n tends to infinity. We can then approximate the expression

$$\frac{1}{d+1} + \frac{1}{d+2} + \cdots + \frac{1}{n-1}$$

with an integral, because as n goes to infinity, the number of terms goes to infinity, and this series is the Riemann sum of the integral

$$\int_d^{n-1} \frac{1}{p} dp = \ln(n-1) - \ln d = \ln n - \ln d = \ln \frac{n}{d}.$$

Remember we can assume $\ln(n-1) = \ln n$ because n is very large, and we proved that when n goes to infinity, the difference goes to zero.

So, plugging back into (4), we get

$$n\Delta P_G = P_A + \ln \frac{n}{d} - 1 \quad (6)$$

As n increases, when we vary d the changes in ΔP_G will get smaller and smaller. So the maximum of P_G when n is large should be when $\Delta P_G \approx 0$.

Solving the previous equation will enable us to find the optimal strategy.

$$\begin{aligned}
0 &= P_A + \ln n - \ln d - 1 \\
1 - P_A &= \ln n - \ln d \\
e^{1-P_A} &= e^{\ln n - \ln d} \\
e^{1-P_A} &= \frac{n}{d} \\
d &= \frac{n}{e^{1-P_A}}.
\end{aligned}$$

Note that the classic problem occurs when $P_A = 0$, and then $d = n/e$, the correct result.

Now we find the actual probability of success, by plugging into (3)

$$\begin{aligned}
P_G(d) &= \frac{d}{n} \cdot P_A + \frac{1}{n} \left(\frac{d}{d} + \frac{d}{d+1} + \frac{d}{d+2} + \cdots + \frac{d}{n-1} \right) \\
&= \frac{d}{n} \cdot P_A + \frac{d}{n} \int_d^n \frac{1}{p} dp \\
&= \frac{d}{n} \left(P_A + \int_d^n \frac{1}{p} dp \right) \\
&= \frac{d}{n} (P_A + \ln n - \ln d) \\
&= \frac{d}{n} \left(P_A + \ln \frac{n}{d} \right) \\
&= \frac{1}{e^{1-P_A}} \left(P_A + \ln \left(e^{1-P_A} \right) \right) \\
&= \frac{1}{e^{1-P_A}} (P_A + 1 - P_A) \\
&= \frac{1}{e^{1-P_A}}.
\end{aligned}$$

Note that the probability is the same as the rejection point, out of n .

4 The Two-Player Secretary Problem, Dominant vs. Subordinate

After a long period of work, the old secretary decided to retire. The person was left without a good secretary, and again he started looking for a new

one.

However, another intelligent businessman has come into the town, and he is looking for a secretary too! Furthermore, that businessman pays much more money than our poor man, so that if both decide on the same secretary, he will certainly go to the other businessman.

As in the basic problem, the same assumptions are made: there are n secretaries, after a secretary is rejected that's it for him, and previous secretaries will not help the person to predict anything about the following secretaries. There are no ties, and our person and the rival businessman interview their secretaries in the same order simultaneously, and they rate the secretaries identically. So, #1 for our person is #1 for the rival.

Let us first consider the strategy for the rival. We will call him the *dominant* player, and our person the *subordinate* player. If the dominant player wants a certain secretary, he will get it; the subordinate player can in no way interfere with whatever strategy the dominant player uses. Thus, the dominant player can ignore the subordinate player in planning his strategy, so his strategy is the same as our man's strategy in the classic, one-player problem. Let us call the number of people he rejects unconditionally D_D , for Dominant.

The problem is, what is the subordinate player's optimal strategy? This is the only remaining problem.

First, to get some grasp of the problem, let us consider a trivial example: $n = 3$, with three potential secretaries. The dominant player's strategy is of course to reject the first person and take the second person if he is better than the rejected person. This gives him a winning probability of $\frac{1}{2}$. In other words, $D_D = 1$. Let the number of people the subordinate player rejects as a sample be S_D . Let's make a table for each arrangement of secretaries, where the best person is 1, the second-best is 2, and the worst is 3.

| Arrangement | Dominant player | $S_D = 0$ | $S_D = 1$ | $S_D = 2$ |
|-------------|-----------------------------|-----------|-----------|-----------|
| 1 2 3 | Loses (forced to take last) | Wins | Loses | Loses |
| 1 3 2 | Loses (forced to take last) | Wins | Loses | Loses |
| 2 1 3 | Wins | Loses | Loses | Loses |
| 2 3 1 | Wins | Loses | Loses | Loses |
| 3 1 2 | Wins | Loses | Loses | Loses |
| 3 2 1 | Loses (#2, suboptimal) | Loses | Wins | Wins |

In this case, the optimal strategy for the subordinate player is clearly to take the first person he can get; any more and the dominant player completely dominates him, so to speak. In fact, we can state this as a little lemma:

Lemma 2. *The subordinate player's rejection point comes before the dominant player's rejection point.*

Before we prove this, we must derive a few important formulas concerning the probabilities of the subordinate player getting #1.

First, if the subordinate player decides just to reject the same number of people as the dominant player, what happens? The subordinate player wins if there are exactly *two* candidates after their rejection-points.

How can we find the probability of this?

Let's look at a more general problem: assume that d people have been rejected and α people remain. What is the probability that there are exactly k candidates among the remaining α applicants?

We can easily start at $k = 0$. The probability is simply the probability that #1 has been rejected, or $\frac{d}{d+\alpha}$. Let $n = d + \alpha$, which is the number of people.

Let's define the probability as $I_k(d, \alpha)$. So, $I_0(d, \alpha) = \frac{d}{d+\alpha}$.

Now, let us consider this problem generally. For the general case of I_k , we can think of it this way: the last candidate must be #1. So, for each possible position of #1, we can think of all the applicants before #1 as another problem. #1 is always the last candidate, so in those remaining applicants success occurs when there are $k - 1$ candidates. We can think of those remaining applicants as another problem; let the best person in those applicants be #1 for this new problem, and so on. We get another set of values to plug into I_{k-1} . And so on.

Let's now actually calculate the probability. For success, #1 must be in the last α positions.

Assume #1 is the $d + p$ th applicant. Thus, for the new problem, success occurs when there are $k - 1$ candidates from position $d + 1$ to $d + p - 1$. Thus the new problem is $I_{k-1}(d, p - 1)$.

#1 is the $d + p$ th applicant with probability $\frac{1}{d+\alpha}$.

Thus, the result probability is:

$$I_k(d, \alpha) = \frac{1}{d + \alpha} (I_{k-1}(d, 0) + I_{k-1}(d, 1) + \cdots + I_{k-1}(\alpha - 1)) \quad (7)$$

This is a beautiful recursive formula, and using the trivial $I_0(d, \alpha) = \frac{d}{d+\alpha}$ we can obtain the familiar equation

$$\begin{aligned} I_1(d, \alpha) &= \frac{1}{d + \alpha} \left(\frac{d}{d} + \frac{d}{d+1} + \cdots + \frac{d}{d + \alpha - 1} \right) \\ &= \frac{d}{d + \alpha} \left(\frac{1}{d} + \frac{1}{d+1} + \cdots + \frac{1}{d + \alpha - 1} \right). \end{aligned}$$

and the not so familiar

$$\begin{aligned} I_2(d, \alpha) &= \frac{1}{d + \alpha} \left(\frac{1}{d} (0) + \frac{1}{d+1} \left(\frac{d}{d} \right) \right. \\ &\quad \left. + \cdots + \frac{1}{d + \alpha - 1} \left(\frac{d}{d} + \frac{d}{d+1} + \cdots + \frac{d}{d + \alpha - 2} \right) \right) \\ &= \frac{d}{d + \alpha} \left(\frac{1}{d(d+1)} + \frac{1}{d(d+2)} + \cdots \right. \\ &\quad \left. + \frac{1}{(d+1)(d+2)} + \cdots + \frac{1}{(d + \alpha - 2)(d + \alpha - 1)} \right). \end{aligned}$$

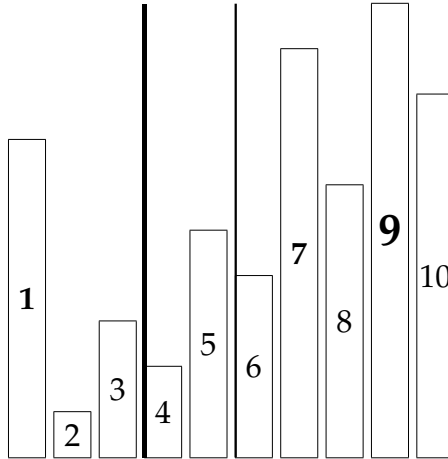
Now, we are all equipped to prove the lemma.

Proof. Consider when the subordinate player's rejection point is after the dominant player's rejection point, and let the subordinate player's rejection point be S_D . We will prove that, when S_D is increased by 1, the probability of success decreases.

First, we must find the success probability of this strategy. The arrangements in which it succeeds can be classified into two cases:

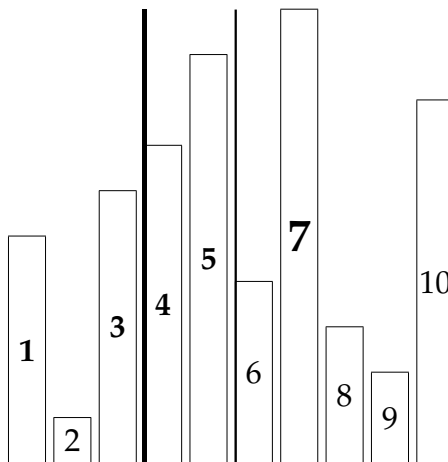
1. #1 is after the subordinate player's rejection point, there are no candidates between the two players' rejection points, and there is one candidate between the subordinate player's rejection point and #1.
2. #1 is after the subordinate player's rejection point, there are one or more candidates between the two players' rejection points, and there are no candidates between the subordinate player's rejection point and #1.

Here are some examples of the two cases. Note that the bolder line is the dominant player's rejection point, and the thinner one is the subordinate player's rejection point.



The first case occurs when #1 is the second candidate after the subordinate player's rejection point and the best applicant before the subordinate player's rejection point is before the dominant player's rejection point. The probability is

$$\frac{D_D}{S_D} I_2(S_D, n - S_D) = \frac{D_D}{n} \left(\frac{1}{S_D(S_D + 1)} + \frac{1}{S_D(S_D + 2)} + \cdots + \frac{1}{(n - 2)(n - 1)} \right).$$



The second case occurs when #1 is after the subordinate player's rejection point, the best applicant before #1 is before the subordinate player's

rejection point, and the best applicant before the subordinate player's rejection point is after the dominant player's rejection point. These two probabilities can be strung together easily; the probability is

$$\frac{S_D - D_D}{n} \left(\frac{1}{S_D} + \frac{1}{S_D + 1} + \cdots + \frac{1}{n-1} \right).$$

Before we continue, note that these formulas degenerate correctly when $S_D = D_D$.

Now, we observe the difference in the success probability when S_D is changed to $S_D + 1$. The probability of case 1 decreases by

$$\begin{aligned} & \frac{D_D}{n} \left(\frac{1}{S_D(S_D + 1)} + \frac{1}{S_D(S_D + 2)} + \cdots + \frac{1}{S_D(n-1)} \right) \\ &= \frac{D_D}{S_D \cdot n} \left(\frac{1}{S_D + 1} + \frac{1}{S_D + 2} + \cdots + \frac{1}{n-1} \right). \end{aligned}$$

The probability of case 2 changes a bit more subtly. $\frac{S_D - D_D}{n}$ increases by $\frac{1}{n}$, but the term $\frac{1}{S_D}$ disappears, so the net gain is

$$\frac{1}{n} \left(\frac{1}{S_D + 1} + \frac{1}{S_D + 2} + \cdots + \frac{1}{n-1} \right) - \frac{S_D - D_D}{S_D \cdot n}.$$

Combine the results, and we find the net gain is

$$\begin{aligned} & \frac{S_D - D_D}{S_D \cdot n} \left(\frac{1}{S_D + 1} + \frac{1}{S_D + 2} + \cdots + \frac{1}{n-1} \right) - \frac{S_D - D_D}{S_D \cdot n} \\ &= \frac{S_D - D_D}{S_D \cdot n} \left(\frac{1}{S_D + 1} + \frac{1}{S_D + 2} + \cdots + \frac{1}{n-1} - 1 \right). \end{aligned}$$

Now, obviously, $\frac{S_D - D_D}{S_D \cdot n}$ is positive (or zero).

$$\left(\frac{1}{S_D + 1} + \frac{1}{S_D + 2} + \cdots + \frac{1}{n-1} - 1 \right).$$

is negative because, in the classical problem, D_D was the largest integer such that

$$\left(\frac{1}{D_D} + \frac{1}{D_D + 1} + \cdots + \frac{1}{n-1} - 1 \right) > 0.$$

Since $S_D \geq D_D$, obviously $S_D + 1 > D_D$, and the above expression must be negative. See (4) if you don't remember. Since the net gain is negative, increasing S_D decreases the probability of success, and thus in an optimal solution $S_D \leq D_D$.

□

Now we can calculate the probability of winning for the subordinate player, when he rejects the first S_D people.

Let's consider what the subordinate player will get before the dominant player's rejection point. There are three possibilities: he wins by getting #1, he loses by getting a suboptimal applicant, or he does not get anybody.

He wins with probability $\frac{S_D}{n} \left(\frac{1}{S_D} + \frac{1}{S_D+1} + \dots + \frac{1}{D_D-1} \right)$, and he does not get anybody if he rejected the best person before the dominant player's rejection point, with probability $\frac{S_D}{D_D}$. Combining these probabilities with the results we have of I_2 , the total probability of the subordinate player winning is:

$$\begin{aligned} P_S &= \frac{S_D}{n} \left(\frac{1}{S_D} + \frac{1}{S_D+1} + \dots + \frac{1}{D_D-1} \right) + \frac{S_D}{D_D} (I_2(D_D, n - D_D)) \\ &= \frac{S_D}{n} \left(\frac{1}{S_D} + \dots + \frac{1}{D_D-1} + C_2 \right), \end{aligned}$$

$$\text{where } C_2 = \frac{1}{D_D(D_D+1)} + \frac{1}{D_D(D_D+2)} + \dots + \frac{1}{(n-2)(n-1)}.$$

Note that the part with a brace under it is constant, so we'll call it C_2 . This monstrous expression, though constant, will likely be very hard to calculate. However, if you have a fairly advanced calculator, you can find that C_2 when $n = 10$ is $\frac{1063}{1440} \approx .7382$. But C_2 for $n = 100$ is about .49415. Does C_2 for large n converge to $\frac{1}{2}$?

The first step is to convert it to a summation and then to an integral; note that one of the limits must be one more/less than the limits of summation because, if you think of the integral as a series of infinitesimal-width rectangles, there needs to be an extra rectangle at one end to represent the last term. Still, the difference of 1 is negligible for big n . The outer integral increases the upper limit, whereas the lower integral decreases the lower

limit in this case. Aside from making the integral easier to evaluate, the resulting integral will be closer to the actual value, because one integral will make the answer slightly higher while the other will make it slightly lower, for large n .

$$\begin{aligned}
C_2 &= \frac{1}{D_D(D_D + 1)} + \frac{1}{D_D(D_D + 2)} + \cdots + \frac{1}{(n-2)(n-1)} \\
&= \sum_{p_1=D_D}^{n-2} \sum_{p_2=p_1+1}^{n-1} \frac{1}{p_1 p_2} \\
&\rightarrow \int_{D_D}^{n-1} \int_{q_1}^{n-1} \frac{1}{q_1 q_2} dq_2 dq_1 \\
&= \int_{D_D}^{n-1} \frac{1}{q_1} (\ln(n-1) - \ln(q_1)) \\
&= \ln(n-1) \ln(q_1) - \frac{(\ln q_1)^2}{2} \Big|_{D_D}^{n-1} \\
&= \ln(n-1) \ln(n-1) - \frac{(\ln(n-1))^2}{2} \\
&\quad - \ln(n-1) \ln(D_D) - \frac{(\ln D_D)^2}{2} \\
&= \frac{1}{2} (\ln(n-1))^2 - \ln D_D (\ln(n-1) - \frac{1}{2} \ln D_D) \\
&= \frac{1}{2} \left((\ln(n-1))^2 - \ln D_D (2 \ln(n-1) - \ln D_D) \right) \\
&= \frac{1}{2} \left((\ln(n-1))^2 - 2 \ln(n-1) \ln D_D + (\ln D_D)^2 \right) \\
&= \frac{1}{2} (\ln(n-1) - \ln D_D)^2 \\
&\rightarrow \frac{1}{2} \left(\ln n - \ln \frac{n}{e} \right)^2 \\
&= \frac{1}{2} \left(\ln \frac{n}{e} \right)^2 \\
&= \frac{1}{2} (\ln e)^2 \\
&= \frac{1}{2}.
\end{aligned}$$

and our suspicions are confirmed.

Now we can use the same trick to find the optimal value: find the difference between successive terms.

$$\begin{aligned}
 P_S &\rightarrow \frac{S_D}{n} \left(\frac{1}{S_D} + \cdots + \frac{1}{D_D - 1} + \frac{1}{2} \right) \\
 P_S(S_D + 1) - P_S(S_D) &= \frac{S_D + 1}{n} \left(\frac{1}{S_D + 1} + \cdots + \frac{1}{D_D - 1} + \frac{1}{2} \right) \\
 &\quad - \frac{S_D}{n} \left(\frac{1}{S_D} + \cdots + \frac{1}{D_D - 1} + \frac{1}{2} \right) \\
 &= \frac{1}{n} \left(\frac{1}{S_D + 1} + \cdots + \frac{1}{D_D - 1} + \frac{1}{2} \right) - \frac{S_D}{n} \frac{1}{S_D} \\
 &= \frac{1}{n} \left(\frac{1}{S_D + 1} + \cdots + \frac{1}{D_D - 1} - \frac{1}{2} \right).
 \end{aligned}$$

Now we can approximate the probability with an integral again! Also, let us denote the difference with Δ .

$$\begin{aligned}
 \Delta P_S &= \frac{1}{n} \left(\int_{S_D + 1}^{D_D} \frac{1}{p} dp - \frac{1}{2} \right) \\
 &= \frac{1}{n} \left(\ln D_D - \ln(S_D + 1) - \frac{1}{2} \right) \\
 &\rightarrow \frac{1}{n} \left(\ln D_D - \ln S_D - \frac{1}{2} \right) \\
 &= \frac{1}{n} \left(\ln \frac{D_D}{S_D} - \frac{1}{2} \right) \\
 n\Delta P_S &= \left(\ln \frac{D_D}{S_D} - \frac{1}{2} \right).
 \end{aligned}$$

Solve for zero:

$$0 = \left(\ln \frac{S_D}{D_D} + \frac{1}{2} \right)$$

Raise e to the power of both sides:

$$\begin{aligned}
1 &= e^{\left(\ln \frac{S_D}{D_D} + \frac{1}{2}\right)} \\
&= e^{\ln \frac{S_D}{D_D}} e^{\frac{1}{2}} \\
&= \frac{S_D}{D_D} e^{\frac{1}{2}} \\
S_D &= D_D e^{-\frac{1}{2}}.
\end{aligned}$$

and plug in how D_D converges to $\frac{n}{e}$:

$$S_D = \frac{n}{e^{\frac{3}{2}}} \quad (8)$$

That's our final result! The ratio is about 22.31%. Finally, let us find the subordinate player's probability of success.

$$\begin{aligned}
P_S &\rightarrow \frac{S_D}{n} \left(\frac{1}{S_D} + \cdots + \frac{1}{D_D - 1} + \frac{1}{2} \right) \\
&\rightarrow \frac{S_D}{n} \left(\int_{S_D}^{D_D} \frac{1}{p} dp + \frac{1}{2} \right) \\
&\rightarrow e^{-\frac{3}{2}} \left(\int_{ne^{-\frac{3}{2}}}^{ne^{-1}} \frac{1}{p} dp + \frac{1}{2} \right) \\
&= e^{-\frac{3}{2}} \left(\ln(ne^{-1}) - \ln(ne^{-\frac{3}{2}}) + \frac{1}{2} \right) \\
&= e^{-\frac{3}{2}} \left(\frac{1}{2} + \frac{1}{2} \right) \\
&= e^{-\frac{3}{2}}.
\end{aligned}$$

Note, interestingly, that the probability of success is the same as the rejection point, just as it was in the classic problem.

5 The Experience Problem

Now, we consider the Experience Problem, which is the result of extending the previous extension again, but giving the subordinate player more advantage. First, the problem statement.

As in the previous problems, there are n secretaries in total, and the subordinate player interviews them one by one. He interviews the first s applicants by himself. After the first s applicants have been interviewed, the dominant player steps in and, together and simultaneously, under the assumptions of the previous extension, they interview the remaining $n - s$ secretaries together. This means that the dominant player will not be able to meet the first s applicants, although he knows of their existence.

First, we must find the strategy of the dominant player, which is no longer as trivial as in the previous extension. The dominant player wins if and only if both these conditions hold:

- #1 is one of the last $n - s$ secretaries that the dominant player can interview.
- The dominant player gets the best applicant in the last $n - s$ secretaries.

If both hold, the best applicant in the $n - s$ would be #1, and the dominant player would receive him. Furthermore, the events are independent, and the dominant player only has control over the second event. To maximize its probability, he would therefore use the classic $\frac{n}{e}$ strategy for the people he can interview, and his rejection point would be $\frac{n+s(e-1)}{e}$.

When does the subordinate player win? Obviously one case occurs when the first candidate he meets is #1 before the dominant player even starts interviewing. Also, #1 could come after both players are looking for the next candidate; however, because the dominant player does not get to see s of the applicants, he may recognize some applicants as candidates that the subordinate player does not. We call these false candidates, and we call the candidates that actually are candidates from both players' point of view true candidates. As a result, to remove the dominant player, there may be one true candidate (note that no false candidates can follow any true candidate), or many false candidates.

- #1 is the first candidate between the two players' rejection points.
- There are no candidates between the two players' rejection points; #1 is after the dominant player's rejection point; and either of:

- Between the dominant player's rejection point and #1, there are at least one false candidate and no true candidates.
- Between the dominant player's rejection point and #1, there are one true candidate and no false candidates. (Any false candidates would have to precede the true candidate and remove the dominant player from the game; then, the subordinate player would take the true, suboptimal candidate.)

The first case, where #1 is the first candidate between the two players' rejection points, is simple, because we can ignore the dominant player:

$$\frac{S_D}{n} \left(\frac{1}{S_D} + \frac{1}{S_D + 1} + \cdots + \frac{1}{D_D - 1} \right) \rightarrow \frac{S_D}{n} (\ln D_D - \ln S_D)$$

The next case is in which the dominant player receives a false candidate. For success in this case, the best applicant before #1 cannot be seen by the dominant player, and must be part of the first s people. Also, the best applicant before #1 and after the first s people must be after the dominant player's rejection point, ensuring that he will see a false candidate. Thus, again summing for each position where #1 could be, we obtain

$$\begin{aligned} & \frac{1}{n} \left(\frac{s}{D_D + 1} \frac{1}{D_D + 1 - s} + \frac{s}{D_D + 2} \frac{2}{D_D + 2 - s} + \frac{s}{n - 1} \frac{n - 1 - D_D}{n - 1 - s} \right) \\ \rightarrow & \frac{s}{n} \int_{D_D + 1}^n \frac{p - D_D}{p(p - s)} dp \\ = & \frac{s}{n} \int_{D_D + 1}^n \frac{\frac{D_D}{s}}{p} + \frac{1 - \frac{D_D}{s}}{p - s} dp \\ \rightarrow & \frac{s}{n} \left(\frac{D_D}{s} (\ln n - \ln D_D) + \left(1 - \frac{D_D}{s} \right) (\ln(n - s) - \ln(D_D - s)) \right) \\ = & \frac{s}{n} \left(\frac{D_D}{s} (\ln n - \ln(n + s(e - 1))) + 1 \right) + \left(1 - \frac{D_D}{s} \right) \\ = & \frac{s}{n} \left(\frac{D_D}{s} (\ln n - \ln(n + s(e - 1))) + 1 \right) \\ = & \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n}. \end{aligned}$$

The last case is the same as in the second extension; two candidates must come after #1, so the probability is also the same:

$$\frac{S_D}{2n}$$

Now, of course, we find the difference between successive terms.

$$\begin{aligned} & \frac{S_D + 1}{n} (\ln D_D - \ln(S_D + 1)) + \frac{S_D + 1}{2n} + \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n} \\ & - \frac{S_D}{n} (\ln D_D - \ln S_D) + \frac{S_D}{2n} + \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n} \\ = & \frac{\ln D_D}{n} + \frac{1}{2n} - \frac{1}{n} ((S_D + 1) \ln(S_D + 1) - S_D \ln S_D) \\ = & \frac{\ln D_D}{n} + \frac{1}{2n} - \frac{1}{n} \ln \left(\frac{(S_D + 1)^{S_D + 1}}{S_D^{S_D}} \right) \\ = & \frac{\ln D_D}{n} + \frac{1}{2n} - \frac{1}{n} \ln \left(\left(\frac{S_D + 1}{S_D} \right)^{S_D} \right) - \frac{1}{n} \\ = & \frac{1}{n} \left(\ln D_D - \ln S_D - \frac{1}{2} \right). \end{aligned}$$

Solving for zero is straightforward.

$$\begin{aligned} 0 &= \frac{1}{n} \left(\ln D_D - \ln S_D - \frac{1}{2} \right) \\ 0 &= \ln D_D - \ln S_D - \frac{1}{2} \\ \ln S_D &= \ln D_D - \frac{1}{2} \\ S_D &= D_D e^{-\frac{1}{2}}. \end{aligned}$$

This is actually the same result that we arrived at in the previous section. Why? The reason is actually quite simple: the only difference in probability is the case where the dominant player meets a false candidate, and the probability of success here is constant. As a result, the optimal rejection point is the same, but not the probability of success. The difference in the results is simply because the dominant player has decided to follow

a strategy that is not optimal from the subordinate player's point of view. Finally, we plug back and solve the Experience problem!

$$\begin{aligned}
& \frac{S_D}{n} (\ln D_D - \ln S_D) + \frac{S_D}{2n} + \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n} \\
&= \frac{D_D e^{-\frac{1}{2}}}{n} (\ln D_D - \ln D_D e^{-\frac{1}{2}}) + \frac{D_D e^{\frac{1}{2}}}{2n} + \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n} \\
&= \frac{D_D e^{-\frac{1}{2}}}{2n} + \frac{D_D e^{-\frac{1}{2}}}{2n} + \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n} \\
&= \frac{D_D e^{-\frac{1}{2}}}{n} + \frac{D_D}{n} (\ln n - \ln(n + s(e - 1))) + \frac{s}{n}.
\end{aligned}$$

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