

FIVE-VARIABLE p -ADIC L -FUNCTIONS FOR $U(3) \times U(2)$

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ABSTRACT. We construct a five-variable p -adic L -function attached to Hida families on the definite unitary groups $U(3)$ and $U(2)$ by using the Ichino-Ikeda formula. The interpolation formula fits into the conjectural shape of p -adic L -functions predicted by Coates and Perrin-Riou.

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1. INTRODUCTION

The aim of this paper is to construct a five-variable p -adic L -function interpolating a square root of the algebraic part of central values of the L -series attached to a pair of Hida families on the definite unitary groups $U(3)$ and $U(2)$. We establish the explicit interpolation formulae, which completely comply with the conjectural framework described in [CPR89, Coa89b, Coa89a].

Let E be an imaginary quadratic field. Throughout this paper we fix a prime number $p > 3$ which *splits* in E and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, where $\overline{\mathbb{Q}_p}$ is a fixed algebraic closure of \mathbb{Q}_p .

1.1. Hida families on $U(n)$ and the associated Galois representations. Fix a finite extension F of \mathbb{Q}_p and denote its maximal compact subring by \mathcal{O} . For each positive integer n , let $T_n \subset \mathrm{GL}_n$ be the diagonal torus. Let

$$\Lambda_n := \mathcal{O}[[T_n(\mathbb{Z}_p)]] = \varprojlim_{m \geq 1} \mathcal{O}[T_n(\mathbb{Z}/p^m\mathbb{Z})]$$

Date: January 9, 2025.

2010 Mathematics Subject Classification. 11F67, 11F33.

be the completed group algebra, and \mathbf{I}_n a local and normal Λ_n -algebra finite and flat over Λ_n . We say that an \mathcal{O} -algebra homomorphism $\underline{Q} : \mathbf{I}_n \rightarrow \overline{\mathbb{Q}}_p$ is locally algebraic if its restriction to $T_n(\mathbb{Z}_p)$ is of the form $\underline{Q}(z_1, \dots, z_n) = \prod_{i=1}^n z_i^{k_{Q_i}} \epsilon_{Q_i}(z_i)$ with $(k_{Q_1}, \dots, k_{Q_n}) \in \mathbb{Z}^n$ and characters $\epsilon_{Q_i} : \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ of finite order. We call $k_{\underline{Q}} = (k_{Q_1}, \dots, k_{Q_n}) \in \mathbb{Z}^n$ the weight of \underline{Q} and $\epsilon_{\underline{Q}} = (\epsilon_{Q_1}, \dots, \epsilon_{Q_n})$ the finite part of \underline{Q} . Let $\mathfrak{X}_{\mathbf{I}_n}$ be the set of locally algebraic points in $\text{Spec } \mathbf{I}_n(\overline{\mathbb{Q}}_p)$. We say that $\underline{Q} \in \mathfrak{X}_{\mathbf{I}_n}$ is dominant if $k_{Q_1} \leq k_{Q_2} \leq \dots \leq k_{Q_n}$, and \underline{Q} is sufficiently regular if $k_{Q_1} < k_{Q_2} < \dots < k_{Q_n}$. Let $\mathfrak{X}_{\mathbf{I}_n}^+$ be the subset of locally algebraic points of dominant weights and $\mathfrak{X}_{\mathbf{I}_n}^{++}$ the subset of points of sufficiently regular weights.

We denote the rings of adèles of \mathbb{Q} and E by \mathbb{A} and \mathbb{E} . Let $x \mapsto x^c$ be the non-trivial automorphism of E . We write \mathfrak{p} for the prime ideal induced by the restriction of ι_p to E . Fix a positive definite Hermitian matrix T in $M_n(E)$. For $g \in M_n(E)$ we define $g^\ddagger := T^{-1} t g^c T$. The definite unitary group $U(n)$ associated with T is the algebraic group defined over \mathbb{Q} by setting

$$U(n)(R) = \{g \in M_n(E \otimes_{\mathbb{Q}} R) \mid g^\ddagger g = \mathbf{1}_n\}$$

for any \mathbb{Q} -algebra R .

We shall make use of Hida theory for definite unitary groups developed in [Ger19, §2]. Let N be a positive integer only divisible by primes $q \neq p$ split in E . Choose an ideal \mathfrak{N} of the ring \mathfrak{r} of integers of E such that $\mathfrak{N}\overline{\mathfrak{N}} = N\mathfrak{r}$. This ideal \mathfrak{N} shall be referred to as the tame level. Hida theory produces a free Λ_n -module $e_{\text{ord}}\mathbf{S}^{U(n)}(\mathfrak{N}, \Lambda_n)$ of finite rank equipped with a faithful action of the universal ordinary Hecke algebra $\mathbf{T}^{(n)}(\mathfrak{N})$ for the unitary group $U(n)$ (See [Ger19, Definition 2.23]). The Λ_n -module $e_{\text{ord}}\mathbf{S}^{U(n)}(\mathfrak{N}, \Lambda_n)$ is referred to the space of ordinary Λ_n -adic forms, which roughly speaking consists of p -adic families of p -ordinary modular forms on $U(n)$ invariant by the mirabolic subgroup of level \mathfrak{N} . An \mathbf{I}_n -adic Hida family \mathbf{f} on $U(n)$ is a non-zero Hecke eigenform in $e_{\text{ord}}\mathbf{S}^{U(n)}(\mathfrak{N}, \mathbf{I}_n) := e_{\text{ord}}\mathbf{S}^{U(n)}(\mathfrak{N}, \Lambda_n) \otimes_{\Lambda_n} \mathbf{I}_n$, which induces a Λ_n -algebra homomorphism $\lambda_{\mathbf{f}} : \mathbf{T}^{(n)}(\mathfrak{N}) \rightarrow \mathbf{I}_n$.

Denote the absolute Galois group of a field L by Γ_L and its cyclotomic character by ε_{cyc} . Let \mathfrak{m} be the maximal ideal of \mathbf{I}_n . To each \mathbf{I}_n -adic Hida family \mathbf{f} , one can associate the residual semisimple Galois representation $\bar{\rho}_{\mathbf{f}} : \Gamma_E \rightarrow \text{GL}_n(\mathbf{I}_n/\mathfrak{m})$ (see [Ger19, Proposition 2.28]). If $\bar{\rho}_{\mathbf{f}}$ is absolutely irreducible, then we can further obtain the Galois representation $\rho_{\mathbf{f}} : \Gamma_E \rightarrow \text{GL}_n(\mathbf{I}_n)$ unramified outside primes dividing Np and primes l where $U(n)(\mathbb{Q}_l)$ is ramified (see [Ger19, Proposition 2.29] for more details). Denote by $V_{\mathbf{f}}$ the free \mathbf{I}_n -module of rank n on which Γ_E acts via $\rho_{\mathbf{f}}$. For each $\underline{Q} \in \mathfrak{X}_{\mathbf{I}_n}^+$, the specialization $V_{\underline{Q}(\mathbf{f})} := V_{\mathbf{f}} \otimes_{\mathbf{I}_n, \underline{Q}} \overline{\mathbb{Q}}_p$ is the geometric p -adic Galois representation associated with some automorphic representation $\pi_{\underline{Q}} \simeq \otimes_v \pi_{Q,v}$ of $U(n)(\mathbb{A})$. Let $\mathfrak{X}_{\mathbf{f}}^{\text{temp}}$ be the set of points $\underline{Q} \in \mathfrak{X}_{\mathbf{I}_n}^+$ such that $\pi_{\underline{Q}}$ is everywhere tempered. Then the representation $V_{\mathbf{f}}$ is *conjugate self-dual* in the sense that

$$V_{\mathbf{f}}^{\vee} \simeq V_{\mathbf{f}}^c \otimes \varepsilon_{\text{cyc}}^{n-1}.$$

Moreover, by the local description of p -adic Galois representations [Ger19, Corollary 2.33] at p combined with [TU99, Lemma 7.2], there exists a filtration $\{\text{Fil}_i(V_{\mathbf{f}}|_{\Gamma_{E_p}})\}_{i=1}^n$ of Γ_{E_p} -stable lattices

$$\{0\} = \text{Fil}_0(V_{\mathbf{f}}|_{\Gamma_{E_p}}) \subset \text{Fil}_1(V_{\mathbf{f}}|_{\Gamma_{E_p}}) \subset \dots \subset \text{Fil}_n(V_{\mathbf{f}}|_{\Gamma_{E_p}}) = V_{\mathbf{f}}|_{\Gamma_{E_p}}$$

such that for every $\underline{Q} \in \mathfrak{X}_{\mathbf{I}_n}^+$, the specialization $V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_p}}$ is Hodge-Tate and each graded piece

$$\mathrm{gr}_i(V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_p}}) := \mathrm{Fil}_i(V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_p}})/\mathrm{Fil}_{i-1}(V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_p}})$$

has Hodge-Tate weights $-k_{Q_i} - i + 1$ for $1 \leq i \leq n$. Here the Hodge-Tate weight of $\mathbb{Q}_p(1)$ is 1 in our convention. Likewise there exists a filtration $\{\mathrm{Fil}_i(V_{\mathbf{f}}|_{\Gamma_{E_{\overline{p}}}})\}_{i=1}^n$ of $\Gamma_{E_{\overline{p}}}$ -stable lattices in $V_{\mathbf{f}}$ such that for every $\underline{Q} \in \mathfrak{X}_{\mathbf{I}_n}^+$, each graded piece

$$\mathrm{gr}_i(V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_{\overline{p}}}}) = \mathrm{Fil}_i(V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_{\overline{p}}}})/\mathrm{Fil}_{i-1}(V_{\underline{Q}(\mathbf{f})}|_{\Gamma_{E_{\overline{p}}}})$$

is Hodge-Tate of weight $k_{Q_{n-i+1}} - i + 1$.

1.2. The algebraicity of central values. Let $\underline{k} = (k_1, k_2, \dots, k_n)$ and $\underline{k}' = (k'_1, k'_2, \dots, k'_{n-1})$ be tuples of integers satisfying the following interlacing relation

$$(1.1) \quad k_1 \leq -k'_{n-1} \leq k_2 \leq \dots \leq -k'_2 \leq k_{n-1} \leq -k'_1 \leq k_n.$$

Let π be an irreducible tempered automorphic representation of $U(n)(\mathbb{A})$ such that π_{∞} has highest weight $-\underline{k}$, and σ an irreducible tempered automorphic representation of $U(n-1)(\mathbb{A})$ such that σ_{∞} has highest weight $-\underline{k}'$.

The *complete* automorphic L -function for the product $\pi \times \sigma$ is defined by

$$L(s, \pi \times \sigma) = L^{\mathrm{GL}}(s, \mathrm{BC}(\pi) \times \mathrm{BC}(\sigma)),$$

where $\mathrm{BC}(\pi)$ (resp. $\mathrm{BC}(\sigma)$) is the functorial lift of π (resp. σ) to an automorphic representation of $\mathrm{GL}_n(\mathbb{E})$ (resp. $\mathrm{GL}_{n-1}(\mathbb{E})$) (cf. [Lab11, Corollaire 5.3]). The L -function in the right hand side has been defined by the Rankin-Selberg convolution whose local and global analytic theories were established by Jacquet, Piatetski-Shapiro and Shalika in [JPSS83]. Let $L(s, \sigma, \mathrm{Ad})$ and $L(s, \pi, \mathrm{Ad})$ be the *complete* adjoint L -functions of σ and π , respectively. These are the Asai and twisted Asai L -functions of $\mathrm{BC}(\sigma)$ and $\mathrm{BC}(\pi)$ (cf. Remark 4.1). The ratio

$$\frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \mathrm{Ad})L(1, \sigma, \mathrm{Ad})}$$

is indeed an algebraic number thanks to [GL21, Theorem C] and [Che23, Corollary 7.9]. Proposition 4.5 proves its refinement for central values by using Shimura's mass formula.

1.3. The period. To make our interpolation formula meaningful, we will give the definition of periods for critical L -values associated with the Galois representation $V_{\mathbf{f}}$. We denote the conductor of $\pi_{\underline{Q}}$ by $\mathfrak{N}_{\pi_{\underline{Q}}}$. In this introductory section we use a simplified period defined by

$$\Omega^{(M)}(V_{\underline{Q}(\mathbf{f})}) = [\mathcal{K} : \mathcal{K}_0(\mathfrak{N}_{\pi_{\underline{Q}}})] 2^{2^* \pi_{\underline{Q}}} L^{(M)}(1, \pi_{\underline{Q}}, \mathrm{Ad}) \mathcal{E}(\pi_{\underline{Q}, p}, \mathrm{Ad}) \mathcal{B}_{\pi_{\underline{Q}, p}}$$

for a positive integer M , where

- $2^{2^* \pi_{\underline{Q}}}$ is the order of the S -group associated to the L -parameter of $\pi_{\underline{Q}}$;
- $L^{(M)}(1, \pi_{\underline{Q}}, \mathrm{Ad})$ is the partial adjoint L -series of $\pi_{\underline{Q}}$ with the archimedean factor but without Euler factors at primes dividing M ,
- $\mathcal{B}_{\pi_{\underline{Q}, p}}$ is the normalized local norm of the essential Whittaker function at p (see Proposition 6.4);
- $\mathcal{E}(\pi_{\underline{Q}, p}, \mathrm{Ad})$ is the modified Euler factor for the adjoint motive attached to $\underline{Q}(\mathbf{f})$ defined in Definition 6.5.

It is proved that the adjoint L -value $L^{(M)}(1, \pi_{\underline{Q}}, \text{Ad})$ is roughly the Petersson norm of the suitably normalized generic form on the quasi-split inner form of $\text{U}(n)$ for $n \leq 3$ in a forthcoming work of Shih-Yu Chen. This period $\Omega^{(M)}(V_{\underline{Q}(\mathbf{f})})$ is canonical in the sense that it only depends on M and the representation $\pi_{\underline{Q}}$ associated with the form $\underline{Q}(\mathbf{f})$.

1.4. **The product L -series for $\text{U}(3) \times \text{U}(2)$.** Fix positive rational numbers t_1 and t_2 . Let $\text{U}(3)$ and $\text{U}(2)$ be the definite unitary groups attached to T and T' , where

$$T = \begin{bmatrix} t_1 & & \\ & 1 & \\ & & t_2 \end{bmatrix}, \quad T' = \begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}.$$

Let Σ_T^- be the finite set consisting of primes q such that $\text{U}(2)(\mathbb{Q}_q)$ is compact. Let N and N' be natural numbers that satisfy the following condition:

(spl t) all the prime factors of NN' are split in E .

To simplify the discussion of the introduction, we assume that

(odd) NN' is odd and $2 \notin \Sigma_T^-$.

Fix a decomposition $N\mathfrak{r} = \mathfrak{N}\overline{\mathfrak{N}}$ and $N'\mathfrak{r} = \mathfrak{N}'\overline{\mathfrak{N}'}$. Let

$$\mathbf{f} \in e_{\text{ord}} \mathbf{S}^{\text{U}(3)}(\mathfrak{N}, \mathbf{I}_3), \quad \mathbf{g} \in e_{\text{ord}} \mathbf{S}^{\text{U}(2)}(\mathfrak{N}', \mathbf{I}_2)$$

be Hida families. We further assume that the residual Galois representations $\bar{\rho}_{\mathbf{f}}$ and $\bar{\rho}_{\mathbf{g}}$ are both absolutely irreducible. Let $V_{\mathbf{f}}$ and $V_{\mathbf{g}}$ be the Galois representation of Γ_E associated with the Hida family \mathbf{f} and \mathbf{g} respectively. Consider the tensor product representation $V_{\mathbf{f}\mathbf{g}} := V_{\mathbf{f}} \otimes V_{\mathbf{g}}$ of rank six over the five variable Iwasawa algebra $\mathbf{I}_3 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2$. Define the induced representation \mathbf{V} of $\Gamma_{\mathbb{Q}}$ by

$$\mathbf{V} := \text{Ind}_{\Gamma_E}^{\Gamma_{\mathbb{Q}}} (V_{\mathbf{f}\mathbf{g}} \otimes \varepsilon_{\text{cyc}}^2).$$

For each prime number q we denote the Weil-Deligne group of \mathbb{Q}_q by $W_{\mathbb{Q}_q}$. For each $\mathcal{Q} = (Q, Q') \in \mathfrak{X}_{\mathbf{I}_3}^+ \times \mathfrak{X}_{\mathbf{I}_2}^+$, let $\mathbf{V}_{\mathcal{Q}}$ be the specialization of \mathbf{V} at \mathcal{Q} and define the complex L -series of the p -adic Galois representation $\mathbf{V}_{\mathcal{Q}}$ by the Euler product

$$L(\mathbf{V}_{\mathcal{Q}}, s) = \prod_q L_q(\mathbf{V}_{\mathcal{Q}}, s)$$

of the local L -factors attached to $\text{WD}_q(\mathbf{V}_{\mathcal{Q}}) \otimes_{\overline{\mathbb{Q}_p}, t_p^{-1}} \mathbb{C}$, where $\text{WD}_q(\mathbf{V}_{\mathcal{Q}})$ is the Weil-Deligne representation of $W_{\mathbb{Q}_q}$ over $\overline{\mathbb{Q}_p}$ associated to $\mathbf{V}_{\mathcal{Q}}$. Putting

$$\begin{aligned} (\lambda_{Q_1}, \lambda_{Q_2}, \lambda_{Q_3}) &= (-k_{Q_1} + 1, -k_{Q_2}, -k_{Q_3} - 1); \\ (\mu_{Q'_1}, \mu_{Q'_2}) &= \left(-k_{Q'_1} + \frac{1}{2}, -k_{Q'_2} - \frac{1}{2} \right), \end{aligned}$$

we define the archimedean L -factor of $\mathbf{V}_{\mathcal{Q}}$ by

$$\Gamma(\mathbf{V}_{\mathcal{Q}}, s) = \prod_{i=1,2,3} \prod_{j=1,2} \Gamma_{\mathbb{C}} \left(s + \frac{1}{2} + |\lambda_{Q_i} + \mu_{Q'_j}| \right),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$. We are interested in the algebraic part of the value of $L(\mathbf{V}_{\mathcal{Q}}, s)$ at $s = 0$. Note that $L(\mathbf{V}_{\mathcal{Q}}, 0)$ are central values as $\mathbf{V}^{\vee} \otimes \varepsilon_{\text{cyc}} \simeq \mathbf{V}$ is self-dual. With the assumption (spl t), the specializations of the Hecke eigensystems

$\lambda_{\underline{Q}(\mathbf{f})} := \underline{Q} \circ \lambda_{\mathbf{f}}$ and $\lambda_{\underline{Q}'(\mathbf{g})} := \underline{Q}' \circ \lambda_{\mathbf{g}}$ determine unique unitary automorphic representations $\underline{\pi}_{\underline{Q}}$ and $\underline{\sigma}_{\underline{Q}'}$ of $U(3)(\mathbb{A})$ and $U(2)(\mathbb{A})$, and we have

$$\Gamma(\mathbf{V}_{\underline{Q}}, s)L(\mathbf{V}_{\underline{Q}}, s) = L\left(s + \frac{1}{2}, \underline{\pi}_{\underline{Q}} \times \underline{\sigma}_{\underline{Q}'}\right).$$

Consider the set of critical points defined by

$$\mathfrak{X}_{\mathbf{V}}^{\text{crit}} = \{(\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{f}}^{\text{temp}} \times \mathfrak{X}_{\mathbf{g}}^{\text{temp}} \mid k_{Q_1} \leq -k_{Q'_2} \leq k_{Q_2} \leq -k_{Q'_1} \leq k_{Q_3}\}.$$

For $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ we view the algebraic number

$$\frac{\Gamma(\mathbf{V}_{\underline{Q}}, 0)L(\mathbf{V}_{\underline{Q}}, 0)}{\Omega^{(NN')}(V_{\underline{Q}(\mathbf{f})})\Omega^{(NN')}(V_{\underline{Q}'(\mathbf{g})})} \sim \frac{L(\frac{1}{2}, \underline{\pi}_{\underline{Q}} \times \underline{\sigma}_{\underline{Q}'})}{L(1, \underline{\pi}_{\underline{Q}}, \text{Ad})L(1, \underline{\sigma}_{\underline{Q}'}, \text{Ad})} \pmod{\overline{\mathbb{Q}}^\times}$$

through the embedding ι_p , as a p -adic number. The purpose of this paper is to understand the p -adic behavior of this ratio when $\underline{Q} \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ varies.

1.5. The modified Euler factor at p . To introduce the modified Euler factor at p , we prepare some notation. We consider the rank three Γ_{E_p} -invariant and $\Gamma_{E_{\bar{p}}}$ -invariant subspaces of $V_{\mathbf{f}\mathbf{g}}$ by

$$\begin{aligned} \text{Fil}_{\mathfrak{p}}^+ \mathbf{V}_{\mathbf{f}\mathbf{g}} &= \text{Fil}_1 V_{\mathbf{f}}|_{\Gamma_{E_p}} \otimes V_{\mathbf{g}}|_{\Gamma_{E_p}} + \text{Fil}_2 V_{\mathbf{f}}|_{\Gamma_{E_p}} \otimes \text{Fil}_1 V_{\mathbf{g}}|_{\Gamma_{E_p}}; \\ \text{Fil}_{\mathfrak{p}}^+ \mathbf{V}_{\mathbf{f}\mathbf{g}} &= \text{Fil}_1 V_{\mathbf{f}}|_{\Gamma_{E_{\bar{p}}}} \otimes V_{\mathbf{g}}|_{\Gamma_{E_{\bar{p}}}} + \text{Fil}_2 V_{\mathbf{f}}|_{\Gamma_{E_{\bar{p}}}} \otimes \text{Fil}_1 V_{\mathbf{g}}|_{\Gamma_{E_{\bar{p}}}}, \end{aligned}$$

and define the six dimensional $\Gamma_{\mathbb{Q}_p}$ -invariant subspace of \mathbf{V} by

$$\text{Fil}^+ \mathbf{V} = \left(\text{Fil}_{\mathfrak{p}}^+ \mathbf{V}_{\mathbf{f}\mathbf{g}} \oplus \text{Fil}_{\mathfrak{p}}^+ \mathbf{V}_{\mathbf{f}\mathbf{g}} \right) \otimes \varepsilon_{\text{cyc}}^2.$$

The pair $(\text{Fil}^+ \mathbf{V}, \mathfrak{X}_{\mathbf{V}}^{\text{crit}})$ satisfies the Panchishkin condition in [Gre94, p. 217] in the sense that all the Hodge-Tate numbers of $\text{Fil}^+ \mathbf{V}_{\underline{Q}}$ are positive but none of the Hodge-Tate numbers of $\mathbf{V}_{\underline{Q}}/\text{Fil}^+ \mathbf{V}_{\underline{Q}}$ is positive for $\underline{Q} \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$.

Let $\psi : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{C}^\times$ be the additive character with the archimedean component $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ and $\psi_p : \mathbb{Q}_p \rightarrow \mathbb{C}^\times$ the local component of ψ at the prime number p . Let dx be the self-dual Haar measure on \mathbb{Q}_p with respect to ψ_p . For each p -adic representation V of $\Gamma_{\mathbb{Q}_p}$, recall that the γ -factor $\gamma(V, s)$ is defined by

$$\gamma(V, s) = \frac{\varepsilon(\text{WD}_p(V), \psi_p, dx)L_p(V^\vee, 1-s)}{L_p(V, s)},$$

where $\varepsilon(\text{WD}_p(V), \psi_p, dx)$ is the local constant defined in [Del73, §4]. The modified Euler factor at p is defined by

$$\mathcal{E}(\text{Fil}^+ \mathbf{V}_{\underline{Q}}) = \frac{1}{\gamma(\text{Fil}_{\mathfrak{p}}^+ \mathbf{V}_{\underline{Q}}, 0)\gamma(\text{Fil}_{\mathfrak{p}}^+ \mathbf{V}_{\underline{Q}}, 0)L_p(\mathbf{V}_{\underline{Q}}, 0)}.$$

1.6. Interpolation formulae. Let $\text{Frac}(\mathbf{I}_3 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2)$ stand for the total ring of fractions of $\mathbf{I}_3 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2$.

Theorem 1.1. *We assume (odd) and (spl t). Then there exists a unique element $L_p(\mathbf{V}) \in \text{Frac}(\mathbf{I}_3 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2)$ which does not have a pole at any critical point $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ and such that*

$$\mathcal{Q}(L_p(\mathbf{V})) = \frac{\Gamma(\mathbf{V}_{\underline{Q}}, 0)L(\mathbf{V}_{\underline{Q}}, 0)}{\Omega^{(NN')}(V_{\underline{Q}(\mathbf{f})})\Omega^{(NN')}(V_{\underline{Q}'(\mathbf{g})})} \mathcal{E}(\text{Fil}^+ \mathbf{V}_{\underline{Q}}).$$

Remark 1.2. (1) We note that the global root number

$$\varepsilon\left(\frac{1}{2}, \pi_{\underline{Q}} \times \sigma_{\underline{Q}'}\right) = +1 \text{ for } (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$$

for $(\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ by (spl), (A.3) and Remark C.2.

- (2) It is important to note that $\mathfrak{X}_{\mathbf{I}_3}^{++} \subset \mathfrak{X}_{\mathbf{f}}^{\text{temp}}$ by the endoscopic classification of cuspidal automorphic representations on $\text{U}(3)$ (cf. [BC04, 3.2.3, p. 618]) combined with the Ramanujan conjecture [Car12, Theorem 1.2]. This explains in particular $\mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ contains the dense subset

$$\{(\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{I}_3}^{++} \times \mathfrak{X}_{\mathbf{I}_2}^{++} \mid k_{Q_1} \leq -k_{Q'_2} \leq k_{Q_2} \leq -k_{Q'_1} \leq k_{Q_3}\}.$$

- (3) Yifeng Liu [Liu] has recently constructed anticyclotomic p -adic L -functions for automorphic representations of $\text{U}(n) \times \text{U}(n-1)$ whose archimedean component is the trivial representation. Despite our result is restricted to $\text{U}(3) \times \text{U}(2)$, this paper works with automorphic representations of general weights at archimedean components and indeed constructs several variable *square root* p -adic L -functions attached to Hida families of modular forms on unitary groups. We also provide more precise interpolation formulae particularly at anticyclotomic characters which are either of infinite order or unramified at p . Our result thus offers a refinement of Liu's work in the particular case $n = 3$. We expect to generalize our construction to $\text{U}(n) \times \text{U}(n-1)$ in the future.
- (4) This paper mainly concerns the p -adic L -functions for the Rankin-Selberg convolution $\text{BC}(\pi_{\underline{Q}}) \times \text{BC}(\sigma_{\underline{Q}'})$ in the *balanced* case in the sense that the weights satisfy the interlacing relation (1.1). This is an analogue of theta elements in [BD96] and p -adic triple product L -functions in the balanced case ([GS20] and [Hsi21]). In a forthcoming joint work with Michael Harris, we construct p -adic L -functions in the *unbalanced* case: the weights of $\text{BC}(\pi_{\underline{Q}}) \times \text{BC}(\sigma_{\underline{Q}'})$ satisfy different interlacing relation

$$k_{Q_1} \leq -k_{Q'_2} \leq k_{Q_2} < k_{Q_3} \leq -k_{Q'_1}.$$

The method uses Hida families of modular forms on non-compact unitary groups $\text{U}(2, 1) \times \text{U}(1, 1)$.

In Definition 2.19, we shall construct the Hecke-equivariant perfect pairing

$$\mathbf{B}_{\mathfrak{N}} : e_{\text{ord}} \mathbf{S}^{\text{U}(n)}(\mathfrak{N}, \chi, \mathbf{I}_n) \times e_{\text{ord}} \mathbf{S}^{\text{U}(n)}(\mathfrak{N}, \chi, \mathbf{I}_n) \rightarrow \mathbf{I}_n,$$

which interpolates the canonical bilinear pairing between automorphic forms on definite unitary groups. Put

$$\eta_{\mathbf{f}} = \mathbf{B}_{\mathfrak{N}}(\mathbf{f}, \mathbf{f}) \in \mathbf{I}_n.$$

The element $\eta_{\mathbf{f}}$ is expected to be related to the congruence number of \mathbf{f} . More precisely, we will construct a *theta element* $\mathcal{L}_{\mathbf{f}, \mathbf{g}} \in \mathbf{I}_3 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2$ associated to a pair of Hida families (\mathbf{f}, \mathbf{g}) on $\text{U}(3) \times \text{U}(2)$, and define

$$L_p(\mathbf{V}) = \frac{\mathcal{L}_{\mathbf{f}, \mathbf{g}}^2}{\eta_{\mathbf{f}} \eta_{\mathbf{g}}}.$$

Therefore, we actually construct the square root for the p -adic L -function for the Galois representation \mathbf{V} .

The proof of Theorem 1.1 is divided into two steps:

- (i) we construct $\mathcal{L}_{f,g}$ via the p -adic interpolation of the period integrals of modular forms on $U(3) \times U(2)$ along $U(2)$ in Section 3;
- (ii) we evaluate those period integrals via the Ikeda-Ichino conjecture explicitly in Section 4.

The second step (ii) may have independent interest in some analytic aspects of the Rankin-Selberg L -values for unitary groups. We begin with a brief outline of the first step (i).

1.7. Construction of $\mathcal{L}_{f,g}$. We denote the finite adèle ring of \mathbb{Q} by $\widehat{\mathbb{Q}}$. Let $G = U(3)$ and $H = U(2)$. For simplicity we suppose that

$$N' = 1, \quad \chi' = 1.$$

We will specify suitable maximal compact subgroups \mathcal{K} of $G(\widehat{\mathbb{Q}})$ and \mathcal{K}' of $H(\widehat{\mathbb{Q}})$ (cf. Appendix B). Define open subgroups of \mathcal{K}' by

$$\mathcal{K}''(1) = \mathcal{K} \cap \mathcal{K}', \quad \mathcal{K}''(p^\ell) = \mathcal{K}''(1) \cap \iota_{\mathfrak{p}}'^{-1} \left(\left\{ \begin{bmatrix} 1 + \mathfrak{p}^\ell & * \\ \mathfrak{p}^\ell & 1 + \mathfrak{p}^\ell \end{bmatrix} \right\} \right)$$

for each positive integer ℓ . We consider the finite sets

$$X_\ell = G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \mathcal{K}_1(p^\ell \mathfrak{N}), \quad X'_\ell = H(\mathbb{Q}) \backslash H(\widehat{\mathbb{Q}}) / \mathcal{K}''(p^\ell), \quad \mathbf{X}_\ell = X_\ell \times X'_\ell.$$

Define $\varsigma^{(p)} = (\varsigma_l^{(p)}) \in G(\widehat{\mathbb{Q}})$ by

$$\varsigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \varsigma_l^{(p)} = \begin{cases} \iota_l^{-1}(\varsigma) & \text{if } l \text{ splits in } E \text{ and differs from } p, \\ \mathbf{1}_3 & \text{otherwise.} \end{cases}$$

Since $\iota(\mathcal{K}''(1)\varsigma^{(p)}) \subset \varsigma^{(p)}\mathcal{K}_1(\mathfrak{N})$, we define the map $j(x) = \iota(x)\varsigma^{(p)}$ induces a map $X'_0 \rightarrow X_0$. In §3.2 we construct a collection of regularized diagonal cycles Δ_ℓ^\dagger of \mathbf{X}_ℓ that are compatible with respect to the projection maps $\mathbf{X}_\ell \rightarrow \mathbf{X}_{\ell-1}$. We therefore obtain the big diagonal cycle

$$\Delta_\infty^\dagger = \lim_{\leftarrow \ell} \Delta_\ell^\dagger \in \lim_{\leftarrow \ell} \text{Pic} \mathbf{X}_\ell \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

We define an $\mathbf{I}_3 \widehat{\otimes} \mathbf{I}_2$ -adic modular form on $G \times H$ by

$$\mathbf{F} = \mathbf{f} \boxtimes \mathbf{g} : \lim_{\leftarrow \ell} \mathbf{X}_\ell \rightarrow \mathbf{I}_3 \widehat{\otimes} \mathbf{I}_2$$

by $\mathbf{F}(x, x') = \mathbf{f}(x)\mathbf{g}(x')$. Then \mathbf{F} naturally induces a map

$$\mathbf{F}_* : \lim_{\leftarrow \ell} \text{Pic} \mathbf{X}_\ell \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \mathbf{I}_3 \widehat{\otimes} \mathbf{I}_2.$$

The element $\mathcal{L}_{f,g}$ equals the theta element $\Theta_{\mathbf{F}}$ attached to \mathbf{F} defined by

$$\Theta_{\mathbf{F}} = \mathbf{F}_*(\Delta_\infty^\dagger) \in \mathbf{I}_3 \widehat{\otimes} \mathbf{I}_2$$

up to some fudge factor (see (4.6)). This theta element is an analogue for $U(2) \times U(3)$ of theta elements constructed by Bertolini and Darmon [BD96] for $SO(2) \times SO(3)$ (cf. [CH18]) and by the first author [Hsi21] for $SO(3) \times SO(4)$.

Let $d_{\mathcal{K}'}h$ be the Haar measure of $H(\mathbb{A})$ giving $H(\mathbb{R})\mathcal{K}'$ volume 1. We then proceed to show in Proposition 3.4 that the evaluation of $\Theta_{\mathbf{F}}$ at $\mathcal{Q} \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ is given

by the normalized period integral

$$\mathcal{P}_{\mathcal{K}'}^*(\Phi^\dagger) := \frac{p^{\frac{5}{2}\ell}}{(\rho_p(p)\mu_p(p)^2\nu_p'(p))^\ell} \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \Phi^\dagger \left(j(h)\iota_p^{-1} \left(\begin{bmatrix} 0 & 0 & -p^{-\ell} \\ 0 & p^\ell & 1 \\ p^{2\ell} & p^\ell & 0 \end{bmatrix} \right), h \right) d_{\mathcal{K}'} h,$$

where $\Phi^\dagger \in \pi_{\underline{Q}} \otimes \sigma_{\underline{Q}'}$ is a suitable $H(\mathbb{R})$ -invariant modular form on $G \times H$.

1.8. An explicit Ichino-Ikeda formula. A key ingredient to evaluate the period integral $\mathcal{P}_{\mathcal{K}'}^*(\Phi^\dagger)$ is the Ichino-Ikeda conjecture which has been first formulated for orthogonal groups in [II10]. Its unitary analogue was formulated for unitary groups in [Har14] and proved in [Zha14, BPLZZ21, BPCZ22] (cf. Theorem 4.2). This Ichino-Ikeda formula relates a square of this period integral to the product of the central value $L(\frac{1}{2}, \pi \times \sigma)$ and local integrals.

To make this formula precise, we need to compute various local integrals for suitable test vectors. Thanks to (spl t), we can apply the local theory of newforms for representations of general linear groups, which was developed in [JPSS81, Mat13]. We will compute these integrals at p in Section 5 and at archimedean and ramified places in Appendices A, B and C. To remove Hypothesis (spl t), one needs to compute the local integral for suitably chosen test vectors at inert primes.

A local key ingredient is the splitting lemma which has been proved in [LM14, Zha14] (see Lemma 5.2). This lemma relates the Ichino-Ikeda local integral to a square of the JPSS integral at split primes. It is well-known that the JPSS integral $Z(s, W_{\pi_l}, W_{\sigma_l})$ of essential vectors coincides with the L -factor $L^{\text{GL}}(s, \pi_l \times \sigma_l)$ when σ_l is unramified.

When $N' > 1$, we need to compute the local integral when σ_l is ramified. When σ_l satisfies the condition (H_3) in §4.10, following the construction [Sch93] of p -adic L -functions for $\text{GL}_3 \times \text{GL}_2$, Section 6 will construct an operator $\mathbf{U}_{\mathfrak{N}'}^{\chi'_l}$ having the following properties:

- the restriction of $\mathbf{U}_{\mathfrak{N}'}^{\chi'_l} W_{\pi_l}$ to $H(\mathbb{Q}_l)$ has an appropriate \mathcal{K}'_l -type;
- $Z(s, \mathbf{U}_{\mathfrak{N}'}^{\chi'_l} W_{\pi_l}, W_{\sigma_l})$ has a simple formula.

Furthermore, we replace the pair (π, σ) by a suitable twist $(\pi \otimes \varrho_{\mathbb{A}}^{-1}, \sigma \otimes \varrho_{\mathbb{A}})$ and replace the pair (\mathbf{f}, \mathbf{g}) by another pair of Hida families $(\mathbf{f}_\varrho, \mathbf{g}_\varrho)$. We apply the step (i) to $\mathbf{U}_{\mathfrak{N}'}^{\chi'_l} \mathbf{f}_\varrho \boxtimes \mathbf{g}_\varrho$. With these local calculations we conclude that

$$\frac{\mathcal{P}_{\mathcal{K}'}^*(\Phi^\dagger)^2}{(\varphi_\pi, \varphi_{\pi^\vee})_{\mathcal{K}}(\varphi_\sigma, \varphi_{\sigma^\vee})_{\mathcal{K}'}} \approx \frac{L(\frac{1}{2}, \pi_{\underline{Q}} \times \sigma_{\underline{Q}'})}{L(1, \pi_{\underline{Q}}, \text{Ad})L(1, \sigma_{\underline{Q}'}, \text{Ad})},$$

where $\varphi_\pi \in \pi_{\underline{Q}}$ and $\varphi_\sigma \in \sigma_{\underline{Q}'}$ are highest weight essential vectors.

Acknowledgement. Hsieh is supported by the NSTC grant 112-2628-M-002-011. Yamana is partially supported by JSPS Grant-in-Aid for Scientific Research (C) 23K03055. This work was partly supported by MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165. We thank Michael Harris for constant help and encouragements, and Shih-Yu Chen for useful discussions on the algebraicity of L -values.

NOTATION

Besides the standard symbols \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} , \mathbb{Z}_q and \mathbb{Q}_q we denote by \mathbb{R}_+^\times and \mathbb{Q}_+^\times the groups of strictly positive real and rational numbers, and by \mathbb{C}^1 the group of complex numbers of absolute value 1. Let \mathbb{A} be the ring of adèles of \mathbb{Q} . We write $\widehat{\mathbb{Q}}$ for the finite part of \mathbb{A} . Put $\widehat{\mathbb{Z}} = \prod_q \mathbb{Z}_q \subset \widehat{\mathbb{Q}}$. Given a place v of \mathbb{Q} , we write \mathbb{Q}_v for the completion of \mathbb{Q} with respect to v . We shall regard \mathbb{Q}_v and \mathbb{Q}_v^\times as subgroups of \mathbb{A} and \mathbb{A}^\times in a natural way. We denote by the formal symbol ∞ the real place of \mathbb{Q} and do not use q, l for the infinite place.

Define $\psi_v : \mathbb{Q}_v \rightarrow \mathbb{C}^1$ by $\psi_\infty(z) = e^{2\pi\sqrt{-1}z}$ for $z \in \mathbb{R}$ and by $\psi_p(x) = \psi_\infty(-y)$ for $x \in \mathbb{Q}_p$ with $y \in \mathbb{Z}[p^{-1}]$ such that $y - x \in \mathbb{Z}_p$. Then $\psi = \prod_v \psi_v$ defines a character of \mathbb{A}/\mathbb{Q} . We associate to $m \in \mathbb{Q}$ the global additive character ψ^m defined by $\psi^m(z) = \psi(mz)$ for $z \in \mathbb{A}$. For $a \in \mathbb{Q}_q$ we define an additive character ψ_q^a of \mathbb{Q}_q by $\psi_q^a(z) = \psi_q(az)$ for $z \in \mathbb{Q}_q$.

Let dz_v be the Haar measure on \mathbb{Q}_v self-dual with respect to the pairing $(z_v, z'_v) \mapsto \psi_v(z_v z'_v)$. Note that $\int_{\mathbb{Z}_q} dz_q = 1$ for each rational prime q and that dz_∞ is the usual Lebesgue measure on \mathbb{R} . Let $dz = \prod_v dz_v$. Then dz is the Haar measures on \mathbb{A} such that \mathbb{A}/\mathbb{Q} has volume 1. Let $d^\times t = \prod_v d^\times t_v$ be the Haar measure on \mathbb{A}^\times , where $d^\times t_v$ is the Haar measure on \mathbb{Q}_v^\times normalized by $\int_{\mathbb{Z}_q^\times} d^\times t_q = 1$ if $v = q < \infty$, and $d^\times t_\infty = \frac{dt_\infty}{|t_\infty|}$.

Let E be an imaginary quadratic field of discriminant $-D_E$ with the integer ring \mathfrak{r} . We write $\epsilon_{E/\mathbb{Q}} = \prod_v \epsilon_{E_v/\mathbb{Q}_v}$ for the quadratic character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ associated to E . Set

$$\mathbb{E} = E \otimes_{\mathbb{Q}} \mathbb{A}, \quad E_v = E \otimes_{\mathbb{Q}} \mathbb{Q}_v, \quad \mathfrak{r}_q = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_q.$$

We denote by $x \mapsto x^c$ the non-trivial automorphism of E . We write ${}^t x \in M_{n,m}(E)$ for the transpose of a matrix $x = (x_{ij}) \in M_{m,n}(E)$ and put

$$x^c = (x_{ij}^c) \in M_{m,n}(E).$$

Let Σ_E^r be the set of prime numbers which are ramified in E .

Once and for all we fix an odd rational prime p that is split in \mathfrak{r} . Fix an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} . We fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota_p : \mathbb{C} \simeq \mathbb{C}_p$, where \mathbb{C}_p is the completion of an algebraic closure of \mathbb{Q}_p . Given an algebraic number field L , we regard L as a subfield in \mathbb{C} (resp. \mathbb{C}_p) via ι_∞ (resp. $\iota_p \circ \iota_\infty$). Let ord_p be the p -adic valuation on \mathbb{C}_p normalized so that $\text{ord}_p p = 1$. We write \mathfrak{p} for the prime ideal of \mathfrak{r} above p that corresponds to the restriction of $\iota_p \circ \iota_\infty$ to E .

2. HIDA FAMILIES ON DEFINITE UNITARY GROUPS

2.1. Unitary groups. We let the base field be the rational field \mathbb{Q} . This assumption simplifies the notation and reduces technicality. We denote by T_n the subgroup of diagonal matrices in GL_n , by B_n the subgroup of upper triangular matrices in GL_n and by N_n the subgroup of upper unitriangular matrices in GL_n . We write $\text{Res}_{E/\mathbb{Q}} \text{GL}_n$ for the general linear group over an imaginary quadratic field E , regarded as an algebraic group over \mathbb{Q} by restricting scalars. We fix a *rational diagonal positive definite* matrix $T \in \text{GL}_n(\mathbb{Z}_{(p)})$ of size n . Define a Hermitian form $(\ , \)_T$ on $W = E^n$ by $(u, v)_T = {}^t u^c T v$ for $u, v \in W$. Let $G = U(T)$ be the unitary group associated with the Hermitian form T . Namely, G is the algebraic group

defined by

$$G = \{g \in \text{Res}_{E/\mathbb{Q}} \text{GL}_n \mid {}^t g^c T g = T\}.$$

Let l be a split prime, i.e., $l\mathfrak{r} = \bar{l}\mathfrak{l}$, $E_l \simeq E_{\mathfrak{l}} \oplus E_{\bar{\mathfrak{l}}}$ and $\mathfrak{r}_l \simeq \mathfrak{r}_{\mathfrak{l}} \oplus \mathfrak{r}_{\bar{\mathfrak{l}}}$. The projection $(g_1, g_2) \mapsto g_1$ gives an isomorphism $\iota_l : G(\mathbb{Q}_l) \xrightarrow{\sim} \text{GL}_n(E_l)$. Fix a maximal compact subgroup $\mathcal{K} = \prod_q \mathcal{K}_q$ of $G(\widehat{\mathbb{Q}})$ such that $\iota_l(\mathcal{K}_l) = \text{GL}_n(\mathfrak{r}_l)$ for every split prime l . Define open subgroups of $\text{GL}_n(\mathfrak{r}_l)$ by

$$\begin{aligned} \mathcal{K}_0^{(n)}(\mathfrak{l}^k) &= \{(g_{ij}) \in \text{GL}_n(\mathfrak{r}_l) \mid g_{nj} \in \mathfrak{l}^k \text{ for } j = 1, 2, \dots, n-1\}, \\ \mathcal{K}_1^{(n)}(\mathfrak{l}^k) &= \{(g_{ij}) \in \mathcal{K}_0^{(n)}(\mathfrak{l}^k) \mid g_{nn} - 1 \in \mathfrak{l}^k\}. \end{aligned}$$

Fix a positive integer N whose prime factors are split in E and which is not divisible by p . We take an ideal \mathfrak{N} of \mathfrak{r} such that $\mathfrak{r}/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$. Let

$$\begin{aligned} \mathcal{K}_0(\mathfrak{N}) &= \{(g_q) \in \mathcal{K} \mid \iota_l(g_l) \in \mathcal{K}_0^{(n)}(N\mathfrak{r}_l) \text{ for } l|N\}, \\ \mathcal{K}_1(\mathfrak{N}) &= \{(g_q) \in \mathcal{K} \mid \iota_l(g_l) \in \mathcal{K}_1^{(n)}(N\mathfrak{r}_l) \text{ for } l|N\} \end{aligned}$$

be open compact subgroups of $G(\widehat{\mathbb{Q}})$.

2.2. Classical modular forms on $U(n)$. For each positive integer ℓ we define open subgroups of $\text{GL}_n(\mathfrak{r}_{\mathfrak{p}})$ by

$$\begin{aligned} \mathcal{I}_0^{(n)}(\mathfrak{p}^\ell) &= \{(g_{ij}) \in \text{GL}_n(\mathfrak{r}_{\mathfrak{p}}) \mid g_{ij} \in \mathfrak{p}^{\ell(i-j)} \text{ for } i > j\}, \\ \mathcal{I}_1^{(n)}(\mathfrak{p}^\ell) &= \{(g_{ij}) \in \mathcal{I}_0^{(n)}(\mathfrak{p}^\ell) \mid g_{ii} - 1 \in \mathfrak{p}^\ell \text{ for } i = 1, 2, \dots, n\} \end{aligned}$$

and open compact subgroups of $G(\widehat{\mathbb{Q}})$ by

$$\begin{aligned} \mathcal{K}_0(p^\ell \mathfrak{N}) &= \{(g_q) \in \mathcal{K}_0(\mathfrak{N}) \mid \iota_{\mathfrak{p}}(g_{\mathfrak{p}}) \in \mathcal{I}_0^{(n)}(\mathfrak{p}^\ell)\}, \\ \mathcal{K}_1(p^\ell \mathfrak{N}) &= \{(g_q) \in \mathcal{K}_1(\mathfrak{N}) \mid \iota_{\mathfrak{p}}(g_{\mathfrak{p}}) \in \mathcal{I}_1^{(n)}(\mathfrak{p}^\ell)\}, \end{aligned}$$

where the open compact subgroups $\mathcal{K}_0(\mathfrak{N})$ and $\mathcal{K}_1(\mathfrak{N})$ of $G(\widehat{\mathbb{Q}})$ are defined in §2.1.

Definition 2.1. Let $\underline{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ be an n -tuple of integers such that $k_1 \leq k_2 \leq \dots \leq k_n$. For a commutative ring A of characteristic 0 we write $\mathcal{L}_{\underline{k}}(A)$ for an A -module on which $\text{GL}_n(A)$ acts as the irreducible representation $\rho_{\underline{k}} : \text{GL}_n(A) \rightarrow \text{GL}(\mathcal{L}_{\underline{k}}(A))$ with the lowest weight \underline{k} with respect to the Borel subgroup of the upper triangular matrices. If $a = \text{diag}(a_1, a_2, \dots, a_n) \in T_n$, then we write $a^{\underline{k}} := a_1^{k_1} a_2^{k_2} \dots a_n^{k_n}$ for simplicity.

We define an embedding $\iota_\infty^T : G(\mathbb{R}) \hookrightarrow \text{GL}_n(\mathbb{C})$ by

$$\iota_\infty^T(g) = \sqrt{T} g \sqrt{T}^{-1}, \quad \sqrt{T} = \begin{bmatrix} \sqrt{t_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \sqrt{t_n} \end{bmatrix}.$$

It is important to note that

$$\iota_\infty^T(g^c) = {}^t \iota_\infty^T(g)^{-1}.$$

We define a representation of $\rho_{\underline{k}, \infty}$ of $G(\mathbb{R})$ on $\mathcal{L}_{\underline{k}}(\mathbb{C})$ by $\rho_{\underline{k}, \infty}(g) = \rho_{\underline{k}}(\iota_\infty^T(g))$.

Definition 2.2. We call an $\mathcal{L}_{\underline{k}}(\mathbb{C})$ -valued function f on $G(\mathbb{A})$ a vector-valued modular form on G of weight \underline{k} and level $p^\ell \mathfrak{N}$ if it satisfies

$$f(\gamma g u_\infty u) = \rho_{\underline{k}, \infty}(u_\infty)^{-1} f(g)$$

for $\gamma \in G(\mathbb{Q})$, $g \in G(\widehat{\mathbb{Q}})$, $u_\infty \in G(\mathbb{R})$, $u \in \mathcal{K}_1(p^\ell \mathfrak{N})$. The space $\mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N})$ consists of vector-valued modular forms on $G(\mathbb{A})$ of weight \underline{k} and level $p^\ell \mathfrak{N}$. We say that $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N})$ is \mathbb{Q} -rational if the restriction of f to $G(\widehat{\mathbb{Q}})$ takes values in $\mathcal{L}_{\underline{k}}(\iota_\infty(\overline{\mathbb{Q}}))$.

Given $g \in G(\mathbb{A})$ and a function \mathcal{F} on $G(\mathbb{Q}) \backslash G(\mathbb{A})$, we define another function $r(g)\mathcal{F}$ on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ by

$$[r(g)\mathcal{F}](h) = \mathcal{F}(hg).$$

Namely, $r(g)\mathcal{F}$ is the right translation of \mathcal{F} by g . For a character χ of $\mathcal{K}_0(p^\ell \mathfrak{N})$ whose restriction to $\mathcal{K}_1(p^\ell \mathfrak{N})$ is trivial we set

$$\mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi) = \{f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}) \mid r(u)f = \chi(u)^{-1} f \text{ for } u \in \mathcal{K}_0(p^\ell \mathfrak{N})\}.$$

Put $\underline{k}^\vee = (-k_n, \dots, -k_2, -k_1)$. Fix a $\mathrm{GL}_n(\mathbb{Q})$ -invariant perfect pairing

$$\ell_{\underline{k}} : \mathcal{L}_{\underline{k}^\vee}(\mathbb{Q}) \otimes \mathcal{L}_{\underline{k}}(\mathbb{Q}) \rightarrow \mathbb{Q}.$$

Let $\mathcal{A}(G)$ denotes the space of scalar valued modular forms on G . We associate to $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N})$ and $\mathbf{v} \in \mathcal{L}_{\underline{k}^\vee}(\mathbb{Q})$ a scalar valued modular form $f_{\mathbf{v}} \in \mathcal{A}(G)$ defined by

$$f_{\mathbf{v}}(g) = \ell_{\underline{k}}(\mathbf{v} \otimes f(g)), \quad g \in G(\mathbb{A}).$$

Given $f' \in \mathcal{A}_{\underline{k}^\vee}^G(p^\ell \mathfrak{N})$ and $\mathbf{u} \in \mathcal{L}_{\underline{k}}(\mathbb{Q})$, we define $\mathbf{u}f' \in \mathcal{A}(G)$ by

$$\mathbf{u}f'(g) = \ell_{\underline{k}}(f'(g) \otimes \mathbf{u}).$$

A $G(\mathbb{R})$ -equivariant map $\mathcal{L}_{\underline{k}^\vee}(\mathbb{C}) \rightarrow \mathcal{A}(G)$ is given by $\mathbf{v} \mapsto f_{\mathbf{v}}$. Let \mathcal{T} be the diagonal torus of G . If $\mathbf{v}_{\underline{k}} \in \mathcal{L}_{\underline{k}^\vee}(\mathbb{C})$ is a highest weight vector, then

$$f_{\mathbf{v}_{\underline{k}}}(gt_\infty) = t_\infty^{-\underline{k}} \cdot f_{\mathbf{v}_{\underline{k}}}(g)$$

for $g \in G(\widehat{\mathbb{Q}})$ and $t_\infty \in \mathcal{T}(\mathbb{R})$.

Let $d_{\mathcal{K}}x$ be the Haar measure on $G(\mathbb{A})$ that gives $G(\mathbb{R})\mathcal{K}$ volume 1. We define the Petersson pairing of $\varphi, \varphi' \in \mathcal{A}(G)$ by

$$(\varphi, \varphi')_{\mathcal{K}} = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(x) \varphi'(x) d_{\mathcal{K}}x.$$

For any function \mathcal{F} on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{R})\mathcal{K}_0(p^\ell \mathfrak{N})$ we have

$$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \mathcal{F}(x) d_{\mathcal{K}}x = \frac{1}{[\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N})]} \sum_{[x] \in G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \mathcal{K}_0(p^\ell \mathfrak{N})} \frac{\mathcal{F}(x)}{\#\Gamma_{p^\ell \mathfrak{N}, x}},$$

where $[x]$ means the double coset $G(\mathbb{Q})x\mathcal{K}_0(p^\ell \mathfrak{N})$ and

$$\Gamma_{p^\ell \mathfrak{N}, x} = G(\mathbb{Q}) \cap x\mathcal{K}_0(p^\ell \mathfrak{N})x^{-1}.$$

We define a perfect pairing

$$(\cdot, \cdot)_{p^\ell \mathfrak{N}} : \mathcal{A}_{\underline{k}^\vee}^G(p^\ell \mathfrak{N}, \chi^{-1}) \times \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi) \rightarrow \mathbb{C}$$

by

$$(f', f)_{p^\ell \mathfrak{N}} = \sum_{[x] \in G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \mathcal{K}_0(p^\ell \mathfrak{N})} \frac{\ell_{\underline{k}}(f'(x) \otimes f(x))}{\#\Gamma_{p^\ell \mathfrak{N}, x}}.$$

The Schur orthogonality relation gives

$$(2.1) \quad (\mathbf{u}f', f_{\mathbf{v}})_{\mathcal{K}} = \frac{\ell_{\underline{k}}(\mathbf{v} \otimes \mathbf{u})}{[\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N})] \dim \mathcal{L}_{\underline{k}}} (f', f)_{p^\ell \mathfrak{N}}.$$

Definition 2.3. Let π be an irreducible automorphic representation of $G(\mathbb{A})$ generated by $f_{\mathbf{v}_{\underline{k}}}$ with a common Hecke eigenform $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N})$ of Hecke operators away from Np . Let $(\lambda_1, \lambda_2, \dots, \lambda_n)$ be the Harish-Chandra parameter of π_∞ . Note that $\lambda_i = -k_i + \frac{n+1}{2} - i$. Assume that π_p is an irreducible principal series $V = I(\mu_1, \mu_2, \dots, \mu_n)$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ with locally algebraic characters $\iota_p \circ \mu_i : \mathbb{Q}_p^\times \rightarrow \mathbb{C}_p^\times$. Put $\alpha_i = \iota_p(\mu_i(p))$. We order μ_i so that $\mathrm{ord}_p \alpha_1 \geq \mathrm{ord}_p \alpha_2 \geq \dots \geq \mathrm{ord}_p \alpha_n$. We say that π is p -ordinary with respect to ι_p if $\mathrm{ord}_p \alpha_i = -\lambda_{n-i+1}$ for $i = 1, 2, \dots, n$ (cf. Conjecture 4.1 of [CPR89]).

Definition 2.4. The operator U_p on $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi)$ is defined by

$$U_p f = \sum_{\substack{u=(u_{i,j}) \in N_n(\mathbb{Q}_p) \\ u_{i,j} \in \mathbb{Z}_p/p^{j-i}\mathbb{Z}_p \text{ for } i < j}} r(\iota_{\mathbf{p}}^{-1}(uD_{n,p}))f,$$

where

$$D_{n,p} = \begin{bmatrix} p^{n-1} & 0 & \dots & 0 & 0 \\ 0 & p^{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Remark 2.5. Put

$$\alpha_{\pi_p} = \prod_{i=1}^n \left(\alpha_i p^{i - \frac{n+1}{2}} \right)^{i-1} = p^{\frac{n(n^2-1)}{12}} \prod_{i=1}^n \alpha_i^{i-1}.$$

The operator U_p acts on the subspace $V^{\mathcal{N}_n} = \{h \in V \mid \pi_p(u)h = h \text{ for } u \in \mathcal{N}_n\}$ by the same formula, where we put $\mathcal{N}_n = N_n(\mathbb{Z}_p)$. If π is p -ordinary with respect to ι_p , then $\mathrm{ord}_p \alpha_i \neq \mathrm{ord}_p \alpha_j$ for $i \neq j$, and so by [Hid04, Theorem 5.3] and Proposition 5.4 below, the subspace

$$V^{\mathrm{ord}} = \{h \in V^{\mathcal{N}_n} \mid U_p h = \alpha_{\pi_p} h\}$$

is spanned by the vector $h_{\pi_p}^{\mathrm{ord}}$ defined in §5.3 (cf. [Hsi11, Remark 6.3], [Ger19, Lemma 5.4]).

Define an automorphism of $\mathrm{GL}_n(A)$ by $g^\vartheta = {}^t g^{-1}$. Let $\vartheta : \mathcal{L}_{\underline{k}}(A) \simeq \mathcal{L}_{\underline{k}^\vee}(A)$ be the isomorphism such that for $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{L}_{\underline{k}}(\mathbb{Q})$

$$(2.2) \quad (\rho_{\underline{k}}(g^\vartheta)\mathbf{u}_1)^\vartheta = \rho_{\underline{k}^\vee}(g)\mathbf{u}_1^\vartheta, \quad \ell_{\underline{k}}(\mathbf{u}_1^\vartheta \otimes \mathbf{u}_2) = \ell_{\underline{k}}(\mathbf{u}_2^\vartheta \otimes \mathbf{u}_1),$$

Remark 2.6. Let $\mathbf{u}_{\underline{k}}$ be the lowest weight vector in $\mathcal{L}_{\underline{k}}(\mathbb{Q})$ that satisfies $\ell_{\underline{k}}(\mathbf{v}_{\underline{k}} \otimes \mathbf{u}_{\underline{k}}) = 1$. Define

$$\varepsilon_{\underline{k}} \in \{\pm 1\} \text{ so that } \mathbf{u}_{\underline{k}}^\vartheta = \varepsilon_{\underline{k}} \mathbf{v}_{\underline{k}}.$$

In our future application, we will have

$$\varepsilon_{\underline{k}} = (-1)^{k_1 - k_2} \text{ if } n = 2; \quad \varepsilon_{\underline{k}} = 1 \text{ if } n = 3$$

(see (3.1), (3.2), §A.2 and §A.4).

Take $\xi_{\mathfrak{N}} = (\xi_{\mathfrak{N},v}) \in G(\widehat{\mathbb{Q}})$ so that $(\xi_{\mathfrak{N}}^{-1} \mathcal{K}_0(\mathfrak{N}) \xi_{\mathfrak{N}})^c = \mathcal{K}_0(\mathfrak{N})$. Define $\xi_{p^\ell \mathfrak{N}} = (\xi_{p^\ell \mathfrak{N},l}) \in G(\widehat{\mathbb{Q}})$ by

$$(2.3) \quad \xi_{p^\ell \mathfrak{N},p} = \iota_p^{-1}(D_{n,p}^{-\ell} T); \quad \xi_{p^\ell \mathfrak{N},l} = \xi_{\mathfrak{N},l} \text{ for } l \neq p.$$

Observe that

$$(2.4) \quad (\xi_{p^\ell \mathfrak{N}}^{-1} \mathcal{K}_0(p^\ell \mathfrak{N}) \xi_{p^\ell \mathfrak{N}})^c = \mathcal{K}_0(p^\ell \mathfrak{N}).$$

Given $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi)$, we define $\mathcal{L}_{\underline{k}^\vee}(\mathbb{C})$ -valued function f^c on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ by $f^c(g) = (f(g^c))^\vartheta$. In view of (2.4), one can obtain $\tau_{p^\ell \mathfrak{N}} f \in \mathcal{A}_{\underline{k}^\vee}^G(p^\ell \mathfrak{N}, \chi^{-1})$ by

$$\tau_{p^\ell \mathfrak{N}} f := r(\xi_{p^\ell \mathfrak{N}}) f^c.$$

It follows from (2.2) that

$$(2.5) \quad \begin{aligned} \mathbf{u}_{\underline{k}}(\tau_{p^\ell \mathfrak{N}} f)(g) &= \ell_{\underline{k}}((r(\xi_{p^\ell \mathfrak{N}}) f(g^c))^\vartheta \otimes \mathbf{u}_{\underline{k}}) \\ &= \ell_{\underline{k}}(\mathbf{u}_{\underline{k}}^\vartheta \otimes r(\xi_{p^\ell \mathfrak{N}}) f(g^c)) = \varepsilon_{\underline{k}} f_{\mathbf{v}_{\underline{k}}}((g \xi_{p^\ell \mathfrak{N}})^c). \end{aligned}$$

2.3. p -ADIC MODULAR FORMS ON $U(n)$. Having fixed the isomorphism $\iota_p : G(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{GL}_n(\mathbb{Q}_p)$ with $\iota_p(\mathcal{K}_p) = \mathrm{GL}_n(\mathbb{Z}_p)$, we define a representation $\rho_{\underline{k},p}$ of $G(\mathbb{Q}_p)$ on $\mathcal{L}_{\underline{k}}(\mathbb{C}_p)$ by

$$\iota_p^T(g) = \sqrt{T} \iota_p(g) \sqrt{T}^{-1}, \quad \rho_{\underline{k},p}(g) = \rho_{\underline{k}}(\iota_p^T(g)).$$

We assume that $\rho_{\underline{k},p}(\mathcal{K}_p)$ acts on $\mathcal{L}_{\underline{k}}(\mathbb{Z}_p)$. By definition ι_p^T is compatible with ι_∞^T in the sense that

$$\iota_p^T(g^c) = \iota_p^T(g)^\vartheta, \quad \iota_p^T(\gamma) = \iota_\infty^T(\gamma)$$

for $g \in G(\mathbb{Q}_p)$ and $\gamma \in G(\mathbb{Q})$.

Definition 2.7. Let A be a flat \mathbb{Z}_p -algebra. Let χ be an A^\times -valued character of $\mathcal{K}_0(p^\ell \mathfrak{N})/\mathcal{K}_1(p^\ell \mathfrak{N})$. The space $\mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, A)$ of p -adic modular forms on G of weight \underline{k} , level $p^\ell \mathfrak{N}$ and nebentypus χ over A consists of vector-valued functions $\widehat{f} : G(\widehat{\mathbb{Q}}) \rightarrow \mathcal{L}_{\underline{k}}(A)$ such that for $\gamma \in G(\mathbb{Q})$, $g \in G(\widehat{\mathbb{Q}})$ and $u \in \mathcal{K}_0(p^\ell \mathfrak{N})$

$$\widehat{f}(\gamma g u) = \chi(u)^{-1} \rho_{\underline{k},p}(u_p)^{-1} \widehat{f}(g),$$

where we denote the p -component of u by u_p .

We proceed to define the ordinary projector on the space of p -adic modular forms. To begin with, given $g \in G(\widehat{\mathbb{Q}})$, we define a twisted action $r_{\underline{k}}(g)$ on the space of functions $\mathcal{F} : G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) \rightarrow \mathcal{L}_{\underline{k}}(A)$ by

$$(2.6) \quad [r_{\underline{k}}(g) \mathcal{F}](h) = \rho_{\underline{k},p}(g_p) \mathcal{F}(hg).$$

Define the Hecke operator U_p on $\widehat{f} \in \mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, A)$ by

$$\begin{aligned} [U_p \widehat{f}](g) &:= \sum_{\substack{u=(u_{i,j}) \in N_n \\ u_{i,j} \in \mathbb{Z}_p/p^{j-i}\mathbb{Z}_p \text{ for } i < j}} r_{\underline{k}}(\iota_p^{-1}(uD_{n,p})) \widehat{f}(g) \\ &= \sum_u \rho_{\underline{k},p}(\iota_p^{-1}(uD_{n,p})) \widehat{f}(g \iota_p^{-1}(uD_{n,p})). \end{aligned}$$

We introduce the normalization of the Hecke operator U_p defined by

$$(2.7) \quad \mathcal{U}_p = D_{n,p}^{-\underline{k}} \cdot U_p = p^{-(n-1)k_1 - (n-2)k_2 - \cdots - k_{n-1}} \cdot U_p.$$

Note that since \underline{k} is the lowest weight, the normalized U_p -operator preserves the A -module $\mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, A)$. If A is p -adically complete, then the ordinary projector $e_{\text{ord}} = \lim_{m \rightarrow \infty} \widehat{U}_p^{m!}$ converges to an idempotent in $\text{End}_A \mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, A)$.

2.4. p -adic avatar of classical modular forms. We associate a function $\widehat{f} : G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) \rightarrow \mathcal{L}_{\underline{k}}(\mathbb{C}_p)$ defined by

$$\widehat{f}(g) = \rho_{\underline{k}, p}(g_p)^{-1} \iota_p(f(g))$$

to a function $f : G(\mathbb{Q}) \backslash G(\mathbb{A}) \rightarrow \mathcal{L}_{\underline{k}}(\mathbb{C})$. Notice that

$$(2.8) \quad \widehat{r(g)f} = r_{\underline{k}}(g)\widehat{f}, \quad f(gu_\infty) = \rho_{\underline{k}, \infty}(u_\infty)^{-1} \iota_p^{-1}(\rho_{\underline{k}, p}(g_p)\widehat{f}(g))$$

for $g \in G(\widehat{\mathbb{Q}})$ and $u_\infty \in G(\mathbb{R})$.

If $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi)$, then $\widehat{f} \in \mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, \mathbb{C}_p)$, and \widehat{f} is called the p -adic avatar of f . On the other hand, we will call f the adèlic lift of \widehat{f} . We define a perfect pairing

$$(\cdot, \cdot)_{p^\ell \mathfrak{N}} : \mathcal{S}_{\underline{k}^\vee}^G(p^\ell \mathfrak{N}, \chi^{-1}, \mathbb{C}_p) \times \mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, \mathbb{C}_p) \rightarrow \mathbb{C}_p$$

by

$$(\widehat{f}', \widehat{f})_{p^\ell \mathfrak{N}} = \sum_{[x] \in G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \mathcal{K}_0(p^\ell \mathfrak{N})} \frac{\ell_{\underline{k}}(\widehat{f}'(x) \otimes \widehat{f}(x))}{\#\Gamma_{p^\ell \mathfrak{N}, x}}.$$

By definition,

$$(2.9) \quad (\widehat{f}', \widehat{f})_{p^\ell \mathfrak{N}} = \iota_p((f', f)_{p^\ell \mathfrak{N}}).$$

To an $\mathcal{L}_{\underline{k}}(\mathbb{C}_p)$ -valued function f on $G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}})$ we associate an $\mathcal{L}_{\underline{k}^\vee}(\mathbb{C}_p)$ -valued function \widehat{f}^c on $G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}})$ defined by $\widehat{f}^c(g) = (\widehat{f}(g^c))^\vartheta$. One easily sees that $\widehat{f}^c = \widehat{f}^c$. We associate to $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi)$ a p -adic modular form $\tau_{p^\ell \mathfrak{N}} \widehat{f}$ of weight \underline{k}^\vee and level $p^\ell \mathfrak{N}$ defined by

$$(2.10) \quad \tau_{p^\ell \mathfrak{N}} \widehat{f} := r_{\underline{k}^\vee}(\xi_{p^\ell \mathfrak{N}}) \widehat{f}^c = \widehat{\tau_{p^\ell \mathfrak{N}} f}.$$

Here $r_{\underline{k}^\vee}$ is the twisted action in (2.6).

Given $\widehat{f} \in \mathcal{S}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi, \mathbb{C}_p)$ and $\mathbf{v} \in \mathcal{L}_{\underline{k}^\vee}(\mathbb{Q})$, we define a scalar valued function $\widehat{f}_{\mathbf{v}}$ on $G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}})$ by

$$\widehat{f}_{\mathbf{v}}(g) = \ell_{\underline{k}}(\mathbf{v} \otimes \widehat{f}(g)), \quad g \in G(\widehat{\mathbb{Q}}).$$

If $\mathbf{v}_{\underline{k}} \in \mathcal{L}_{\underline{k}^\vee}(\mathbb{Q})$ is a highest weight vector invariant by $\iota_p^{-1}(N_n(\mathbb{Z}_p))$, then $\widehat{f}_{\mathbf{v}_{\underline{k}}}$ is a function on $G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \iota_p^{-1}(N_n(\mathbb{Z}_p))$, and

$$\widehat{f}_{\mathbf{v}_{\underline{k}}}(gt_p) = \chi(t_p)^{-1} \iota_p(t_p)^{-\underline{k}} \widehat{f}_{\mathbf{v}_{\underline{k}}}(g)$$

for $g \in G(\widehat{\mathbb{Q}})$ and $\iota_p(t_p) \in T_n(\mathbb{Z}_p^\times)$.

Definition 2.8. The Hecke operator U_p on the space of functions on $G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \iota_p^{-1}(N_n(\mathbb{Z}_p))$ is defined by

$$[U_p \mathcal{F}](g) = \sum_{\substack{u=(u_{i,j}) \in N_n(\mathbb{Q}_p) \\ u_{i,j} \in \mathbb{Z}_p / p^{j-i} \mathbb{Z}_p \text{ for } i < j}} \mathcal{F}(gt_p^{-1}(uD_{n,p})).$$

Remark 2.9. Observe that

$$(2.11) \quad \widehat{U_p f} = U_p \widehat{f}, \quad [\mathcal{U}_p f]_{\mathbf{v}_{\underline{k}}} = U_p \widehat{f}_{\mathbf{v}_{\underline{k}}}.$$

We verify the second relation. By definition

$$\begin{aligned} [U_p \widehat{f}_{\mathbf{v}_{\underline{k}}}] (g) &= \sum_u \ell_{\underline{k}}(\rho_{\underline{k}^\vee, p}(g_p \iota_{\mathbf{p}}^{-1}(uD_{n,p}))_{\mathbf{v}_{\underline{k}}} \otimes \iota_p(f(g_p \iota_{\mathbf{p}}^{-1}(uD_{n,p}))) \\ &= p^{-(n-1)k_1 - (n-2)k_2 - \dots - k_{n-1}} \ell_{\underline{k}}(\rho_{\underline{k}^\vee, p}(g_p)_{\mathbf{v}_{\underline{k}}} \otimes \iota_p([U_p f](g))). \end{aligned}$$

If π is a p -ordinary irreducible automorphic representation of $G(\mathbb{A})$ with respect to ι_p , then Proposition 5.4 below gives an eigenform $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N})$ of U_p attached to π such that $\widehat{f}_{\mathbf{v}_{\underline{k}}}$ is an eigenform of \mathcal{U}_p with the p -adic unit eigenvalue

$$\alpha_f = p^{-(n-1)k_1 - (n-2)k_2 - \dots - k_{n-1}} \alpha_{\pi_p}.$$

2.5. Review of Hida theory for $U(n)$. We define $X_1^G(p^\ell \mathfrak{N})$ as the finite set

$$X_1^G(p^\ell \mathfrak{N}) = G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \mathcal{K}_1(p^\ell \mathfrak{N})$$

for each positive integer ℓ . Recall that \mathcal{O} is the ring of the integers of a finite extension F of \mathbb{Q}_p . Let $\mathcal{O}[X_1^G(p^\ell \mathfrak{N})] = \bigoplus_{x \in X_1^G(p^\ell \mathfrak{N})} \mathcal{O}x$ be the finitely generated \mathcal{O} -module spanned by divisors of $X_1^G(p^\ell \mathfrak{N})$. Put

$$X_1^G(p^\infty \mathfrak{N}) := \varprojlim_{\ell} X_1^G(p^\ell \mathfrak{N}).$$

We retain the notation from the introduction. Write $z \mapsto [z]_{\Lambda_n}$ for the inclusion of group-like elements $T_n(\mathbb{Z}_p) \rightarrow \mathcal{O}[[T_n(\mathbb{Z}_p)]]^\times = \Lambda_n^\times$. Let \mathcal{R} be a normal ring finite flat over $\Lambda_n = \mathcal{O}[[T_n(\mathbb{Z}_p)]]$. For $\underline{Q} = (Q_1, Q_2, \dots, Q_n) \in \mathfrak{X}_{\mathcal{R}}$ we put

$$k_{\underline{Q}} = (k_{Q_1}, k_{Q_2}, \dots, k_{Q_n}), \quad \ell = \max\{1, c(\epsilon_{Q_1}), c(\epsilon_{Q_2}), \dots, c(\epsilon_{Q_n})\}$$

and define finite order characters of $\mathcal{K}_0(p^\ell \mathfrak{N})$ by

$$\epsilon_{\underline{Q}}(t_p) = \epsilon_{Q_1}(t_{p,1}) \epsilon_{Q_2}(t_{p,2}) \cdots \epsilon_{Q_n}(t_{p,n}),$$

where $\iota_{\mathbf{p}}(t_p) = \text{diag}(t_{p,1}, t_{p,2}, \dots, t_{p,n})$ with $t_{p,i} \in \mathbb{Z}_p^\times$ for $i = 1, 2, \dots, n$. Let $\wp_{\underline{Q}}$ be the ideal of \mathcal{R} corresponding to \underline{Q} and $\mathcal{R}(\underline{Q})$ the image of \mathcal{R} under \underline{Q} . Let $P_\ell^{(n)}$ be the ideal of Λ_n generated by $[t]_{\Lambda_n}^{p^\ell} - 1$ for $t \in T_n(\mathbb{Z}_p)$. Let Λ_n act on $\mathcal{O}[X_1^G(p^\ell \mathfrak{N})]$ by

$$[t]_{\Lambda_n} x := x \cdot \iota_{\mathbf{p}}^{-1}(t), \quad t \in T_n(\mathbb{Z}_p).$$

Definition 2.10. Put $\Delta = (\mathfrak{r}/\mathfrak{N})^\times$. For $d \in \Delta$ the diamond operator σ_d acts on the module $\mathcal{O}[X_1^G(p^\ell \mathfrak{N})]$ by

$$\sigma_d x := x \prod_{l|\mathfrak{N}} \iota_l^{-1} \left(\begin{bmatrix} \mathbf{1}_{n-1} & \\ & \tilde{d}_l \end{bmatrix} \right),$$

where $\tilde{d} = (\tilde{d}_l) \in \prod_{l|N} \mathbb{Z}_l^\times$ is a lift of d .

Thus $\mathcal{O}[X_1^G(p^\ell \mathfrak{N})]$ is a finitely generated $\Lambda_n[\Delta]$ -module.

Definition 2.11. The module $\mathcal{O}[X_1^G(p^\ell \mathfrak{N})]$ is equipped with the operator U_p defined by

$$U_p x = \sum_{u_{i,j} \in \mathbb{Z}_p/p^{j-i}\mathbb{Z}_p} x \cdot \begin{bmatrix} p^{n-1} & p^{n-2}u_{1,2} & \dots & p^2u_{1,n-2} & pu_{1,n-1} & u_{1,n} \\ 0 & p^{n-2} & \dots & p^2u_{2,n-2} & pu_{2,n-1} & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & p^2 & pu_{n-2,n-1} & u_{n-2,n} \\ 0 & 0 & \dots & 0 & p & u_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix},$$

where we define the action of $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ on $x \in X_1^G(p^\ell \mathfrak{N})$ by $x \cdot g = x \iota_p^{-1}(g)$. The limit

$$e_{\mathrm{ord}} = \lim_{m \rightarrow \infty} U_p^{m!}$$

converges to an idempotent in $\mathrm{End}_{\Lambda_n} \mathcal{O}[X_1^G(p^\ell \mathfrak{N})]$.

Definition 2.12. A Λ_n -adic modular form on G of tame level \mathfrak{N} is a *continuous* function $\mathbf{f} : X_1^G(p^\infty \mathfrak{N}) \rightarrow \Lambda_n$ which satisfies

$$\mathbf{f}(x \cdot \iota_p^{-1}(t)) = \mathbf{f}(x)[t]_{\Lambda_n}^{-1}, \quad t \in T_n(\mathbb{Z}_p).$$

Let $\mathbf{S}^G(\mathfrak{N}, \Lambda_n)$ be the space of Λ_n -adic modular forms on G of tame level \mathfrak{N} .

Recall that \mathbf{f} is continuous if for any $m_1, m_2 > 0$, there exists sufficiently large ℓ such that the function $\mathbf{f} \pmod{(P_{m_1}^{(n)}, p^{m_2})} : X_1^G(p^\infty \mathfrak{N}) \rightarrow \Lambda_n/(P_{m_1}^{(n)}, p^{m_2})$ factors through $X_1^G(p^\ell \mathfrak{N})$.

The Λ_n -module $\mathbf{S}^G(\mathfrak{N}, \Lambda_n)$ is equipped with the natural actions of Hecke and diamond operators given by

$$U_p \mathbf{f}(x) = \mathbf{f}(U_p x), \quad \sigma_d \mathbf{f}(x) = \mathbf{f}(\sigma_d x).$$

The ordinary projector $e_{\mathrm{ord}} = \lim_{m \rightarrow \infty} U_p^{m!}$ converges in $\mathrm{End}_{\Lambda_n} \mathbf{S}^G(\mathfrak{N}, \Lambda_n)$. For a character $\chi : \Delta \rightarrow \mathcal{O}^\times$ we put

$$\mathbf{S}^G(\mathfrak{N}, \chi, \Lambda_n) = \{\mathbf{f} \in \mathbf{S}^G(\mathfrak{N}, \Lambda_n) \mid \sigma_d \mathbf{f} = \chi(d)^{-1} \mathbf{f} \text{ for } d \in \Delta\}.$$

For a normal ring \mathcal{R} finite flat over Λ_n we set

$$\begin{aligned} \mathbf{S}^G(\mathfrak{N}, \mathcal{R}) &= \mathbf{S}^G(\mathfrak{N}, \Lambda_n) \otimes_{\Lambda_n} \mathcal{R}, \\ \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R}) &= \mathbf{S}^G(\mathfrak{N}, \chi, \Lambda_n) \otimes_{\Lambda_n} \mathcal{R}. \end{aligned}$$

Theorem 2.13. Put $N_\chi = \sum_{d \in \Delta} \chi(d) \sigma_d \in \mathcal{O}[\Delta]$. Let P_χ be the ideal of $\mathcal{R}[\Delta]$ generated by $\{\chi(d) \sigma_d - 1\}_{d \in \Delta}$. Suppose that $p > 3$.

- (1) $e_{\mathrm{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$ is a free \mathcal{R} -module, and the norm map

$$N_\chi : e_{\mathrm{ord}} \mathbf{S}^G(\mathfrak{N}, \mathcal{R}) / P_\chi \simeq e_{\mathrm{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$$

is an isomorphism.

- (2) For every $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+$ we have a Hecke equivariant isomorphism

$$\begin{aligned} e_{\mathrm{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R} / \wp_{\underline{Q}} &\simeq e_{\mathrm{ord}} \mathbf{S}_{k_{\underline{Q}}}^G(p^\ell \mathfrak{N}, \chi \epsilon_{\underline{Q}}, \mathcal{R}(\underline{Q})) \\ \mathbf{f} \pmod{\wp_{\underline{Q}}} &\mapsto \mathbf{f}_{\underline{Q}}, \end{aligned}$$

where $\mathbf{f}_{\underline{Q}}$ is the unique p -adic modular form such that

$$\underline{Q}(\mathbf{f}(x)) = (\rho_{k_{\underline{Q}}}(\sqrt{T})^{-1} \mathbf{f}_{\underline{Q}})_{\mathbf{v}_{k_{\underline{Q}}}}(x).$$

Remark 2.14. Put $\hat{f}' = \rho_{k_{\underline{Q}}}(\sqrt{T})^{-1} \underline{f}'_{\underline{Q}}$. Then

$$\hat{f}'(\gamma gu) = (\chi_{\epsilon_{\underline{Q}}}(u))^{-1} \rho_{k_{\underline{Q}}}(\iota_{\mathfrak{p}}(u_{\mathfrak{p}}))^{-1} \hat{f}'(g), \quad \hat{f}'_{\mathfrak{v}_{k_{\underline{Q}}}} = \sqrt{T}^{-k_{\underline{Q}}}(\underline{f}'_{\underline{Q}})_{\mathfrak{v}_{k_{\underline{Q}}}}$$

for $\gamma \in G(\mathbb{Q})$, $g \in G(\widehat{\mathbb{Q}})$ and $u \in \mathcal{K}_0(p^\ell \mathfrak{N})$.

Proof. This is essentially proved in [Ger19] by adapting the arguments in [Hid88] in the case of $GL(2)$ to the case of unitary groups (cf. [Hsi21, Theorem 4.2]). For any abelian group A , let $\mathcal{S}_0(\alpha, A)$ be the space of A -valued functions on the finite set $X_1^G(p^\alpha \mathfrak{N})$. Let $\mathcal{V}^{\text{ord}}(\mathfrak{N}) := \varprojlim_{\beta} \varinjlim_{\alpha} e_{\text{ord}} \mathcal{S}_0(\alpha, p^{-\beta} \mathcal{O}/\mathcal{O})$ be the discrete Λ_n -module. Note that $\mathcal{V}^{\text{ord}}(\mathfrak{N})$ is nothing but the space $\mathcal{S}_{0, \{\chi_v\}}^{\text{ord}}(U(I^\infty), K/\mathcal{O})$ in [Ger19, p. 1358]. Let

$$V^{\text{ord}}(\mathfrak{N}) = \varprojlim_{\alpha} e_{\text{ord}} \mathcal{O}[X_1^G(p^\alpha \mathfrak{N})]$$

be the Pontryagin dual of $\mathcal{V}^{\text{ord}}(\mathfrak{N})$. Then we have

$$\mathbf{S}^G(\mathfrak{N}, \Lambda_n) = \text{Hom}_{\Lambda_n}(V^{\text{ord}}(\mathfrak{N}), \Lambda_n).$$

Therefore part (1) follows from [Ger19, Lemma 2.6, Proposition 2.20], and part (2) is proved in [Ger19, Proposition 2.22, Lemma 2.25]. \square

Definition 2.15 (Hida families). A non-zero \mathcal{R} -adic modular form $\mathbf{f} \in e_{\text{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$ is an \mathcal{R} -adic Hida family if $\underline{f}_{\underline{Q}}$ is a simultaneous eigenform of Hecke operators away from Np for $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+$. Let $\mathfrak{X}'_{\mathbf{f}} = \{\underline{Q} \in \mathfrak{X}_{\mathbf{f}}^{\text{temp}} \mid \underline{Q}(\mathbf{f}) \neq 0\}$ be a Zariski dense subset of the rigid generic fiber of $\text{Spf} \mathcal{R}$.

Lemma 2.16. *Assume that the space of automorphic forms on G has multiplicity one. If $\mathbf{f} \in e_{\text{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$ is a \mathcal{R} -adic Hida family, then $\underline{Q}(\mathbf{f})$ generates an irreducible p -ordinary automorphic representation of $G(\mathbb{A})$ for $\underline{Q} \in \mathfrak{X}'_{\mathbf{f}}$.*

Proof. Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible constituent of the automorphic representation generated by $\underline{Q}(\mathbf{f})$. If l and Np are coprime, then π_l is determined by the Hecke eigenvalues of $\underline{Q}(\mathbf{f})$. Thus π belongs to the A -packet associated to these eigenvalues. Moreover, π_q is uniquely determined for each prime factor q of N by the assumption on N as the associated local A -packet is a singleton for each split prime. Therefore the equivalence class π is determined by $\underline{Q}(\mathbf{f})$. Thus $\underline{Q}(\mathbf{f})$ generates an irreducible representation of $G(\mathbb{A})$ by the multiplicity one for G . \square

If \mathbf{f} is a Hida family, then $\underline{Q}(\mathbf{f})$ is an eigenform of the operator \mathcal{U}_p with unit eigenvalue $\alpha_{\underline{f}_{\underline{Q}}}$ by Remarks 2.5 and 2.9 for $\underline{Q} \in \mathfrak{X}'_{\mathbf{f}}$, and we denote by $\pi_{\underline{Q}}$ the automorphic representation of $G(\mathbb{A})$ associated to $\underline{f}_{\underline{Q}}$, which is p -ordinary with respect to ι_p .

2.6. A pairing on the space of ordinary \mathcal{R} -adic modular forms. In this subsection, we do the p -adic interpolation of the bilinear pairing. We first introduce the regularized diagonal cycles for $U(n) \times U(n)$.

Define the finite sets

$$\mathcal{X}_\ell = X_0^G(p^\ell \mathfrak{N}), \quad \mathbb{X}_\ell = \mathcal{X}_\ell \times \mathcal{X}_\ell$$

for each positive integer ℓ . Given $x, y \in G(\widehat{\mathbb{Q}})$, we write $[(x, y)] \in \mathbb{X}_\ell$ for the double coset represented by (x, y) . The following definition makes sense in view of (2.4).

Definition 2.17. Let $\diamond_\ell \in \mathbb{Z}_p[\mathbb{X}_\ell]$ be the twisted diagonal cycle defined by

$$\diamond_\ell = \sum_{[x] \in \mathcal{X}_\ell} [(x, (x\xi_{p^\ell \mathfrak{N}})^c)].$$

The element $\xi_{p^\ell \mathfrak{N}}$ is defined in (2.3). Note that $(x\xi_{p^\ell \mathfrak{N}})_p^c = \iota_{\mathfrak{p}}^{-1}(T^{-1} \iota_{\mathfrak{p}}(x_p D_{n,p}^{-\ell})^{-1})$.

The homomorphism

$$N_{\ell+1,\ell} : \mathbb{Z}_p[\mathbb{X}_{\ell+1}] \rightarrow \mathbb{Z}_p[\mathbb{X}_\ell]$$

is induced by the projection $\mathbb{X}_{\ell+1} \rightarrow \mathbb{X}_\ell$.

Lemma 2.18. For $\ell \geq 1$ we have

$$N_{\ell+1,\ell}(\diamond_{\ell+1}) = (1 \otimes U_p)\diamond_\ell.$$

Proof. Fix a complete set S_ℓ of representatives for $\mathbb{Z}_p/p^\ell \mathbb{Z}_p$. Denote the unipotent radical of the Borel subgroup opposite to B_n by N_n^- . Put

$$\iota_{\mathfrak{p}}(\Sigma_\ell) := \{(p^{(i-j)\ell} v_{ij}) \in N_n^-(\mathbb{Q}_p) \mid v_{ij} \in S_{i-j} \text{ for } i > j\}.$$

Then Σ_ℓ is a complete set of representatives for $\mathcal{K}_0(p^\ell \mathfrak{N})/\mathcal{K}_0(p^{\ell+1} \mathfrak{N})$. We may assume that $\Gamma_{p^\ell \mathfrak{N}, x} = \mathfrak{r}^\times$ for every $x \in G(\widehat{\mathbb{Q}})$ (see the proof of Lemma 4.4 of [Hsi21]). Then $\mathcal{X}_{\ell+1}$ consists of elements represented by xk with $x \in \mathcal{X}_\ell$ and $k \in \Sigma_\ell$. Since

$$(2.12) \quad \begin{aligned} \iota_{\mathfrak{p}}((\xi_{p^\ell \mathfrak{N}, p}^{-1} \Sigma_\ell \xi_{p^\ell \mathfrak{N}, p})^c) &= {}^t(D_{n,p}^\ell \iota_{\mathfrak{p}}(\Sigma_\ell) D_{n,p}^{-\ell})^{-1} \\ &= \{u \in N_n(\mathbb{Q}_p) \mid u_{ij} \in S_{j-i} \text{ for } i < j\}, \end{aligned}$$

we get the distribution property stated above. \square

Define the regularized diagonal cycle by $\diamond_\ell^\dagger = (1 \otimes U_p^{-\ell})e_{\text{ord}}\diamond_\ell$. Lemma 2.18 says that $N_{\ell+1,\ell}(\diamond_{\ell+1}^\dagger) = \diamond_\ell^\dagger$. We can therefore define

$$\diamond_\infty^\dagger = \lim_{\leftarrow \ell} \diamond_\ell^\dagger \in \lim_{\leftarrow \ell} \mathbb{Z}_p[\mathbb{X}_\ell].$$

Definition 2.19. Let $\mathbf{f}, \mathbf{g} \in e_{\text{ord}}\mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$. We define an \mathcal{R} -adic modular form on $G \times G$ by

$$\mathcal{F} = \mathbf{f} \boxtimes \mathbf{g} : G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) \times G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) \rightarrow \mathcal{R}$$

by $\mathcal{F}(x, y) = \mathbf{f}(x)\mathbf{g}(y)$. Then \mathcal{F} naturally induces a Λ_n -linear map

$$\mathcal{F}_* : \lim_{\leftarrow \ell} \mathcal{O}[\mathbb{X}_\ell] \rightarrow \mathcal{R}.$$

Define an \mathcal{R} -bilinear pairing

$$\mathbf{B}_{\mathfrak{N}} : e_{\text{ord}}\mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R}) \times e_{\text{ord}}\mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R}) \rightarrow \mathcal{R}$$

by

$$\mathbf{B}_{\mathfrak{N}}(\mathbf{f}, \mathbf{g}) = \mathcal{F}_*(\diamond_\infty^\dagger) \in \mathcal{R}.$$

The following result generalizes [Hsi21, Lemma 4.4].

Proposition 2.20. For each $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+$ and sufficiently large ℓ we have

$$\underline{Q}(\mathbf{B}_{\mathfrak{N}}(\mathbf{f}, \mathbf{g})) = \varepsilon_{k_{\underline{Q}}} \cdot (\tau_{p^\ell \mathfrak{N}}[U_p^{-\ell} \underline{g}_{\underline{Q}}], \underline{f}_{\underline{Q}})_{p^\ell \mathfrak{N}}.$$

Here $\varepsilon_{k_{\underline{Q}}} \in \{\pm 1\}$ is the sign defined in Remark 2.6.

Proof. We shall adapt the the proof of [Hsi21, Lemma 4.4]. To lighten notation, we put

$$\begin{aligned} w_E &= \sharp \mathfrak{r}^\times, & \underline{k} &= k_Q, & \kappa &= (n-1)k_{Q_1} + (n-2)k_{Q_2} + \cdots + k_{Q_{n-1}}, \\ \mathcal{O} &= \mathcal{R}(\underline{Q}), & \widehat{f} &= \underline{f}_Q, & \widehat{g} &= \underline{g}_Q \in e_{\text{ord}} \mathcal{S}_{\underline{k}}^G(p^e \mathfrak{N}, \chi \in \underline{Q}, \mathcal{O}). \end{aligned}$$

We first claim that the value $(\tau_{p^l \mathfrak{N}}[U_p^{-l} \widehat{g}], \widehat{f})_{p^l \mathfrak{N}}$ is independent of any sufficiently large integer l . To see the claim, we note that if l is sufficiently large, then $\Gamma_{p^l \mathfrak{N}, x} = \mathfrak{r}^\times$ for every $x \in G(\widehat{\mathbb{Q}})$, and by (2.10)

$$\begin{aligned} & w_E (\tau_{p^{l+1} \mathfrak{N}}[\mathcal{U}_p^{-l-1} \widehat{g}], \widehat{f})_{p^{l+1} \mathfrak{N}} \\ &= \sum_{[x] \in \mathcal{X}_{l+1}} \ell_{\underline{k}}(\rho_{\underline{k}^\vee, p}(\xi_{p^{l+1} \mathfrak{N}, p})) ([\mathcal{U}_p^{-l-1} \widehat{g}]((x \xi_{p^{l+1} \mathfrak{N}})^c))^\vartheta \otimes \widehat{f}(x) \\ &= \sum_{[x] \in \mathcal{X}_l} \sum_{v \in \mathcal{S}_l} \ell_{\underline{k}}(\rho_{\underline{k}^\vee, p}(\xi_{p^{l+1} \mathfrak{N}, p})) ([\mathcal{U}_p^{-l-1} \widehat{g}]((xv \xi_{p^{l+1} \mathfrak{N}})^c))^\vartheta \otimes \rho_{\underline{k}, p}(v)^{-1} \widehat{f}(x) \\ &= \sum_{[x] \in \mathcal{X}_l} \sum_{v \in \mathcal{S}_l} \ell_{\underline{k}}((\rho_{\underline{k}, p}((v \xi_{p^l \mathfrak{N}, p})^c \iota_{\mathfrak{p}}^{-1}(D_{n,p}))) [\mathcal{U}_p^{-l-1} \widehat{g}]((xv \xi_{p^l \mathfrak{N}})^c \iota_{\mathfrak{p}}^{-1}(D_{n,p})))^\vartheta \otimes \widehat{f}(x). \end{aligned}$$

By (2.12), we see that the last inner sum is

$$\begin{aligned} & \sum_{\substack{u=(u_{ij}) \in N_n \\ u_{ij} \in S_{j-i} \text{ for } i < j}} \ell_{\underline{k}}((\rho_{\underline{k}, p}(\xi_{p^l \mathfrak{N}, p}^c \iota_{\mathfrak{p}}^{-1}(uD_{n,p}))) [\mathcal{U}_p^{-l-1} \widehat{g}]((x \xi_{p^l \mathfrak{N}})^c \iota_{\mathfrak{p}}^{-1}(uD_{n,p})))^\vartheta \otimes \widehat{f}(x) \\ &= \ell_{\underline{k}}((\rho_{\underline{k}, p}(\xi_{p^l \mathfrak{N}, p}^c) [U_p \mathcal{U}_p^{-l-1} \widehat{g}]((x \xi_{p^l \mathfrak{N}})^c))^\vartheta \otimes \widehat{f}(x)). \end{aligned}$$

Since $U_p \mathcal{U}_p^{-1} = p^\kappa$ by (2.7), we conclude that

$$(\tau_{p^{l+1} \mathfrak{N}}[\mathcal{U}_p^{-l-1} \widehat{g}], \widehat{f})_{p^{l+1} \mathfrak{N}} = p^\kappa (\tau_{p^l \mathfrak{N}}[\mathcal{U}_p^{-l} \widehat{g}], \widehat{f})_{p^l \mathfrak{N}}.$$

This proves the claim.

Recall that $Q(\mathbf{f}) = \sqrt{T}^{-k} \widehat{f}_{\mathbf{v}_{\underline{k}}}$ and $Q(\mathbf{g}) = \sqrt{T}^{-k} \widehat{g}_{\mathbf{v}_{\underline{k}}}$ by Theorem 2.13 (2). For an arbitrarily large integer m there exists a sufficiently larger integer $l \gg m$ such that

$$(2.13) \quad \underline{Q}(\mathbf{B}_{\mathfrak{N}}(\mathbf{f}, \mathbf{g})) \equiv T^{-k} \sum_{[x] \in \mathcal{X}_l} [U_p^{-l} \widehat{g}_{\mathbf{v}_{\underline{k}}}]((x \xi_{p^l \mathfrak{N}})^c) \widehat{f}_{\mathbf{v}_{\underline{k}}}(x) \pmod{p^m \mathcal{O}}.$$

Take a basis $\mathcal{B}_{\underline{k}^\vee} = \{\mathbf{v}_i\}$ of $\mathcal{L}_{\underline{k}^\vee}(\mathbb{Q})$ which consists of weight vectors and contains the highest weight vector $\mathbf{v}_{\underline{k}}$. Let $\mathcal{B}_{\underline{k}} = \{\mathbf{u}_i\}$ be a basis of $\mathcal{L}_{\underline{k}}(\mathbb{Q})$ dual to $\mathcal{B}_{\underline{k}^\vee}$ with $\mathbf{u}_{\underline{k}}$ in Remark 2.6.. Then we can write

$$\mathcal{U}_p^{-l} \widehat{g} = \sum_i [\mathcal{U}_p^{-l} \widehat{g}]_{\mathbf{v}_i} \cdot \mathbf{u}_i.$$

If l is sufficiently larger than m , then

$$\begin{aligned} \rho_{\underline{k}}(D_{n,p}^l) [U_p^{-l} \widehat{g}] &= \sum_i [\mathcal{U}_p^{-l} \widehat{g}]_{\mathbf{v}_i} \cdot p^{-\kappa l} \rho_{\underline{k}}(D_{n,p}^l) \mathbf{u}_i \\ &\equiv [\mathcal{U}_p^{-l} \widehat{g}]_{\mathbf{v}_{\underline{k}}} \cdot \mathbf{u}_{\underline{k}} \pmod{p^m \mathcal{L}_{\underline{k}}(\mathcal{O})}. \end{aligned}$$

It therefore follows that

$$(\tau_{p^l \mathfrak{N}}[U_p^{-l} \widehat{g}], \widehat{f})_{p^l \mathfrak{N}} = \sum_{[x] \in \mathcal{X}_l} \ell_{\underline{k}}((\rho_{\underline{k}, p}(\xi_{p^l \mathfrak{N}, p}^c) [U_p^{-l} \widehat{g}]((x \xi_{p^l \mathfrak{N}})^c))^\vartheta \otimes \widehat{f}(x))$$

$$\equiv T^{-k} \sum_{[x] \in \mathcal{X}_l} [U_p^{-l} \widehat{g}_{\mathbf{v}_k}]((x \xi_{p^l \mathfrak{N}})^c) \ell_{\underline{k}}(\mathbf{u}_{\underline{k}}^\vartheta \otimes \widehat{f}(x)) \pmod{p^m \mathcal{O}}.$$

In the last equality, we have used $\rho_{k,p}(\xi_{p^l \mathfrak{N}}^c) = \rho_{\underline{k}}(T^{-1} D_{n,p}^l)$ and (2.11). Combining (2.13) and Remark 2.6, we find that

$$(\tau_{p^l \mathfrak{N}}[U_p^{-l} \widehat{g}], \widehat{f})_{p^l \mathfrak{N}} = (\tau_{p^l \mathfrak{N}}[U_p^{-l} \widehat{g}], \widehat{f})_{p^l \mathfrak{N}} \equiv \varepsilon_{k_Q} Q(\mathbf{B}_{\mathfrak{N}}(\mathbf{f}, \mathbf{g})) \pmod{p^m \mathcal{O}}$$

for arbitrarily large m , from which the formula follows. \square

Definition 2.21. Let $\mathbf{f} \in e_{\text{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$ be an \mathcal{R} -adic Hida family. We define $\eta_{\mathbf{f}} \in \mathcal{R}$ by

$$\eta_{\mathbf{f}} = \mathbf{B}_{\mathfrak{N}}(\mathbf{f}, \mathbf{f}).$$

Proposition 2.22. *Suppose that \mathbf{f} is an eigenvector of the U_p -operator with the eigenvalue $\alpha_{\mathbf{f}} \in \mathcal{R}^\times$. Put*

$$\alpha_{\pi_Q} = p^{(n-1)k_{Q_1} + (n-2)k_{Q_2} + \cdots + k_{Q_{n-1}}} \underline{Q}(\alpha_{\mathbf{f}}).$$

For $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+$ and $\ell \gg 0$, we have

$$\underline{Q}(\eta_{\mathbf{f}}) = [\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N}_{\pi_Q})] \alpha_{\pi_Q}^{-\ell} \iota_p((f_{\mathbf{v}_{k_Q}}^\vee, f_{\mathbf{v}_{k_Q}})_{\mathcal{K}}) \dim \mathcal{L}_{k_Q}.$$

Here the modular form $f_{\mathbf{v}_{k_Q}}^\vee$ is defined by $f_{\mathbf{v}_{k_Q}}^\vee(g) = f_{\mathbf{v}_{k_Q}}((g \xi_{p^\ell \mathfrak{N}})^c)$.

Proof. If ℓ is sufficiently large, then Proposition 2.20 gives

$$\begin{aligned} \varepsilon_{k_Q} \underline{Q}(\eta_{\mathbf{f}}) &= (\tau_{p^\ell \mathfrak{N}}[U_p^{-\ell} \underline{\mathbf{f}}_Q], \underline{\mathbf{f}}_Q)_{p^\ell \mathfrak{N}} \\ &= \iota_p((\tau_{p^\ell \mathfrak{N}}[U_p^{-\ell} f], f)_{p^\ell \mathfrak{N}}) = \alpha_{\pi_Q}^{-\ell} \iota_p((\tau_{p^\ell \mathfrak{N}} f, f)_{p^\ell \mathfrak{N}}) \end{aligned}$$

by (2.9), (2.10), (2.11) and Remark 2.9. We can rewrite this identity as

$$\begin{aligned} \varepsilon_{k_Q} \underline{Q}(\eta_{\mathbf{f}}) &= [\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N})] \alpha_{\pi_Q}^{-\ell} \iota_p((\mathbf{u}_{k_Q}(\tau_{p^\ell \mathfrak{N}} f), f_{\mathbf{v}_{k_Q}})_{\mathcal{K}}) \dim \mathcal{L}_{k_Q} \\ &= [\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N})] \alpha_{\pi_Q}^{-\ell} \varepsilon_{k_Q} \iota_p((f_{\mathbf{v}_{k_Q}}^\vee, f_{\mathbf{v}_{k_Q}})_{\mathcal{K}}) \dim \mathcal{L}_{k_Q} \end{aligned}$$

by (2.1) and (2.5). \square

3. REGULARIZED DIAGONAL CYCLES AND THETA ELEMENTS

3.1. Definite unitary groups in two and three variables. We can let

$$T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t_2 \end{bmatrix}, \quad T' = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$$

by multiplying T by an appropriate constant with positive $t_i \in \mathbb{Z}_{(p)}$. We view $H = \mathbf{U}(T')$ as a subgroup of $G = \mathbf{U}(T)$ via the embedding

$$\iota \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}.$$

We will frequently add $'$ to the notation for various objects to indicate that they are attached to H . Fix normal rings $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3, \mathbf{I}'_1, \mathbf{I}'_2$ finite flat over $\Lambda = \mathcal{O}[[\mathbb{Z}_p^\times]]$. Put

$$\mathcal{R} = \mathbf{I}_1 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_2 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_3, \quad \mathcal{R}' = \mathbf{I}'_1 \widehat{\otimes}_{\mathcal{O}} \mathbf{I}'_2.$$

The finite set Σ_T^- consists of non-split rational primes q at which $(\ , \)_{T'}$ does not split, i.e., $q \in \Sigma_T^-$ if and only if $\epsilon_{E_q/\mathbb{Q}_q}(-\det T) = -1$. For simplicity we assume that if $q \in \Sigma_T^-$, then q is odd and σ_q is the trivial representation of the compact unitary group $H(\mathbb{Q}_q)$.

If $q \notin \Sigma_T^-$, then $G(\mathbb{Q}_q)$ and $H(\mathbb{Q}_q)$ are quasi split, and $\mathcal{K}_q, \mathcal{K}'_q$ are their special good maximal bounded subgroups in the sense of §1.1 of [Cas80]. When $q \in \Sigma_T^- \cap \Sigma_E^r$, we define maximal compact subgroups by

$$\mathcal{K}_q = G(\mathbb{Q}_q) \cap \mathrm{GL}_3(\mathfrak{r}_q), \quad \mathcal{K}'_q = H(\mathbb{Q}_q),$$

assuming that $t_1, t_2 \in \mathbb{Z}_q^\times$. For each $q \in \Sigma_T^- \setminus \Sigma_E^r$ we take maximal compact subgroups

$$\mathcal{K}_q = G(\mathbb{Q}_q) \cap \mathrm{GL}(L_q), \quad \mathcal{K}'_q = H(\mathbb{Q}_q),$$

where L_q is a maximal integral lattice of the Hermitian space $(W \otimes \mathbb{Q}_q, q(\ , \)_T)$ which contains $e_2 := {}^t(0, 1, 0)$ and such that $q \cdot (e_2, L_q)_T = \mathfrak{r}_q$ (see §C.1 for details). We take $\xi_{\mathfrak{N}} = (\xi_{\mathfrak{N}, l}) \in G(\widehat{\mathbb{Q}})$ and $\xi'_{\mathfrak{N}'} = (\xi'_{\mathfrak{N}', l}) \in H(\widehat{\mathbb{Q}})$ defined by

$$\xi_{\mathfrak{N}, l} = \begin{cases} \iota_l^{-1} \left(\begin{bmatrix} N\mathbf{1}_2 & \\ & 1 \end{bmatrix}^{-1} T \right) & \text{if } l|N, \\ \mathbf{1}_3 & \text{otherwise,} \end{cases}$$

$$\xi'_{\mathfrak{N}', l} = \begin{cases} \iota_l'^{-1} \left(\begin{bmatrix} N' & 0 \\ 0 & 1 \end{bmatrix}^{-1} T' \right) & \text{if } l|N', \\ \mathbf{1}_2 & \text{otherwise.} \end{cases}$$

We define $\varsigma^{(p)} = (\varsigma_l^{(p)}) \in G(\widehat{\mathbb{Q}})$, where

$$\varsigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \varsigma_l^{(p)} = \begin{cases} \iota_l^{-1}(\varsigma) & \text{if } l \text{ splits in } E \text{ and differs from } p, \\ \mathbf{1}_3 & \text{otherwise.} \end{cases}$$

3.2. Twisted diagonal cycles for $U(3) \times U(2)$. The embedding $\iota' : \mathrm{GL}_2(\mathbb{Q}_l) \hookrightarrow \mathrm{GL}_3(\mathbb{Q}_l)$ is defined by $\iota'(g) = \begin{bmatrix} g & \\ & 1 \end{bmatrix}$. For each prime factor l of NN' we fix an open compact subgroup \mathcal{J}_l of $G(\mathbb{Q}_l)$ which contains the subgroup $\iota_l^{-1}(\iota'(\mathcal{K}_0^{(2)}(N'\mathfrak{r}_l)))$. Put

$$\mathcal{J}_\ell = \iota_{\mathfrak{p}}^{-1}(\mathcal{I}_1^{(3)}(\mathfrak{p}^\ell)) \times \prod_{q \nmid pNN'} \mathcal{K}_q \times \prod_{l|NN'} \mathcal{J}_l.$$

We will specify a natural number $N'' = M^2$ whose prime factors split in E in §3.4. Fix an ideal \mathfrak{N}'' of \mathfrak{r} such that $\mathfrak{r}/\mathfrak{N}'' \simeq \mathbb{Z}/N''\mathbb{Z}$. We define $\mathcal{K}'_0(\mathfrak{N}'')$ with respect to $\mathcal{K}' = \prod_q \mathcal{K}'_q$. Define an open compact subgroup $\mathcal{K}'_{01}(p^\ell \mathfrak{N}'')$ of $H(\widehat{\mathbb{Q}})$ by

$$\mathcal{K}'_{01}(p^\ell \mathfrak{N}'') = \{(h_i) \in \mathcal{K}'_0(\mathfrak{N}'') \mid \iota_{\mathfrak{p}}(h_{\mathfrak{p}}) \in \mathcal{I}_1^{(2)}(\mathfrak{p}^\ell), h_q \in \mathcal{K}''_q \text{ for } q \in \Sigma_T^-\},$$

where $\mathcal{K}''_q = \mathcal{K}_q \cap \mathcal{K}'_q$. We consider the projective systems of the finite sets

$$X_\ell = G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / \mathcal{J}_\ell, \quad X'_\ell = H(\mathbb{Q}) \backslash H(\widehat{\mathbb{Q}}) / \mathcal{K}'_{01}(p^\ell \mathfrak{N}''), \quad \mathbf{X}_\ell = X_\ell \times X'_\ell.$$

Consider the finitely generated \mathcal{O} -module $\mathcal{O}[\mathbf{X}_\ell]$ equipped with the operator $\mathbb{U}_p := U_p \otimes U_p$ and the ordinary projector $\mathbf{e}_{\mathrm{ord}} := e_{\mathrm{ord}} \otimes e_{\mathrm{ord}}$. Given $x \in G(\widehat{\mathbb{Q}})$ and

$x' \in H(\widehat{\mathbb{Q}})$, we write $[(x, x')] \in \mathbf{X}_\ell$ for the double coset represented by (x, x') . We define the embedding

$$j : H(\widehat{\mathbb{Q}}) \hookrightarrow G(\widehat{\mathbb{Q}}), \quad j(x') = \iota(x')\varsigma^{(p)}.$$

Set

$$\Upsilon_\ell = \iota_{\mathfrak{p}}^{-1} \left(\begin{bmatrix} p^{2\ell} & p^\ell & 0 \\ 0 & p^\ell & 1 \\ 0 & 0 & 1 \end{bmatrix} \right) \in G(\mathbb{Q}_p), \quad \tau_\ell = \iota_{\mathfrak{p}}'^{-1} \left(\begin{bmatrix} 0 & 1 \\ -p^\ell & 0 \end{bmatrix} \right) \in H(\mathbb{Q}_p).$$

For $z \in \mathbb{Q}_p$ we put $\mathbf{n}(z) = \iota_{\mathfrak{p}}'^{-1} \left(\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \right) \in H(\mathbb{Q}_p)$ and

$$\mathcal{I}_1^-(\mathfrak{p}^\ell) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{I}_1^{(2)}(\mathfrak{p}^\ell) \mid b = 0 \right\}.$$

Definition 3.1. Let $\Delta_\ell \in \mathbb{Z}_p[\mathbf{X}_\ell]$ be the twisted diagonal cycle defined by

$$\Delta_\ell = \sum_{[x'] \in X'_\ell} \frac{1}{\#\Gamma_{\ell, x'}^H} \sum_{z \in \mathbb{Z}_p/p^{2\ell}\mathbb{Z}_p} [(j(x'\mathbf{n}(z))\Upsilon_\ell, x'\tau_\ell)],$$

where $\Gamma_{\ell, x'}^H := H(\mathbb{Q}) \cap x'\mathcal{K}'_{01}(p^\ell\mathfrak{N}'')x'^{-1}$. Since $\iota(\mathcal{K}''(1))\varsigma^{(p)} \subset \varsigma^{(p)}\mathcal{K}_1(\mathfrak{N})$, this definition makes sense in view of the fact that for each $\gamma \in \mathcal{I}_1^-(\mathfrak{p}^\ell)$ there is $z \in \mathbb{Z}_p/p^{2\ell}\mathbb{Z}_p$ such that $\Upsilon_\ell^{-1}\iota(\iota_{\mathfrak{p}}^{-1}(\gamma\mathbf{n}(z)))\Upsilon_\ell \in \iota_{\mathfrak{p}}^{-1}(\mathcal{I}_1^{(3)}(\mathfrak{p}^\ell))$.

3.3. Regularized diagonal cycles for $U(3) \times U(2)$. The homomorphism

$$\mathbf{N}_{\ell+1, \ell} : \mathbb{Z}_p[\mathbf{X}_{\ell+1}] \rightarrow \mathbb{Z}_p[\mathbf{X}_\ell]$$

is induced by the projection $\mathbf{X}_{\ell+1} \rightarrow \mathbf{X}_\ell$.

Lemma 3.2 (Distribution property). *For $\ell \geq 1$ we have*

$$\mathbf{N}_{\ell+1, \ell}(\Delta_{\ell+1}) = \mathbb{U}_p \Delta_\ell.$$

Proof. The proof is similar to [Hsi21, Lemma 4.7]. Let S_ℓ be a complete set of representatives for $\mathbb{Z}_p/p^\ell\mathbb{Z}_p$. Since $\ell \geq 1$,

$$\Sigma_\ell := \left\{ \begin{bmatrix} 1 + p^\ell u & 0 \\ 0 & 1 + p^\ell v \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^\ell b & 1 \end{bmatrix} \mid u, v, b \in S_1 \right\}$$

is a complete set of representatives for $\mathcal{K}'_{01}(p^\ell\mathfrak{N}'')/\mathcal{K}'_{01}(p^{\ell+1}\mathfrak{N}'')$. By an argument similar to the proof of [Hsi21, Lemma 4.4], we may assume that $\Gamma_{\ell, x'}^H = \{1\}$ for every $x' \in H(\widehat{\mathbb{Q}})$. Then $X'_{\ell+1}$ consists of elements represented by $x'k'$ with $x' \in X'_\ell$ and $k' \in \Sigma_\ell$. Given $b, z \in \mathbb{Z}_p$, we put

$$w = \frac{z}{1 + p^\ell bz}, \quad s = \frac{1}{1 + p^\ell bz}, \quad t = \frac{1 - p^\ell bw}{1 - p^\ell bt}, \quad r = 1 + p^\ell b(st - w).$$

Then

$$\begin{aligned} \iota \left(\iota_{\mathfrak{p}}'^{-1} \left(\begin{bmatrix} 1 & 0 \\ p^\ell b & 1 \end{bmatrix} \right) \mathbf{n}(z) \right) \Upsilon_{\ell+1} &= \iota_{\mathfrak{p}}^{-1} \left(\begin{bmatrix} p^{2\ell+2} & p^{\ell+1}t & w \\ 0 & p^{\ell+1} & s \\ 0 & 0 & 1 \end{bmatrix} \gamma_{b,z} \right), \\ \gamma_{b,z} &= \begin{bmatrix} r & 0 & 0 \\ -p^{2\ell+1}bs & 1 - p^\ell bs & 0 \\ p^{3\ell+2}b & p^{2\ell+1}b & 1 + p^\ell bz \end{bmatrix} \in \mathcal{I}_1^{(3)}(\mathfrak{p}^\ell). \end{aligned}$$

Since $s, t, r \in 1 + p^\ell \mathbb{Z}_p$, we find that $\mathbf{N}_{\ell+1, \ell}(\Delta_{\ell+1})$ equals

$$\begin{aligned}
 & \sum_{[x'] \in X'_{\ell+1}} \sum_{z \in S_{2\ell}} \sum_{z_1 \in S_2} [(j(x' \mathbf{n}(z + p^{2\ell} z_1)) \Upsilon_{\ell+1}, x' \tau_{\ell+1})] \\
 &= \sum_{[x'] \in X'_\ell} \sum_{k' \in \Sigma_\ell} \sum_{z \in S_{2\ell}} \sum_{z_1 \in S_2} [(j(x' k' \mathbf{n}(z + p^{2\ell} z_1)) \Upsilon_{\ell+1}, x' k' \tau_{\ell+1})] \\
 &= \sum_{[x'] \in X'_\ell} \sum_{z \in S_{2\ell}} \sum_{z_1 \in S_2} \sum_{u, v, b \in S_1} \left[\left(j(x') \begin{bmatrix} p^{2\ell+2} & p^{\ell+1}(1+p^\ell u) & z + p^{2\ell} z_1 \\ 0 & p^{\ell+1} & 1 + p^\ell v \\ 0 & 0 & 1 \end{bmatrix}, x' \tau_\ell \begin{bmatrix} p & b \\ 0 & 1 \end{bmatrix} \right) \right] \\
 &= \sum_{[x'] \in X'_\ell} \sum_{z \in S_{2\ell}} \sum_{z_1 \in S_2} \sum_{u, v \in S_1} (\mathbf{1} \otimes \mathcal{U}'_p) \left[\left(j(x' \mathbf{n}(z)) \Upsilon_\ell \begin{bmatrix} p^2 & pu & z_1 \\ 0 & p & v \\ 0 & 0 & 1 \end{bmatrix}, x' \tau_\ell \right) \right] \\
 &= \mathbb{U}_p \Delta_\ell
 \end{aligned}$$

by Definition 2.11. □

Definition 3.3. Define the regularized diagonal cycle by $\Delta_\ell^\dagger = \mathbb{U}_p^{-\ell} \mathbf{e}_{\text{ord}} \Delta_\ell$.

Since $\mathbf{N}_{\ell+1, \ell}(\Delta_{\ell+1}^\dagger) = \Delta_\ell^\dagger$ by Lemma 3.2, we can define

$$\Delta_\infty^\dagger = \lim_{\leftarrow \ell} \Delta_\ell^\dagger \in \lim_{\leftarrow \ell} \mathbb{Z}_p[\mathbf{X}_\ell].$$

3.4. Theta elements. Let χ' be a Dirichlet character of $(\mathfrak{r}/\mathfrak{N}')^\times$ of conductor M . Recall that a character of $\mathcal{K}'_0(\mathfrak{N}')$ is associated to χ' by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \chi'(d)$$

(cf. Definition 2.10). Take a divisor \mathfrak{M} of \mathfrak{N}' such that $\mathfrak{r}/\mathfrak{M} \simeq \mathbb{Z}/M\mathbb{Z}$.

Let $\mathbf{f} \in e_{\text{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R})$ and $\mathbf{g} \in e_{\text{ord}} \mathbf{S}^H(\mathfrak{N}', \chi', \mathcal{R}')$ be Hida families. Recall that $w_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We define the tame twisting operator $\mathbf{U}_{\mathfrak{N}'}^{\chi'}$ on an \mathcal{R} -adic modular form $\mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}$ on G by

$$\begin{aligned}
 \mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}(x) &= \sum_{i, j \in (\mathbb{Z}/M\mathbb{Z})^\times} \sum_{y \in \mathbb{Z}/M^2\mathbb{Z}} \chi'(ij)^{-1} \mathbf{f} \left(x \cdot \begin{bmatrix} \xi_{\mathfrak{N}'}^i w_2 & \\ & 1 \end{bmatrix} \Delta_M(g_{M, i, j, y}) \right), \\
 g_{M, i, j, y} &= \begin{bmatrix} 1 & \frac{i}{M} & \frac{y}{M^2} \\ 0 & 1 & \frac{j}{M} \\ 0 & 0 & 1 \end{bmatrix},
 \end{aligned}$$

where we write $\Delta_M : \text{GL}_3(\mathbb{Q}) \hookrightarrow \prod_{l|M} \text{GL}_3(\mathbb{Q}_l)$ for the diagonal embedding and define the action of $g = (g_l) \in \prod_{l|M} \text{GL}_3(\mathbb{Q}_l)$ on $x \in G(\widehat{\mathbb{Q}})$ by

$$x \cdot g = x \prod_{l|M} \iota_l^{-1}(g_l).$$

Proposition 6.7 below shows that for $u \in \mathcal{K}'_0(\mathfrak{M}^2)$

$$r(\iota(u)) \mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f} = \chi(u) \mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}.$$

We hereafter assume the following hypothesis:

(H'_3) the square of the conductor \mathfrak{M} of χ' is divisible by \mathfrak{N}' .

We construct the regularized diagonal cycle Δ_∞^\dagger by letting $\mathfrak{N}'' = \mathfrak{M}^2$. Put $\mathbf{G} = G \times H$. We define an $\mathcal{R} \widehat{\otimes} \mathcal{R}'$ -adic modular form on \mathbf{G} by

$$\mathbf{F} = \mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f} \boxtimes \mathbf{g} : \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\widehat{\mathbb{Q}}) \rightarrow \mathcal{R} \widehat{\otimes} \mathcal{R}'$$

by

$$\mathbf{F}(x, x') = \mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}(x) \mathbf{g}(x').$$

Then \mathbf{F} naturally induces a $\Lambda_3 \boxtimes \Lambda_2$ -linear map

$$\mathbf{F}_* : \lim_{\leftarrow \ell} \mathcal{O}[\mathbf{X}_\ell] \rightarrow \mathcal{R} \widehat{\otimes} \mathcal{R}'.$$

The theta element $\Theta_{\mathbf{F}}$ attached to the product \mathbf{F} is then defined by the evaluation of \mathbf{F}_* at the regularized diagonal cycle, namely,

$$\Theta_{\mathbf{F}} = \mathbf{F}_*(\Delta_\infty^\dagger) \in \mathcal{R} \widehat{\otimes} \mathcal{R}'.$$

3.5. Period integrals. The set $\mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ consists of pairs $(\underline{Q}, \underline{Q}')$ with

$$\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathbf{f}}^{\text{temp}}, \quad \underline{Q}' = (Q'_1, Q'_2) \in \mathfrak{X}_{\mathbf{g}}^{\text{temp}}$$

such that

$$k_{Q_1} \leq -k_{Q'_2} \leq k_{Q_2} \leq -k_{Q'_1} \leq k_{Q_3}.$$

Fix $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$. We denote the automorphic representation of $G(\mathbb{A})$ associated to $\underline{Q}(\mathbf{f})$ by $\pi_{\underline{Q}}$ and the automorphic representation of $H(\mathbb{A})$ attached to $\underline{Q}'(\mathbf{g})$ by $\sigma_{\underline{Q}'}$. We here abbreviate

$$\pi = \pi_{\underline{Q}}, \quad \sigma = \sigma_{\underline{Q}'}, \quad \Pi = \pi \otimes \sigma, \quad k_i = k_{Q_i}, \quad k'_j = k_{Q'_j}$$

for $i = 1, 2, 3$ and $j = 1, 2$. Put

$$a = k_3 - k_2, \quad b = k_2 - k_1, \quad n = k_3 + k'_1, \quad l = -k'_2 - k_1.$$

We use the notation in §A.6 to note that

$$\pi_\infty = \rho_{k_{\underline{Q}}} \simeq \mathfrak{H}_{b,a} \otimes (\det)^{-k_2}, \quad \sigma_\infty = \rho_{k_{\underline{Q}'}} \simeq \varrho_{(-k'_1, -k'_2)}.$$

Take highest weight vectors

$$(3.1) \quad \mathbf{v}_{k_{\underline{Q}}} = x_1^b y_3^a \in \pi_\infty, \quad \mathbf{v}_{k_{\underline{Q}'}} = X_1^{a-n+b-l} \in \sigma_\infty.$$

Then the lowest weight dual vectors are

$$(3.2) \quad \mathbf{u}_{k_{\underline{Q}}} = y_1^b x_3^a \in \pi_\infty^\vee, \quad \mathbf{u}_{k_{\underline{Q}'}} = (-Y_1')^{a-n+b-l} \in \sigma_\infty^\vee$$

(see Remark 2.6).

Let $f \in \mathcal{A}_{\underline{k}}^G(p^\ell \mathfrak{N}, \chi)$ and $g \in \mathcal{A}_{\underline{k}'}^H(p^\ell \mathfrak{N}', \chi')$ be the adèlic lifts of

$$\begin{aligned} \widehat{f} &= \mathbf{f}_{\underline{Q}} \in \mathcal{S}_{k_{\underline{Q}}}^G(p^\ell \mathfrak{N}, \chi \in_{\underline{Q}}, \mathcal{R}(\underline{Q})), \\ \widehat{g} &= \mathbf{g}_{\underline{Q}'} \in \mathcal{S}_{k_{\underline{Q}'}}^H(p^\ell \mathfrak{N}', \chi' \in_{\underline{Q}'}, \mathcal{R}'(\underline{Q}')) \end{aligned}$$

(see (2.8)). Recall that f and g are eigenforms of the operators U_p and U'_p with eigenvalues α_{π_p} and α_{σ_p} by [Hid04, Theorem 5.3] and Proposition 5.4 below. Define $\mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}$ by replacing \mathbf{f} with f in the definition of $\mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}$. Clearly,

$$\underline{Q}(\mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}) = \sqrt{T}^{-k_{\underline{Q}}} (\widehat{\mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}})_{\mathbf{v}_{k_{\underline{Q}}}}.$$

Define the vector-valued modular form $F : \mathbf{G}(\mathbb{A}) \rightarrow \Pi_\infty^\vee$ by

$$F(x, x') = \mathbf{U}_{\mathfrak{N}'}^{\chi'} f(x) \otimes g(x') \quad (x \in G(\mathbb{A}), x' \in H(\mathbb{A})).$$

Define a scalar-valued modular form $\Phi_F : G(\mathbb{A}) \times H(\mathbb{A}) \rightarrow \mathbb{C}$ by

$$\Phi_F(x, x') = \ell_{\Pi_\infty}(\mathbf{W}_{\Pi_\infty}^H \otimes F(x, x')),$$

where $\mathbf{W}_{\Pi_\infty}^H \in \Pi_\infty$ is an $H(\mathbb{R})$ -invariant vector defined in §A.6. Proposition A.6(4) shows that

$$\Phi_F(x\iota(u_\infty), x'u_\infty) = \Phi_F(x, x')$$

for $u_\infty \in H(\mathbb{R})$. Define $\mathbf{t}_\ell \in G(\mathbb{Q}_p)$ by

$$\mathbf{t}_\ell = \iota_p^{-1} \left(\begin{bmatrix} 0 & 0 & -p^{-\ell} \\ 0 & p^\ell & 1 \\ p^{2\ell} & p^\ell & 0 \end{bmatrix} \right).$$

We consider the following period integral

$$\mathcal{P}_{\mathcal{K}'}(\Phi) = \int_{H(F) \backslash H(\mathbb{A})} \Phi(h, h) d_{\mathcal{K}'} h$$

for scalar-valued modular forms Φ on $G \times H$.

Proposition 3.4. *Let $\mathcal{Q} \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$. Then*

$$\mathcal{Q}(\Theta_{\mathbf{F}}) = \frac{(-1)^{k_2 - k_1}}{\zeta_p(1)\zeta_p(2)} [\mathcal{K}' : \mathcal{K}'_0(\mathfrak{M}^2)] \frac{\mathcal{P}_{\mathcal{K}'}(\pi(\mathbf{t}_\ell \zeta^{(p)}) \Phi_F)}{(p^{-5} \alpha_{\pi_p} \alpha_{\sigma_p})^\ell} \prod_{q \in \Sigma_T^- \setminus \Sigma_E} (q+1)$$

for sufficiently large ℓ .

Proof. The proof is similar to that of [Hsi21, Proposition 4.9] (cf. [CH18, Lemma 4.4]). Recall that $\underline{\mathcal{Q}}(\mathbf{f}) = \sqrt{T}^{-k_{\mathcal{Q}}} \widehat{f}_{\mathbf{v}_{k_{\mathcal{Q}}}}$ and $\underline{\mathcal{Q}}'(\mathbf{g}) = \sqrt{T'}^{-k_{\mathcal{Q}'}} \widehat{g}_{\mathbf{v}_{k_{\mathcal{Q}'}}}$ are eigenforms of the operators \mathcal{U}_p and \mathcal{U}'_p with unit eigenvalues α_f and α_g . Thus

$$\mathcal{Q}(\mathbf{F}) = \underline{\mathcal{Q}}(\mathbf{U}_{\mathfrak{N}'}^{\chi'} \mathbf{f}) \boxtimes \underline{\mathcal{Q}}'(\mathbf{g}) = \sqrt{T}^{-k_{\mathcal{Q}}} \sqrt{T'}^{-k_{\mathcal{Q}'}} (\widehat{\mathbf{U}_{\mathfrak{N}'}^{\chi'} f})_{\mathbf{v}_{k_{\mathcal{Q}}}} \boxtimes g_{\mathbf{v}_{k_{\mathcal{Q}'}}}$$

is an eigenform of \mathbb{U}_p with unit eigenvalue

$$\alpha_F = \alpha_f \alpha_g = p^{-2k_1 - k_2 - k'_1} \alpha_{\pi_p} \alpha_{\sigma_p}$$

by Remark 2.9. Let $\widehat{F} = \widehat{\mathbf{U}_{\mathfrak{N}'}^{\chi'} f} \otimes \widehat{g}$ be a p -adic modular form on \mathbf{G} . Then

$$\mathcal{Q}(\mathbf{F})(x, x') = \ell_{\Pi_\infty} (x_1^b y_3^a X_1^{a-n+b-l} \otimes \Pi_\infty^\vee(\sqrt{T}, \sqrt{T'})^{-1} \widehat{F}(x, x'))$$

for $x \in G(\widehat{\mathbb{Q}})$ and $x' \in H(\widehat{\mathbb{Q}})$. Recall that $\Pi_\infty^\vee = \rho_{k_{\mathcal{Q}}} \otimes \rho_{k_{\mathcal{Q}'}}$.

By definition we have

$$\begin{aligned} \mathcal{Q}(\Theta_{\mathbf{F}}) &= \mathcal{Q}(\mathbf{F})(\mathbb{U}_p^{-\ell} \mathbf{e}_{\text{ord}} \Delta_\ell) \\ &= \alpha_F^{-\ell} \sum_{[x'] \in X'_\ell} \sum_{z \in \mathbb{Z}_p/p^{2\ell} \mathbb{Z}_p} \mathcal{Q}(\mathbf{F})(j(x' \mathbf{n}(z)) \Upsilon_\ell, x' \tau_\ell) \end{aligned}$$

for sufficiently large ℓ , we see that $\alpha_F^\ell \mathcal{Q}(\Theta_{\mathbf{F}})$ equals

$$\sum_{[x'] \in X'_\ell} \sum_{z \in \mathbb{Z}_p/p^{2\ell} \mathbb{Z}_p} \ell_{\Pi_\infty} (x_1^b y_3^a X_1^{a-n+b-l} \otimes \Pi_\infty^\vee(\sqrt{T}, \sqrt{T'})^{-1} \widehat{F}(j(x' \mathbf{n}(z)) \Upsilon_\ell, x' \tau_\ell)).$$

Since

$$F(x, x') = \iota_p^{-1}(\Pi_\infty^\vee(\iota_p^T(x_p), \iota_p^{T'}(x'_p))\widehat{F}(x, x'))$$

by (2.8), we see that for $x \in G(\widehat{\mathbb{Q}})$ and $x' \in H(\widehat{\mathbb{Q}})$

$$\Phi_F(x, x') = \ell_{\Pi_\infty}(\mathbf{W}_{\Pi_\infty}^H \otimes \iota_p^{-1}(\Pi_\infty^\vee(\iota_p^T(x_p), \iota_p^{T'}(x'_p))\widehat{F}(x, x')))$$

Proposition A.6(4) shows that for $x' \in H(\mathbb{Q}_p)$, $g \in G(\mathbb{Q}_p)$ and $h \in H(\mathbb{Q}_p)$

$$\begin{aligned} & \ell_{\Pi_\infty}(\mathbf{W}_{\Pi_\infty}^H \otimes F(j(x')g, x'h)) \\ &= \ell_{\Pi_\infty}(\mathbf{W}_{\Pi_\infty}^H \otimes \Pi_\infty^\vee(\iota_p^T(\iota(x'_p)), \iota_p^{T'}(x'_p))^{-1}F(j(x')g, x'h)) \\ &= \iota_p^{-1}(\ell_{\Pi_\infty}(\Pi_\infty(\iota(\sqrt{T'}), \sqrt{T'})\mathbf{W}_{\Pi_\infty}^H \otimes \Pi_\infty^\vee(\iota_p^T(\iota(g)), \iota_p^{T'}(h))\widehat{F}(j(x')g, x'h))) \\ &= \iota_p^{-1}(\ell_{\Pi_\infty}(\Pi_\infty(\iota_p(g), \iota'_p(h))^{-1}\mathbf{W}_{\Pi_\infty}^H \otimes \Pi_\infty^\vee(\sqrt{T}, \sqrt{T'})^{-1}\widehat{F}(j(x')g, x'h))). \end{aligned}$$

Let $g = \iota(\mathbf{n}(z))\Upsilon_\ell$ and $h = \tau_\ell$. Observe that

$$\begin{aligned} \iota_p(g^{-1}\iota(h)) &= \frac{1}{p^{2\ell}} \begin{bmatrix} 1 & -1 & 1-z \\ 0 & p^\ell & -p^\ell \\ 0 & 0 & p^{2\ell} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -p^\ell & 0 & 0 \end{bmatrix} = \frac{1}{p^{2\ell}} \begin{bmatrix} (z-1)p^\ell & -1 & 1 \\ p^{2\ell} & p^\ell & 0 \\ -p^{3\ell} & 0 & 0 \end{bmatrix}, \\ \iota_p(\iota(h)^{-1}g) &= \begin{bmatrix} 0 & 0 & -p^{-\ell} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p^{2\ell} & p^\ell & z \\ 0 & p^\ell & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -p^{-\ell} \\ 0 & p^\ell & 1 \\ p^{2\ell} & p^\ell & z \end{bmatrix}. \end{aligned}$$

Recall that

$$\mathbf{W}_H^H \equiv \det \begin{bmatrix} x_1 & x_3 \\ X_2 & Y_2 \end{bmatrix}^{b-l} \det \begin{bmatrix} X_2 & Y_2 \\ y_3 & -y_1 \end{bmatrix}^{a-n} x_2^l y_2^n \pmod{\mathcal{T}_{b,a}(\mathbb{Q})}.$$

Since

$$\begin{aligned} (x_1, x_2, x_3)\iota_p(g^{-1}\iota(h)) &= p^{-2\ell}(x_1(z-1)p^\ell + x_2p^{2\ell} - x_3p^{3\ell}, x_2p^\ell - x_1, x_1), \\ (y_1, y_2, y_3)\iota_p(\iota(h)^{-1}g) &= (-y_3p^{-\ell}, y_3 + y_2p^\ell, y_3z + y_2p^\ell + y_1p^{2\ell}), \end{aligned}$$

we see that

$$\begin{aligned} & \Pi_\infty(\iota_p(\iota(\mathbf{n}(z))\Upsilon_\ell), \iota'_p(\tau_\ell))^{-1}\mathbf{W}_{\Pi_\infty}^H \\ &= \frac{1}{p^{2\ell(b-l)}} \det \begin{bmatrix} x_1(z-1)p^\ell + x_2p^{2\ell} - x_3p^{3\ell} & x_1 \\ X_2 & Y_2 \end{bmatrix}^{b-l} \\ & \quad \times \det \begin{bmatrix} X_2 & Y_2 \\ y_3z + y_2p^\ell + y_1p^{2\ell} & y_3p^{-\ell} \end{bmatrix}^{a-n} \\ & \quad \times p^{-2\ell}(x_2p^\ell - x_1)^l (y_3 + y_2p^\ell)^n \det(g^{-1}\iota(h))^{-k_2}, \end{aligned}$$

from which we find the following congruence relation

$$\begin{aligned} & (p^{2k_1+k_2+k'_1})^{-\ell} \Pi_\infty(\iota_p(j(\mathbf{n}(z))\Upsilon_\ell), \iota'_p(\tau_\ell))^{-1}\mathbf{W}_{\Pi_\infty}^H \\ & \equiv (-x_1)^b y_3^a X_2^{b-l+a-n} \pmod{p^s} \end{aligned}$$

if ℓ is sufficiently larger than s . We conclude that

$$\begin{aligned} & (p^{2k_1+k_2+k'_1})^{-\ell} \iota_p(\ell_{\Pi_\infty}(\mathbf{W}_{\Pi_\infty}^H \otimes F(j(x'\mathbf{n}(z))\Upsilon_\ell, x'\tau_\ell))) \\ &= \ell_{\Pi_\infty}((-x_1)^b y_3^a X_2^{b-l+a-n} \otimes \Pi_\infty^\vee(\sqrt{T}, \sqrt{T'})^{-1}\widehat{F}(j(x'\mathbf{n}(z))\Upsilon_\ell, x'\tau_\ell)). \end{aligned}$$

Substituting this expression, we see that

$$(p^{2k_1+k_2+k'_1}\alpha_F)^\ell \mathcal{Q}(\Theta_F)$$

$$\begin{aligned}
 &= (-1)^b \sum_{[x'] \in X'_\ell} \sum_{z \in \mathbb{Z}_p/p^{2\ell}\mathbb{Z}_p} \Phi_F(j(x'\mathbf{n}(z))\Upsilon_\ell, x'\tau_\ell) \\
 &= (-1)^b [\mathcal{K}' : \mathcal{K}'_{01}(p^\ell \mathfrak{M}^2)] \sum_{z \in \mathbb{Z}_p/p^{2\ell}\mathbb{Z}_p} \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Phi_F(j(h\mathbf{n}(z))\Upsilon_\ell, h\tau_\ell) d_{\mathcal{K}'} h.
 \end{aligned}$$

Observe that

$$\begin{aligned}
 &\int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Phi_F(\iota(h\mathbf{n}(z))\Upsilon_\ell, h\tau_\ell) d_{\mathcal{K}'} h \\
 &= \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Phi_F(\iota(h)\Upsilon_\ell, h\mathbf{n}(-z)\tau_\ell) d_{\mathcal{K}'} h = \mathcal{P}_{\mathcal{K}'}(\Pi_p(\Upsilon_\ell, \tau_\ell)\Phi_F).
 \end{aligned}$$

Since $[\mathcal{K}'_q : \mathcal{K}''_q] = q + 1$ for $q \in \Sigma_T^- \setminus \Sigma_E^r$ by Lemma 3.14 of [Shi08], we have

$$[\mathcal{K}' : \mathcal{K}'_{01}(p^\ell \mathfrak{M}^2)] = p^{3\ell}(1 - p^{-2})(1 - p^{-1})[\mathcal{K}' : \mathcal{K}'_0(\mathfrak{M}^2)] \prod_{q \in \Sigma_T^- \setminus \Sigma_E^r} (q + 1).$$

Since $\iota(\tau_\ell^{-1})\Upsilon_\ell = \mathbf{t}_\ell$, we obtain the stated formula. \square

4. THE CENTRAL VALUE FORMULAE

4.1. The Ichino-Ikeda formula. Let G be the unitary group of the Hermitian form $(\ , \)_T$ on $W = E^n$. Let H be the unitary group of a subspace W' of W of dimension $n - 1$ on which $(\ , \)_T$ is non-degenerate. We view H as a subgroup of G . Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ and $\sigma \simeq \otimes'_v \sigma_v$ an irreducible cuspidal automorphic representation of $H(\mathbb{A})$. Set

$$\mathbf{G} = G \times H, \quad \Pi = \pi \otimes \sigma, \quad \Pi_v = \pi_v \otimes \sigma_v.$$

We define the complete L -series associated to π and σ by

$$L(s, \pi \times \sigma) = L^{\text{GL}}(s, \text{BC}(\pi) \times \text{BC}(\sigma)),$$

where $\text{BC}(\pi)$ (resp. $\text{BC}(\sigma)$) is the functorial lift of π (resp. σ) to an automorphic representation of $\text{GL}_n(\mathbb{E})$ (resp. $\text{GL}_{n-1}(\mathbb{E})$) (see [Lab11, Corollaire 5.3]). The right hand side is the L -factor defined by Jacquet, Piatetski-Shapiro and Shalika in [JPSS83]. Let $L(s, \pi, \text{Ad})$ denote the complete adjoint L -series for π . Assume that both π and σ are tempered. Put

$$\begin{aligned}
 \mathcal{L}(\pi \times \sigma) &= \frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \text{Ad})L(1, \sigma, \text{Ad})} \prod_{i=1}^n L(i, \epsilon_{E/\mathbb{Q}}^i), \\
 \mathcal{L}(\pi_v \times \sigma_v) &= \frac{L(\frac{1}{2}, \pi_v \times \sigma_v)}{L(1, \pi_v, \text{Ad})L(1, \sigma_v, \text{Ad})} \prod_{i=1}^n L(i, \epsilon_{E_v/\mathbb{Q}_v}^i).
 \end{aligned}$$

Remark 4.1. If n is even, then $L(s, \pi, \text{Ad}) = L(s, \text{BC}(\pi), \text{As})$ is the Asai L -series for $\text{BC}(\pi)$ while if n is odd, then $L(s, \pi, \text{Ad}) = L(s, \text{BC}(\pi), \text{As}^-)$ is the twisted Asai L -series by Proposition 7.4 of [GGP12].

We define the Petersson pairing by

$$(\Phi, \Phi') = \int_{\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})} \Phi(g, h)\Phi'(g, h) d^\tau g d^\tau h$$

for cusp forms Φ and Φ' on \mathbf{G} , where $d^\tau g$ and $d^\tau h$ are the Tamagawa measures on $G(\mathbb{A})$ and $H(\mathbb{A})$. Given a cusp form Φ on \mathbf{G} , we consider the integral

$$\mathcal{P}(\Phi) = \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Phi((h, h)) d^\tau h.$$

Fix a local perfect pairing

$$\langle\langle \cdot, \cdot \rangle\rangle_v : \Pi_v \otimes \Pi_v^\vee \rightarrow \mathbb{C}.$$

If Π_v is tempered, then the integral

$$I(\mathbf{W}_1, \mathbf{W}_2) = \int_{H(\mathbb{Q}_v)} \langle\langle \Pi_v((h_v, h_v)) \mathbf{W}_1, \mathbf{W}_2 \rangle\rangle_v dh_v$$

is convergent for $\mathbf{W}_1 \in \Pi_v$ and $\mathbf{W}_2 \in \Pi_v^\vee$ by Proposition 2.1 of [Har14]. The local Haar measure dh_v is defined so that a maximal compact subgroup \mathcal{K}'_v of $H(\mathbb{Q}_v)$, which we will specify later for $n = 3$, has volume 1. Let C_H be the ratio between the Tamagawa measure and the product measure of local measures. Namely C_H is defined so that $d^\tau h = C_H \prod_v dh_v$.

Ichino and Ikeda [II10] have refined the global Gross-Prasad conjecture and predicted an explicit relation between the central value and the period for orthogonal groups. The analogue of the Ichino-Ikeda conjecture for unitary groups was formulated in [Har14] and proved by [BPLZZ21] in the stable case and by [BPCZ22] in the endoscopic case.

Theorem 4.2 ([Zha14, BPLZZ21, BPCZ22]). *Let Π be an irreducible tempered cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$. If $\Phi = \otimes'_v \mathbf{W}_v \in \Pi$ and $\Phi' = \otimes'_v \mathbf{W}'_v \in \Pi^\vee$ are factorizable, then*

$$\frac{\mathcal{P}(\Phi)\mathcal{P}(\Phi')}{\langle\langle \Phi, \Phi' \rangle\rangle} = C_H \frac{\mathcal{L}(\pi \times \sigma)}{2^{2^\pi + 2^\sigma}} \prod_v \frac{I(\mathbf{W}_v, \mathbf{W}'_v)}{\mathcal{L}(\pi_v \times \sigma_v) \langle\langle \mathbf{W}_v, \mathbf{W}'_v \rangle\rangle_v},$$

where 2^{2^π} (resp. 2^{2^σ}) is the order of the component group associated to the L -parameter of π (resp. σ).

Roughly speaking, this theorem tells us that the product of global period integrals is a product of the Rakin-Selberg central value $\mathcal{L}(\pi \times \sigma)$ and the local zeta integrals $I(\mathbf{W}_v, \mathbf{W}'_v)$. Therefore, by Proposition 3.4 the task of obtaining the interpolation formula of $\Theta_{\mathbf{F}}$ boils down to (i) choices of test vectors \mathbf{W}_v and \mathbf{W}'_v and (ii) the explicit evaluation of $I(\mathbf{W}_v, \mathbf{W}'_v)$. The purpose of this section is to carry out the step (i) and give the explicit formula of the relevant local zeta integrals $I(\mathbf{W}_v, \mathbf{W}'_v)$. The details of the step (ii) are left to §5, §6 and Appendices.

4.2. Shimura's mass formula. We now assume that T is positive definite. Fix an open compact subgroup $\mathcal{K} = \prod_q \mathcal{K}_q$ of $G(\widehat{\mathbb{Q}})$. The space $\mathcal{A}(G)$ of automorphic forms on G consists of left $G(\mathbb{Q})$ -invariant, right $G(\mathbb{R})\mathcal{K}$ -finite functions on $G(\mathbb{A})$. We normalize the Haar measure dh_∞ by $\int_{H(\mathbb{R})} dh_\infty = 1$. Actually, it is more suitable for our later application to use the Haar measure $d_{\mathcal{K}'} h = \prod_v dh_v$, which gives the maximal compact subgroup $H(\mathbb{R})\mathcal{K}'$ volume 1. We similarly define the Haar measure $d_{\mathcal{K}} g$ and choose the constants C_H and C_G so that

$$d^\tau h = C_H d_{\mathcal{K}'} h, \quad d^\tau g = C_G d_{\mathcal{K}} g.$$

We normalize the period integrals by

$$\begin{aligned} (\Phi, \Phi')_{\mathcal{K} \times \mathcal{K}'} &= \int_{\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})} \Phi(g, h) \Phi'(g, h) d_{\mathcal{K}} g d_{\mathcal{K}'} h, \\ \mathcal{P}_{\mathcal{K}'}(\Phi) &= \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Phi((h, h)) d_{\mathcal{K}'} h \end{aligned}$$

for $\Phi, \Phi' \in \mathcal{A}(\mathbf{G})$. We rewrite the formula in Theorem 4.2 as

$$\frac{\mathcal{P}_{\mathcal{K}'}(\Phi) \mathcal{P}_{\mathcal{K}'}(\Phi')}{(\Phi, \Phi')_{\mathcal{K} \times \mathcal{K}'}} = C_G \frac{\mathcal{L}(\pi \times \sigma)}{2^{\varkappa_\pi + \varkappa_\sigma}} \prod_v \frac{I(\mathbf{W}_v, \mathbf{W}'_v)}{\mathcal{L}(\pi_v \times \sigma_v) \langle \langle \mathbf{W}_v, \mathbf{W}'_v \rangle \rangle_v}.$$

When \mathcal{K}_q is the stabilizer of a maximal lattice, Shimura [Shi97] has explicitly computed the mass

$$\frac{2}{C_G} = \sum_{g \in G(\mathbb{Q}) \backslash G(\hat{\mathbb{Q}}) / \mathcal{K}} \frac{1}{\sharp(G(\mathbb{Q}) \cap g \mathcal{K} g^{-1})} = 2^{1-n} \prod_{i=1}^n L_{\mathbf{f}}(1-i, \epsilon_{E/\mathbb{Q}}^i) \prod_v \lambda_v$$

(see Propositions 4.4 and 4.5 of [GHY01]), where $L_{\mathbf{f}}(s, \epsilon_{E/\mathbb{Q}}^i)$ is the *non-complete* Dirichlet L -series associated to the Dirichlet character $\epsilon_{E/\mathbb{Q}}^i$. Observe that

$$(4.1) \quad \gamma_G := \pi^{-\frac{n(n+1)}{2}} C_G \prod_{i=1}^n L_{\mathbf{f}}(i, \epsilon_{E/\mathbb{Q}}^i) \in \sqrt{D_E}^a \mathbb{Q}^\times,$$

where D_E is the absolute value of the discriminant of E , and $a = 0$ if $n \equiv 0, 3 \pmod{4}$ and $a = 1$ if $n \equiv 1, 2 \pmod{4}$. When we replace \mathcal{K}_q by another open compact subgroup of $G(\mathbb{Q}_q)$, we have only to multiply λ_q by a rational constant.

We hereafter suppose that

$$(H_2) \quad \text{Hom}_{H(\mathbb{R})}(\Pi_\infty, \mathbb{C}) \neq \{0\}.$$

Remark 4.3. The relation (1.1) is equivalent to (H_2) by [He17, Theorem 1.1].

Namely, there are $H(\mathbb{R})$ -invariant vectors $\mathbf{W}_\infty^H \in \Pi_\infty$ and $\mathbf{W}_\infty^{H'} \in \Pi_\infty^\vee$. Then

$$I(\mathbf{W}_\infty^H, \mathbf{W}_\infty^{H'}) = \langle \langle \mathbf{W}_\infty^H, \mathbf{W}_\infty^{H'} \rangle \rangle_\infty.$$

It follows from (A.1) that

$$c_\infty := \frac{L(1, \pi_\infty, \text{Ad}) L(1, \sigma_\infty, \text{Ad})}{L(\frac{1}{2}, \pi_\infty \times \sigma_\infty)} \pi^{\frac{n(n+1)}{2}} \in \mathbb{Q}^\times.$$

Corollary 4.4. Assume that T is positive definite. Let Π be an irreducible tempered cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$ whose archimedean part Π_∞ satisfies (H_2) . Let $\Phi = \otimes'_v \mathbf{W}_v \in \Pi$ and $\Phi' = \otimes'_v \mathbf{W}'_v \in \Pi^\vee$ be factorizable. If $\mathbf{W}_\infty = \mathbf{W}_\infty^H$ and $\mathbf{W}'_\infty = \mathbf{W}_\infty^{H'}$, then

$$\frac{\mathcal{P}_{\mathcal{K}'}(\Phi) \mathcal{P}_{\mathcal{K}'}(\Phi')}{(\Phi, \Phi')_{\mathcal{K} \times \mathcal{K}'}} = \frac{\gamma_G}{2^{\varkappa_\pi + \varkappa_\sigma}} c_\infty \frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \text{Ad}) L(1, \sigma, \text{Ad})} \prod_l \frac{I(\mathbf{W}_l, \mathbf{W}'_l)}{\mathcal{L}(\pi_l \times \sigma_l) \langle \langle \mathbf{W}_l, \mathbf{W}'_l \rangle \rangle_l}.$$

The product in the right-hand side is taken over all rational primes l and is actually a finite product by Theorem 2.12 of [Har14].

4.3. Rationality of central values. We assume that tempered automorphic representations of $G(\mathbb{A})$ occur in $\mathcal{A}(G)$ with multiplicity one. This can be seen from Theorems 1.7.1 and 1.6.1(6) of [KMSW]. The proofs for these theorems are still work in progress. For $n = 3$ the endoscopic classification was completed for all inner forms in [Rog90], and hence our results are unconditional at least for $n = 3$.

We write $\text{Aut}(\mathbb{C})$ for the group of automorphisms of \mathbb{C} . Let $\tau \in \text{Aut}(\mathbb{C})$. For a complex representation Π of a group \mathcal{G} on the space V_Π of Π , let ${}^\tau\Pi$ be the representation of \mathcal{G} defined by ${}^\tau\Pi(g) = t \circ \Pi(g) \circ t^{-1}$, where $t : V_\Pi \rightarrow V_\Pi$ is a τ -linear isomorphism. Note that the isomorphism class of ${}^\tau\Pi$ is independent of the choice of t . Given $\varphi \in \mathcal{A}(G)$, we define ${}^\tau\varphi \in \mathcal{A}(G)$ by ${}^\tau\varphi(g) = \tau(\varphi(g))$ for $g \in G(\mathbb{A})$. The representation ${}^\tau\pi$ is realized in the space $\{{}^\tau\varphi \mid \varphi \in \pi\}$ by the multiplicity one for unitary groups. Similarly, ${}^\tau\sigma$ is an automorphic representation of $H(\mathbb{A})$.

Proposition 4.5. *Suppose that every irreducible tempered automorphic representation of $\mathbf{G}(\mathbb{A})$ occurs in the decomposition of the space $\mathcal{A}(\mathbf{G})$ with multiplicity one. Let π and σ be irreducible tempered automorphic representations of definite unitary groups $G(\mathbb{A})$ and $H(\mathbb{A})$. If $\text{Hom}_{H(\mathbb{Q}_v)}(\pi_v \otimes \sigma_v, \mathbb{C}) \neq \{0\}$ for all v , then for every $\tau \in \text{Aut}(\mathbb{C})$*

$$\tau \left(\frac{L(\frac{1}{2}, \pi \times \sigma)}{\sqrt{D_E^a} L(1, \pi, \text{Ad}) L(1, \sigma, \text{Ad})} \right) = \frac{L(\frac{1}{2}, {}^\tau\pi \times {}^\tau\sigma)}{\sqrt{D_E^a} L(1, {}^\tau\pi, \text{Ad}) L(1, {}^\tau\sigma, \text{Ad})},$$

where $a = 0$ if $n \equiv 0, 3 \pmod{4}$ and $a = 1$ if $n \equiv 1, 2 \pmod{4}$.

Proof. It is evident that

$$\tau((\Phi, \Phi')_{\mathcal{K} \times \mathcal{K}'}) = ({}^\tau\Phi, {}^\tau\Phi')_{\mathcal{K} \times \mathcal{K}'}, \quad \tau(\mathcal{P}_{\mathcal{K}'}(\Phi)) = \mathcal{P}_{\mathcal{K}'}({}^\tau\Phi)$$

(cf. (2.1)). We take \underline{k} and \underline{k}' so that $\pi_\infty \simeq \mathcal{L}_{\underline{k}^\vee}(\mathbb{C})$ and $\sigma_\infty \simeq \mathcal{L}_{\underline{k}'^\vee}(\mathbb{C})$. It is easy to see that π is spanned by $f_{\mathbf{v}}$ with $\mathbf{v} \in \mathcal{L}_{\underline{k}^\vee}(\mathbb{Q})$ and $\overline{\mathbb{Q}}$ -rational $\mathcal{L}_{\underline{k}}(\mathbb{C})$ -valued modular forms f defined in Definition 2.2 (see [Gro99]). One may therefore assume that modular forms Φ and Φ' are $\overline{\mathbb{Q}}$ -rational, namely, they have values in $\overline{\mathbb{Q}}$. Then $(\Phi, \Phi')_{\mathcal{K} \times \mathcal{K}'}$, $\mathcal{P}_{\mathcal{K}'}(\Phi)$ and $\mathcal{P}_{\mathcal{K}'}(\Phi')$ are algebraic numbers.

Given a matrix coefficient ϕ_l of Π_l , we define the matrix coefficient ${}^\tau\phi_l$ of ${}^\tau\Pi_l$ by ${}^\tau\phi_l(g_l) = \tau(\phi_l(g_l))$ for $g_l \in G(\mathbb{Q}_l)$. Put

$$\phi_l(g) = \langle \Pi_l(g_l) \mathbf{W}_l, \mathbf{W}'_l \rangle_l.$$

Since $\text{Hom}_{H(\mathbb{Q}_l)}(\Pi_l, \mathbb{C}) \neq \{0\}$ if and only if I is not zero on $\Pi_l \otimes \Pi_l^\vee$ by Théorème 14.3.1 of [BP16], the assumption allows us to choose $\Phi = \otimes_v \mathbf{W}_v$ and $\Phi' = \otimes_v \mathbf{W}'_v$ so that

$$I(\phi_l) := I(\mathbf{W}_l, \mathbf{W}'_l) \neq 0$$

for all rational primes l . If we write

$$(\Pi(g)\Phi, \Phi')_{\mathcal{K} \times \mathcal{K}'} = \langle \mathbf{W}_\infty^H, \mathbf{W}'_\infty^{H'} \rangle_\infty \prod_l \phi_l(g_l),$$

then $({}^\tau\Pi(g){}^\tau\Phi, {}^\tau\Phi')_{\mathcal{K} \times \mathcal{K}'} = \langle \mathbf{W}_\infty^H, \mathbf{W}'_\infty^{H'} \rangle_\infty \prod_l {}^\tau\phi_l(g_l)$ for $g = (g_l) \in \mathbf{G}(\widehat{\mathbb{Q}})$. One can easily show that $\tau(I(\phi_l)) = I({}^\tau\phi_l)$ (cf. the proofs of [Gro18, Theorem A] and [Liu, Lemma 2.6]). Since $\tau(\mathcal{L}(\pi_l, \sigma_l)) = \mathcal{L}({}^\tau\pi_l, {}^\tau\sigma_l)$ by [Liu, Lemma 2.4], we get

$$\tau \left(\frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \text{Ad}) L(1, \sigma, \text{Ad})} \right) = \frac{\mathcal{P}_{\mathcal{K}'}({}^\tau\Phi) \mathcal{P}_{\mathcal{K}'}({}^\tau\Phi')}{({}^\tau\Phi, {}^\tau\Phi')_{\mathcal{K} \times \mathcal{K}'}} c_\infty^{-1} \frac{2^{\varkappa_\pi + \varkappa_\sigma}}{\tau(\gamma_G)} \prod_l \frac{\mathcal{L}({}^\tau\pi_l, {}^\tau\sigma_l) {}^\tau\varphi_{\Pi_l}(1)}{I({}^\tau\phi_l)}$$

$$= \frac{\gamma_G L\left(\frac{1}{2}, \tau\pi \times \tau\sigma\right)}{\tau(\gamma_G) L(1, \tau\pi, \text{Ad}) L(1, \tau\sigma, \text{Ad})},$$

applying τ to the formula in Corollary 4.4. The proof is complete by (4.1). \square

4.4. An outer involution. To relate the central value to the square of the period, we introduce an involution of automorphic forms on unitary groups. Since T has entries in \mathbb{Q} , we can define an outer automorphism of G by

$${}^c : g \mapsto g^c = T^{-1} {}^t g^{-1} T.$$

Take $\xi_{\mathfrak{N}} = (\xi_{\mathfrak{N},v}) \in G(\widehat{\mathbb{Q}})$ so that

$$(\xi_{\mathfrak{N}}^{-1} \mathcal{K}_0(\mathfrak{N}) \xi_{\mathfrak{N}})^c = \mathcal{K}_0(\mathfrak{N}).$$

For a function φ on $G(\mathbb{A})$, we define a function $\varphi^c : G(\mathbb{A}) \rightarrow \mathbb{C}$ by $\varphi^c(g) = \varphi(g^c)$.

Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible automorphic representation of $G(\mathbb{A})$ such that π_q admits a \mathcal{K}_q -invariant vector for every non-split prime q . For each split prime l we denote the conductor of π_l by $c(\pi_l)$ in the sense of (6.1). Let $N_\pi = \prod_l l^{c(\pi_l)}$ be the conductor of π and $N = \prod_{l \neq p} l^{c(\pi_l)}$ the tame conductor of π . Fix ideals \mathfrak{N}_π and \mathfrak{N} of \mathfrak{r} such that $\mathfrak{r}/\mathfrak{N}_\pi \simeq \mathbb{Z}/N_\pi \mathbb{Z}$ and $\mathfrak{r}/\mathfrak{N} \simeq \mathbb{Z}/N \mathbb{Z}$.

Let $\pi^\vee \simeq \otimes'_v \pi_v^\vee$ denote the contragredient representation of π . We define representations π^c of $G(\mathbb{A})$ and π_v^c of $G(\mathbb{Q}_v)$ as the twists $\pi^c(g) = \pi(g^c)$ and $\pi_v^c(g_v) = \pi(g_v^c)$ by c for $g \in G(\mathbb{A})$ and $g_v \in G(\mathbb{Q}_v)$. It is well-known that $\pi_v^\vee \simeq \pi_v^c$ (see [MVW87]). Since

$$\varphi^c(gh) = \varphi((gh)^c) = (\pi^c(h)\varphi)^c(g),$$

and $\pi^\vee \simeq \pi^c$, we have $\{\bar{\varphi} \mid \varphi \in \pi\} = \{\varphi^c \mid \varphi \in \pi\}$ by the global multiplicity one for unitary groups ([Rog90, Theorem 13.3.1]), where the automorphic form $\bar{\varphi}$ is defined by $\bar{\varphi}(g) = \varphi(g)$. Let $\varphi_\pi = \otimes_v \varphi_v \in \pi$ be a highest weight essential vector with respect to $\mathcal{K}_0(\mathfrak{N}_\pi)$, namely, φ_∞ is a highest weight vector, φ_q is a \mathcal{K}_q -invariant vector for inert primes q , and φ_l is an essential vector for split primes l (see Definition 6.1). Then $\pi^\vee(\xi_{\mathfrak{N}_\pi})\varphi_\pi^c \in \pi^\vee$ is a lowest weight essential vector with respect to $\mathcal{K}_0(\mathfrak{N}_\pi)$ up to scalar by Proposition 6.2.

Assume that $H^c = H$. Similarly we associate $\varphi^c \in \mathcal{A}(H)$, $\Phi^c \in \mathcal{A}(\mathbf{G})$ and the automorphic representation $\sigma^c \simeq \otimes'_v \sigma_v^c$ to automorphic forms $\varphi \in \mathcal{A}(H)$, $\Phi \in \mathcal{A}(\mathbf{G})$ and an automorphic representation $\sigma \simeq \otimes'_v \sigma_v$ of $H(\mathbb{A})$.

4.5. A factorization of the dual representation. Define the longest Weyl element $w_n \in \text{GL}_n(F)$ by

$$w_1 = 1, \quad w_n = \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ w_{n-1} & & & \end{bmatrix} \quad (n \geq 2).$$

Let l be a split rational prime. We view π_l as a representation of $\text{GL}_n(\mathbb{Q}_l)$ via ι_l . We identify π_l with its Whittaker model $\mathcal{W}_{\psi_l}(\pi)$ with respect to ψ_l and identify π_l^\vee with its Whittaker model $\mathcal{W}_{\psi_l^{-1}}(\pi^\vee)$ with respect to ψ_l^{-1} . Let $W_{\pi_l} \in \pi_l$ be the normalized essential Whittaker vector with respect to ψ_l , and $W_{\pi_l^\vee} \in \pi_l^\vee$ the normalized essential Whittaker vector with respect to ψ_l^{-1} . For $W \in \pi_l$ we define $W^c \in \pi_l^\vee$ by

$$W^c(g) = \pi_l^\vee(T^{-1})\widetilde{W}(g) = W(w_n {}^t g^{-1} T),$$

where $\widetilde{W}(g) = W(w_n {}^t g^{-1})$. It is important to note that

$$\pi_l^\vee(h)W^c(g) = W^c(gh) = W((gh)^c) = (\pi_l^c(h)W)^c(g).$$

When q remains a prime in \mathfrak{r} , we fix \mathcal{K}_q -invariant vectors $W_{\pi_q} \in \pi_q$ and $W_{\pi_q^\vee} \in \pi_q^\vee$. Fix a highest weight vector $W_{\pi_\infty} \in \pi_\infty$ and a lowest weight vector $W_{\pi_\infty^\vee} \in \pi_\infty^\vee$. Let ${}^c : \pi_v^c \simeq \pi_v^\vee$ be the $G(\mathbb{Q}_v)$ -equivariant isomorphism determined by $W_{\pi_v}^c = W_{\pi_v^\vee}$ for inert places v .

To apply Corollary 4.4 to $\Phi' = \Phi^c$, we need explicate the factorization of Φ^c . Fix isomorphisms $\pi \simeq \otimes'_v \pi_v$ and $\pi^\vee \simeq \otimes'_v \pi_v^\vee$ so that

$$\varphi_\pi = \otimes'_v W_{\pi_v}, \quad \varphi_\pi^c = \otimes'_v W_{\pi_v}^c.$$

Using this factorization $\pi^\vee \simeq \otimes'_v \pi_v^\vee$, we define a cusp form $\varphi_{\pi^\vee} \in \pi^\vee$ by

$$\varphi_{\pi^\vee} = \otimes'_v W_{\pi_v^\vee}.$$

Lemma 4.6. *If $\varphi = \otimes'_v W_v \in \pi$ is factorizable and if $W_v = W_{\pi_v}$ for all non-split places v , then $\varphi^c = \otimes'_v W_v^c$.*

Proof. Define a finite set $\mathfrak{S}_\varphi = \{v \mid W_v \neq W_{\pi_v}\}$. For $v \in \mathfrak{S}_\varphi$ there are $c_{v,i} \in \mathbb{C}$ and $g_{v,i} \in G(\mathbb{Q}_v)$ such that

$$W_v = \mathcal{U}_v W_{\pi_v}, \quad \mathcal{U}_v = \sum_i c_{v,i} \pi_v(g_{v,i}).$$

We have $\varphi = \otimes_{v \in \mathfrak{S}_\varphi} \mathcal{U}_v \cdot \varphi_\pi$. Put $\mathcal{V}_v = \sum_i c_{v,i} \pi_v^\vee(g_{v,i})$. Then

$$\varphi^c = \otimes_{v \in \mathfrak{S}_\varphi} \mathcal{V}_v \cdot \varphi_\pi^c = (\otimes_{v \in \mathfrak{S}_\varphi} \mathcal{V}_v W_{\pi_v}^c) \otimes (\otimes_{v \notin \mathfrak{S}_\varphi} W_{\pi_v}^c).$$

Lemma 4.6 follows from the observation that $\mathcal{V}_v W_{\pi_v}^c = (\mathcal{U}_v W_{\pi_v})^c = W_v^c$. \square

Let $\sigma \simeq \otimes'_v \sigma_v$ be an irreducible automorphic representation of $H(\mathbb{A})$ such that σ_q admits \mathcal{K}_q' -invariant vector for every non-split prime q . Put $N' = \prod_{l \neq p} l^{c(\sigma_l)}$. We take an ideal \mathfrak{N}' of \mathfrak{r} such that $\mathfrak{r}/\mathfrak{N}' \simeq \mathbb{Z}/N'\mathbb{Z}$. Define an open compact subgroup $\mathcal{K}'_0(p^b \mathfrak{N}')$ of $H(\widehat{\mathbb{Q}})$ and take $\xi'_{\mathfrak{N}'} \in H(\mathbb{A})$ in a similar way. Let $\varphi_\sigma \in \sigma$ be a highest weight essential vector with respect to $\mathcal{K}'_0(\mathfrak{N}')$. Fix the factorizations $\sigma \simeq \otimes'_v \sigma_v$ and $\sigma^\vee \simeq \otimes'_v \sigma_v^\vee$ such that

$$\varphi_\sigma = \otimes'_v W_{\sigma_v}, \quad \varphi_\sigma^c = \otimes'_v W_{\sigma_v}^c.$$

Using the factorization, we define a cusp form $\varphi_{\sigma^\vee} \in \sigma^\vee$ by $\varphi_{\sigma^\vee} = \otimes'_v W_{\sigma_v^\vee}$.

For each place v we put

$$\mathbf{W}_{\Pi_v} = W_{\pi_v} \otimes W_{\sigma_v} \in \Pi_v, \quad \mathbf{W}_{\Pi_v^\vee} = W_{\pi_v^\vee} \otimes W_{\sigma_v^\vee} \in \Pi_v^\vee$$

and define the $G(\mathbb{Q}_v)$ -equivariant isomorphism ${}^c : \Pi_v^c \simeq \Pi_v^\vee$ by

$$(W \otimes W')^c = W^c \otimes W'^c.$$

Put

$$\Phi_\Pi = \varphi_\pi \otimes \varphi_\sigma, \quad \Phi_{\Pi^\vee} = \varphi_{\pi^\vee} \otimes \varphi_{\sigma^\vee}, \quad I_\infty = c_\infty \frac{I(\mathbf{W}_\infty^H, \mathbf{W}_\infty^{Hc})}{\langle\langle \mathbf{W}_{\Pi_\infty}, \mathbf{W}_{\Pi_\infty^\vee} \rangle\rangle_\infty}.$$

Since $\mathcal{P}(\Phi^c) = \mathcal{P}(\Phi)$, one can deduce the following formula from Lemma 4.6 and Corollary 4.4.

Corollary 4.7. Let Π be an irreducible tempered cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$ such that Π_q is unramified for all non-split rational primes q and whose archimedean part Π_∞ satisfies (H_2) . If $\Phi = \otimes'_v \mathbf{W}_v \in \Pi$ is factorizable and $\mathbf{W}_\infty = \mathbf{W}_\infty^H$, then

$$\frac{\mathcal{P}_{\mathcal{K}'}(\Phi)^2}{(\Phi_\Pi, \Phi_{\Pi^\vee})_{\mathcal{K} \times \mathcal{K}'}} = \frac{\gamma_G}{2^{\kappa_\pi + \kappa_\sigma}} I_\infty \frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \text{Ad})L(1, \sigma, \text{Ad})} \prod_l \frac{I(\mathbf{W}_l, \mathbf{W}_l^c)}{\mathcal{L}(\pi_l \times \sigma_l) \langle \langle \mathbf{W}_{\Pi_l}, \mathbf{W}_{\Pi_l^\vee} \rangle \rangle_l}.$$

4.6. Local integrals at ∞ , Σ_E^r and Σ_T^- . From now on we suppose that G and H are definite unitary groups in three and two variables. We retain the notation in §3.2. Proposition A.7 below gives

$$I_\infty = (-1)^{k'_1 - k'_2} 2^{-2} (\dim \pi_\infty) (\dim \sigma_\infty).$$

If $q \notin \Sigma_T^-$, if q and pNN' are coprime, and if $\mathbf{W}_q = \mathbf{W}_{\Pi_q}$ is spherical, then since $\mathbf{W}_q^c = \mathbf{W}_{\Pi_q^\vee}$, we have

$$I(\mathbf{W}_q, \mathbf{W}_q^c) = \mathcal{L}(\pi_q \times \sigma_q) \langle \langle \mathbf{W}_{\Pi_q}, \mathbf{W}_{\Pi_q^\vee} \rangle \rangle_q$$

by Theorem 2.12 of [Har14] and Proposition B.2 below (cf. Remark 5.3).

When $q \in \Sigma_T^-$, we let $\mathbf{W}_q = \mathbf{W}_{\pi_q}$ be a spherical spherical vector. Recall that σ_q is the trivial representation of $H(\mathbb{Q}_q)$. Put

$$I_- = \prod_{q \in \Sigma_T^-} \frac{I(\mathbf{W}_q, \mathbf{W}_q^c)}{\mathcal{L}(\pi_q \times \sigma_q) \langle \langle \mathbf{W}_{\Pi_q}, \mathbf{W}_{\Pi_q^\vee} \rangle \rangle_q}.$$

Proposition C.1 shows that

$$(4.2) \quad I_- = \prod_{q \in \Sigma_T^- \setminus \Sigma_E^r} L(1, \epsilon_{E_q/\mathbb{Q}_q})^2.$$

Since λ_q is either $\frac{1}{2}$ or 1 according as q is ramified in E or not, we have

$$\gamma_G = D_E^{-3} 2^{5+t_E},$$

where t_E is the number of prime numbers which are ramified in E .

For each split prime l we put

$$\begin{aligned} B_{\pi_l} &= \langle W_{\pi_l}, W_{\pi_l^\vee} \rangle_l, & B_{\sigma_l} &= \langle W_{\sigma_l}, W_{\sigma_l^\vee} \rangle'_l, \\ \mathcal{B}_{\pi_l} &= \frac{\zeta_l(3)}{L^{\text{GL}}(1, \pi_l \times \pi_l^\vee)} B_{\pi_l}, & \mathcal{B}_{\sigma_l} &= \frac{\zeta_l(2)}{L^{\text{GL}}(1, \sigma_l \times \sigma_l^\vee)} B_{\sigma_l} \end{aligned}$$

(cf. (5.3)), where $W_{\sigma_l} \in \sigma_l$ is the normalized essential Whittaker vector with respect to ψ_l^{-1} , $W_{\sigma_l^\vee} \in \sigma_l^\vee$ the normalized essential Whittaker vector with respect to ψ_l , and the local pairings $\langle \cdot, \cdot \rangle_l$ and $\langle \cdot, \cdot \rangle'_l$ are constructed by (5.1). To avoid possible confusion, we recall that

$$L(s, \pi_l, \text{Ad}) = L^{\text{GL}}(s, \pi_l \times \pi_l^\vee), \quad L(s, \sigma_l, \text{Ad}) = L^{\text{GL}}(s, \sigma_l \times \sigma_l^\vee).$$

We regard π_l and σ_l as representations of unitary groups in the left hand side and representations of general linear groups in the right hand side. We denote the Petersson pairings with respect to $d_{\mathcal{K}'}g$ and $d_{\mathcal{K}'}h$ by $(\cdot, \cdot)_{\mathcal{K}}$ and $(\cdot, \cdot)_{\mathcal{K}'}$.

Corollary 4.8. Assumption being as in Corollary 4.4, if $\mathbf{W}_\infty = \mathbf{W}_\infty^H$ and if $\mathbf{W}_l = \mathbf{W}_{\Pi_l}$ is a spherical vector unless pNN' is divisible by l , then

$$\begin{aligned} & \frac{\mathcal{P}_{\mathcal{K}'}(\Phi)^2}{(\varphi_\pi, \varphi_{\pi^\vee})_{\mathcal{K}} \dim \pi_\infty \cdot (\varphi_\sigma, \varphi_{\sigma^\vee})_{\mathcal{K}'} \dim \sigma_\infty} \\ = & (-1)^{k'_1 - k'_2} \frac{2^{3+t_E}}{2^{\varkappa_\pi + \varkappa_\sigma} D_E^3} I_- \frac{L\left(\frac{1}{2}, \pi \times \sigma\right)}{L(1, \pi, \text{Ad})L(1, \sigma, \text{Ad})} \prod_{l|pNN'} \frac{I(\mathbf{W}_l, \mathbf{W}_l^c)}{\mathcal{L}(\pi_l \times \sigma_l) B_{\pi_l} B_{\sigma_l}}. \end{aligned}$$

4.7. An application of the splitting lemma. Fix a split rational prime l . We regard π_l and σ_l as representations of general linear groups via ι_l and ι'_l . Put $\zeta_l(s) = (1 - l^{-s})^{-1}$. Then the splitting lemma stated in §5.2 gives

$$I(W_l \otimes W'_l, W_l^c \otimes W_l'^c) = \zeta_l(1) Z\left(\frac{1}{2}, \pi_l(\varsigma) W_l, W'_l\right) Z\left(\frac{1}{2}, \pi_l^\vee(\varsigma) W_l^c, W_l'^c\right),$$

where

$$\varsigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We regard $H(\mathbb{Q}_p)$ as a subgroup of $G(\mathbb{Q}_p)$ via the embedding ι while we use the embedding $\iota' : \text{GL}_2(\mathbb{Q}_p) \hookrightarrow \text{GL}_3(\mathbb{Q}_p)$ defined by $\iota'(g) = \begin{bmatrix} g & & \\ & & \\ & & 1 \end{bmatrix}$ to define the JPSS integral. Since $\varsigma T^{-1} = \begin{bmatrix} T'^{-1} & & \\ & & \\ & & 1 \end{bmatrix} \varsigma$, we have

$$\begin{aligned} Z\left(\frac{1}{2}, \pi_l^\vee(\varsigma) W_l^c, W_l'^c\right) &= Z\left(\frac{1}{2}, \pi_l^\vee(\varsigma T^{-1}) \widetilde{W}_l, \sigma_l^\vee(T'^{-1}) \widetilde{W}_l'\right) \\ &= Z\left(\frac{1}{2}, \pi_l^\vee(\varsigma) \widetilde{W}_l, \widetilde{W}_l'\right) \\ &= \gamma^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l, \psi_l\right) Z\left(\frac{1}{2}, \pi_l(\varsigma) W_l, W_l'\right) \end{aligned}$$

by the invariance and the functional equation (5.2) of the JPSS integrals. We can rewrite the identity above as

$$\frac{I(W_l \otimes W'_l, W_l^c \otimes W_l'^c)}{\mathcal{L}(\pi_l \times \sigma_l) B_{\pi_l} B_{\sigma_l}} = \frac{\mathcal{I}(W_l \otimes W'_l)}{\mathcal{B}_{\pi_l} \mathcal{B}_{\sigma_l}},$$

where

$$\mathcal{I}(W_l \otimes W'_l) = \gamma^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l, \psi_l\right) \frac{Z\left(\frac{1}{2}, \pi_l(\varsigma) W_l, W_l'\right)^2}{L\left(\frac{1}{2}, \pi_l \times \sigma_l\right)}.$$

Definition 4.9. Put

$$\begin{aligned} \text{Pet}(\pi) &= 2^{\varkappa_\pi} L(1, \pi, \text{Ad}) \prod_{l|pN} \mathcal{B}_{\pi_l}; & \text{Pet}(\sigma) &= 2^{\varkappa_\sigma} L(1, \sigma, \text{Ad}) \prod_{l|pN'} \mathcal{B}_{\sigma_l}, \\ \eta_{\varphi_\pi} &= (\varphi_\pi, \varphi_{\pi^\vee})_{\mathcal{K}} \dim \pi_\infty; & \eta_{\varphi_\sigma} &= (\varphi_\sigma, \varphi_{\sigma^\vee})_{\mathcal{K}'} \dim \sigma_\infty. \end{aligned}$$

Since we can replace pN (resp. pN') by pNN' in the definition of $\text{Pet}(\pi)$ (resp. $\text{Pet}(\sigma)$) by (5.3), we can rewrite Corollary 4.8 in the following way:

Corollary 4.10. Assumption being as in Corollary 4.8, we have

$$\frac{\mathcal{P}_{\mathcal{K}'}(\Phi)^2}{\eta_{\varphi_\pi} \eta_{\varphi_\sigma}} = (-1)^{k'_1 - k'_2} \frac{2^{3+t_E}}{D_E^3} I_- \frac{L\left(\frac{1}{2}, \pi \times \sigma\right)}{\text{Pet}(\pi)\text{Pet}(\sigma)} \prod_{l|pNN'} \mathcal{I}(\mathbf{W}_l).$$

4.8. The local integral at a ramified split prime l . Let $l \neq p$ be a rational prime that is split in E . We write ω_{σ_l} for the central character of σ_l , which is viewed as a character of \mathbb{Q}_l^\times via ι_l . Observe that

$$\mathcal{I}(W_l \otimes W'_l) = \varepsilon^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l, \psi_l\right) \left(\frac{Z\left(\frac{1}{2}, \pi_l(\varsigma)W_l, W'_l\right)}{L^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l\right)}\right)^2.$$

If σ_l is unramified, then Théorème on p. 208 of [JPSS81] gives

$$(4.3) \quad Z(s, W_{\pi_l}, W_{\sigma_l}) = L^{\text{GL}}(s, \pi_l \times \sigma_l).$$

Put $\varsigma_l = \iota_l^{-1}(\varsigma)$. We see that

$$\mathcal{I}(\pi_l(\varsigma_l)W_{\pi_l} \otimes W_{\sigma_l}) = \varepsilon^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l, \psi_l\right) =: \mathfrak{f}_{\pi_l, \sigma_l}.$$

If σ_l is ramified, then we put $f_l = c(\omega_{\sigma_l})$ and define $\mathbf{U}^{\omega_{\sigma_l}} W_{\pi_l} \in \mathcal{W}_{\psi_l}(\pi_l)$ by

$$\begin{aligned} \mathbf{U}^{\omega_{\sigma_l}} W_{\pi_l} &= \sum_{i, j \in (\mathbb{Z}_l / l^{f_l} \mathbb{Z}_l)^\times} \sum_{y \in \mathbb{Z}_l / l^{2f_l} \mathbb{Z}_l} \omega_{\sigma_l}(ij) \pi_l \left(\begin{bmatrix} 1 & \frac{i}{l^{f_l}} & \frac{y}{l^{2f_l}} \\ 0 & 1 & \frac{j}{l^{f_l}} \\ 0 & 0 & 1 \end{bmatrix} \right) W_{\pi_l}, \\ \mathbf{U}^{\omega_{\sigma_l}} W_{\pi_l} &= \pi_l(l'(\xi_{\mathfrak{N}'}^l w_2)) \mathbf{U}^{\omega_{\sigma_l}} W_{\pi_l}. \end{aligned}$$

Proposition 6.8 below shows that if $c(\sigma_l) \leq 2c(\omega_{\sigma_l})$, then

$$Z(s, \mathbf{U}^{\omega_{\sigma_l}} W_{\pi_l}, W_{\sigma_l}) = \frac{l^{3c(\omega_{\sigma_l})} \omega_{\sigma_l}(l)^{2c(\omega_{\sigma_l})} \varepsilon\left(\frac{1}{2}, \sigma_l, \psi_l\right)}{\varepsilon\left(\frac{1}{2}, \omega_{\sigma_l}, \psi_l\right)^2 [\text{GL}_2(\mathbb{Z}_l) : \mathcal{K}_0^{(2)}(l^{2c(\omega_{\sigma_l})} \mathbb{Z}_l)]}.$$

Put

$$\mathfrak{f}_{\pi_l, \sigma_l} = \varepsilon^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l, \psi_l\right) \frac{l^{6c(\omega_{\sigma_l})} \omega_{\sigma_l}(l)^{4c(\omega_{\sigma_l})} \varepsilon\left(\frac{1}{2}, \sigma_l, \psi_l\right)^2}{\varepsilon\left(\frac{1}{2}, \omega_{\sigma_l}, \psi_l\right)^4 L^{\text{GL}}\left(\frac{1}{2}, \pi_l \times \sigma_l\right)^2}.$$

It follows that

$$(4.4) \quad \mathcal{I}(\pi_l(\varsigma_l) \mathbf{U}^{\omega_{\sigma_l}} W_{\pi_l}, W_{\sigma_l}) = \frac{\mathfrak{f}_{\pi_l, \sigma_l}}{[\text{GL}_2(\mathbb{Z}_l) : \mathcal{K}_0^{(2)}(l^{2c(\omega_{\sigma_l})} \mathbb{Z}_l)]^2}.$$

4.9. The local integral at p and the modified p -factor. Assume that π and σ are an irreducible p -ordinary automorphic representations with respect to ι_p , namely, π_p is the irreducible generic constituent of a principal series $I(\nu_p, \rho_p, \mu_p)$, and σ_p is the irreducible generic constituent of $I(\mu'_p, \nu'_p)$, where $\nu_p, \rho_p, \mu_p; \mu'_p, \nu'_p$ are $\overline{\mathbb{Q}}^\times$ -valued smooth characters of \mathbb{Q}_p^\times , such that

$$p^{-k_3-1} \nu_p(p), \quad p^{-k_2} \rho_p(p), \quad p^{-k_1+1} \mu_p(p); \quad p^{-k_2-\frac{1}{2}} \mu'_p(p), \quad p^{-k_1+\frac{1}{2}} \nu'_p(p)$$

are p -units with respect to ι_p .

Definition 4.11. Define the modified p -factor $\mathcal{E}(\pi_p, \sigma_p)$ by

$$\mathcal{E}(\pi_p, \sigma_p)^{-1} = L\left(\frac{1}{2}, \pi_p \times \sigma_p\right) \gamma\left(\frac{1}{2}, \mu_p \nu'_p, \psi_p\right) \gamma\left(\frac{1}{2}, \rho_p \nu'_p, \psi_p\right) \gamma\left(\frac{1}{2}, \mu_p \mu'_p, \psi_p\right)$$

$$\times \gamma\left(\frac{1}{2}, (\nu_p \nu'_p)^{-1}, \psi_p^{-1}\right) \gamma\left(\frac{1}{2}, (\rho_p \mu'_p)^{-1}, \psi_p^{-1}\right) \gamma\left(\frac{1}{2}, (\nu_p \mu'_p)^{-1}, \psi_p^{-1}\right).$$

Let $W_{\pi_p}^{\text{ord}} \in \pi_p$ and $W_{\sigma_p}^{\text{ord}} \in \sigma_p$ the normalized p -ordinary Whittaker functions in (5.5). These are eigenvectors of the U_p -operators with specified eigenvalues (see Proposition 5.4, Remarks 2.5 and 2.9 below). Put

$$\alpha_{\pi_p} = p^2 \rho_p(p) \mu_p(p)^2, \quad \alpha_{\sigma_p} = p^{1/2} \nu'_p(p).$$

Then

$$U_p W_{\pi_p}^{\text{ord}} = \alpha_{\pi_p} W_{\pi_p}^{\text{ord}}, \quad U_p W_{\sigma_p}^{\text{ord}} = \alpha_{\sigma_p} W_{\sigma_p}^{\text{ord}}$$

by Proposition 5.4. Define an element $\mathbf{t}_\ell \in G(\mathbb{Q}_p)$ by

$$\mathbf{t}_\ell = i_p^{-1} \left(\begin{bmatrix} 0 & 0 & -p^{-\ell} \\ 0 & p^\ell & 1 \\ p^{2\ell} & p^\ell & 0 \end{bmatrix} \right).$$

If ℓ is sufficiently large, then Proposition 5.10 below gives

$$(4.5) \quad \begin{aligned} & \left(\frac{\zeta_p(1)}{\zeta_p(2)(p^{-5}\alpha_{\pi_p}\alpha_{\sigma_p})^\ell} \right)^2 \mathcal{S}(\pi_p(\mathbf{t}_\ell) W_{\pi_p}^{\text{ord}} \otimes W_{\sigma_p}^{\text{ord}}) \\ &= \frac{\gamma^{\text{GL}}(\frac{1}{2}, \pi_p \times \sigma_p, \psi_p)}{L(\frac{1}{2}, \pi_p \times \sigma_p) (\gamma(\frac{1}{2}, \mu_p \nu'_p, \psi_p) \gamma(\frac{1}{2}, \rho_p \nu'_p, \psi_p) \gamma(\frac{1}{2}, \mu_p \mu'_p, \psi_p))^2} \\ &= \frac{\gamma(\frac{1}{2}, \nu_p \nu'_p, \psi_p) \gamma(\frac{1}{2}, \rho_p \mu'_p, \psi_p) \gamma(\frac{1}{2}, \nu_p \mu'_p, \psi_p)}{L(\frac{1}{2}, \pi_p \times \sigma_p) \gamma(\frac{1}{2}, \mu_p \nu'_p, \psi_p) \gamma(\frac{1}{2}, \rho_p \nu'_p, \psi_p) \gamma(\frac{1}{2}, \mu_p \mu'_p, \psi_p)} \\ &= (\rho_p \nu'_p)(-1) \mathcal{E}(\pi_p, \sigma_p) \end{aligned}$$

in view of the multiplicativity and functional equation of the gamma factor.

4.10. An explicit central value formula. We say that an irreducible representation of $G(\mathbb{Q}_q)$ (resp. $H(\mathbb{Q}_q)$) is unramified if it admits a non-zero \mathcal{K}_q (resp. \mathcal{K}'_q) invariant vector. Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible tempered automorphic representation of $G(\mathbb{A})$ and $\sigma \simeq \otimes'_v \sigma_v$ that of $H(\mathbb{A})$ satisfying the following conditions:

- (H₀) if $q \in \Sigma_T^-$, then q is odd and σ_q is the trivial representation;
- (H₁) π_q is unramified for every non-split rational prime q ;
 σ_q is unramified for every non-split rational prime $q \notin \Sigma_T^-$;
- (H₂) π_∞ and σ_∞ are discrete series such that

$$\text{Hom}_{H(\mathbb{R})}(\pi_\infty \otimes \sigma_\infty, \mathbb{C}) \neq \{0\};$$

- (H₃) $c(\sigma_l) \leq 2c(\omega_{\sigma_l})$ for every split rational prime $l \neq p$;
- (H₄) π_p is a generic constituent of a principal series $I(\nu_p, \rho_p, \mu_p)$;
 σ_p is a generic constituent of a principal series $I(\mu'_p, \nu'_p)$.

Recall that $N = \prod_{l \neq p} l^{c(\pi_l)}$ and $N' = \prod_{l \neq p} l^{c(\sigma_l)}$, where we set $c(\pi_l) = c(\sigma_l) = 0$ for non-split rational primes l . Put $M = \prod_{l|N'} l^{c(\omega_{\sigma_l})}$. Take a divisor \mathfrak{M} of \mathfrak{N}' such that $\mathfrak{r}/\mathfrak{M} \simeq \mathbb{Z}/M\mathbb{Z}$. Let $\Phi^\dagger \in \Pi$ be an ordinary $H(\mathbb{R})$ -invariant essential vector. Define $\mathbf{U}_{\mathfrak{N}'}^{\omega_\sigma} \Phi^\dagger \in \Pi$ by

$$\mathbf{U}_{\mathfrak{N}'}^{\omega_\sigma} \Phi^\dagger = \mathbf{W}_\infty^H \otimes (W_{\pi_p}^{\text{ord}} \otimes W_{\sigma_p}^{\text{ord}}) \otimes (\otimes_{q|pN'} \mathbf{W}_{\Pi_q}) \otimes (\otimes_{l|N'} \mathbf{U}_{\mathfrak{N}'}^{\omega_{\sigma_l}} W_{\pi_l} \otimes W_{\sigma_l}).$$

Recall the element $\zeta^{(p)} \in G(\widehat{\mathbb{Q}})$ defined in §3.1. We combine Corollary 4.10 and (4.4), (4.5) to obtain the following formula.

Proposition 4.12. *Notations and assumptions being as above, if ℓ is sufficiently large, then*

$$\begin{aligned} & \frac{1}{\eta_{\varphi\pi}\eta_{\varphi\sigma}} \left(\frac{\zeta_p(1)}{\zeta_p(2)} [\mathcal{K}' : \mathcal{K}'_0(\mathfrak{M}^2)] \frac{\mathcal{P}_{\mathcal{K}'}(\pi(\mathbf{t}_\ell \zeta^{(p)})) \mathbf{U}_{\mathfrak{M}'}^{\omega_\sigma} \Phi^\dagger}{(p^{-5}\alpha_{\pi_p}\alpha_{\sigma_p})^\ell} \right)^2 \\ &= \frac{L(\frac{1}{2}, \pi \times \sigma)}{\text{Pet}(\pi)\text{Pet}(\sigma)} (-1)^{k'_1 - k'_2} (\rho_p \nu'_p) (-1) \mathcal{E}(\pi_p, \sigma_p) \frac{2^{3+t_E}}{D_E^3} L_- \prod_{l|NN'} f_{\pi_l, \sigma_l}. \end{aligned}$$

4.11. Statement of the main result.

Definition 4.13 (p -modified period). Let $\mathbf{f} \in e_{\text{ord}} \mathbf{S}^{U(n)}(\mathfrak{N}, \chi, \mathcal{R})$ be a Hida family. Letting $\mu_1, \mu_2, \dots, \mu_n$ be chosen for $\pi_{\underline{Q}}$ as in Definition 2.3, we define the modified adjoint p -factor $\mathcal{E}(\pi_{\underline{Q}, p}, \text{Ad}) = \mathcal{E}(\pi_{\underline{Q}, p}, \text{Ad}, \psi_p)$ as in Definition 6.5 for $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+$. The subset $\mathfrak{Y}_{\mathbf{f}}$ consists of $\underline{Q} \in \mathfrak{X}_{\mathbf{f}}$ such that $\underline{Q}(\mathbf{f})$ is new outside p . For $\underline{Q} \in \mathfrak{Y}_{\mathbf{f}}$ we define the p -modified period by

$$\begin{aligned} \Omega_{\underline{Q}(\mathbf{f})}^\dagger &= [\mathcal{K} : \mathcal{K}_0(\mathfrak{N}_{\pi_{\underline{Q}}})] \text{Pet}(\pi_{\underline{Q}}) \mathcal{E}(\pi_{\underline{Q}, p}, \text{Ad}) \\ &= [\mathcal{K} : \mathcal{K}_0(\mathfrak{N}_{\pi_{\underline{Q}}})] 2^{2\pi_{\underline{Q}}} L(1, \pi_{\underline{Q}}, \text{Ad}) \mathcal{E}(\pi_{\underline{Q}, p}, \text{Ad}) \prod_{l|pN} \mathcal{B}_{\pi_{\underline{Q}, l}}, \end{aligned}$$

and define the Gross period by

$$\Omega_{\underline{Q}(\mathbf{f})} = \frac{\Omega_{\underline{Q}(\mathbf{f})}^\dagger}{Q(\eta_{\mathbf{f}})}.$$

Let

$$\mathbf{f} \in e_{\text{ord}} \mathbf{S}^G(\mathfrak{N}, \chi, \mathcal{R}); \quad \mathbf{g} \in e_{\text{ord}} \mathbf{S}^H(\mathfrak{N}', \chi', \mathcal{R}')$$

be Hida families. For $\underline{Q} \in \mathfrak{X}_{\mathbf{f}}$ and $\underline{Q}' \in \mathfrak{X}_{\mathbf{g}}$, let $\pi_{\underline{Q}}$ and $\sigma_{\underline{Q}'}$ be the automorphic representations of $G(\mathbb{A})$ and $H(\mathbb{A})$ associated with $\underline{Q}(\mathbf{f})$ and $\underline{Q}'(\mathbf{g})$ respectively. Choose $\nu_p, \rho_p, \mu_p; \mu'_p, \nu'_p$ for $\pi_{\underline{Q}, p}$ and $\sigma_{\underline{Q}', p}$ as in §4.9. We define the modified p -factor by

$$\mathcal{E}(\text{Fil}^+ \mathbf{V}_{\underline{Q}}) = \mathcal{E}(\pi_{\underline{Q}, p}, \sigma_{\underline{Q}', p}).$$

Theorem 4.14. *Assume that q is odd whenever $H(\mathbb{Q}_q)$ is compact. Let M be the conductor of χ' . Suppose that*

$$(H_3) \quad M^2 \text{ is divisible by } N'.$$

Then there exists a unique element $\mathcal{L}_{\mathbf{f}, \mathbf{g}} \in \text{Frac}(\mathcal{R} \widehat{\otimes}_{\mathcal{O}} \mathcal{R}')$ such that

$$Q(\mathcal{L}_{\mathbf{f}, \mathbf{g}})^2 = \frac{\Gamma(0, \mathbf{V}_{\underline{Q}}) L(0, \mathbf{V}_{\underline{Q}})}{\Omega_{\underline{Q}(\mathbf{f})} \Omega_{\underline{Q}'(\mathbf{g})}} \mathcal{E}(\text{Fil}^+ \mathbf{V}_{\underline{Q}})$$

for $\underline{Q} = (Q, Q') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}} \cap (\mathfrak{Y}_{\mathbf{f}} \times \mathfrak{Y}_{\mathbf{g}})$.

Remark 4.15. The denominator of $\mathcal{L}_{\mathbf{f}, \mathbf{g}}$ is simply a product of explicit local L -factors. For each prime factor l of N' let $P_{\pi_l, \sigma_l} \in \mathcal{R} \widehat{\otimes}_{\mathcal{O}} \mathcal{R}'$ be an element such that $Q(P_{\pi_l, \sigma_l}) = L^{\text{GL}}(\frac{1}{2}, \pi_{\underline{Q}, l} \times \sigma_{\underline{Q}', l})^{-1}$ for $\underline{Q} = (Q, Q') \in \mathfrak{X}_{\mathcal{R}}^+ \times \mathfrak{X}_{\mathcal{R}'}^+$. It follows from the definition (4.6) that

$$\mathcal{L}_{\mathbf{f}, \mathbf{g}} \cdot \prod_{l|N'} P_{\pi_l, \sigma_l} \in \mathcal{R} \widehat{\otimes}_{\mathcal{O}} \mathcal{R}'.$$

4.12. Proof of Theorem 1.1. This subsection supposes that NN' is odd. Let M be an *odd* integer divisibly only by split primes. Fix a primitive character ϱ of $(\mathfrak{t}/\mathfrak{M})^\times$ such that $\chi'\varrho^{-2}$ has conductor M . We extend ϱ to an automorphic character $\varrho_{\mathbb{A}} = \prod_v \varrho_v$ of $\mathrm{U}(1)(\mathbb{A})$.

Since $\underline{Q}'(\mathfrak{g})$ has tame level \mathfrak{N}' for $\underline{Q}' \in \mathfrak{X}_{\mathcal{R}'}^+$, there exists a positive integer A such that if $m \geq A$ and M is divisible by N'^m , then $\sigma_{\underline{Q}'} \otimes \varrho_{\mathbb{A}}$ has tame conductor M^2 (cf. Remark 6.9 and (6.1)) and $(\sigma_{\underline{Q}',l} \otimes \varrho_l)_u$ is trivial for all prime factors l of N' and $\underline{Q}' \in \mathfrak{X}_{\mathcal{R}'}^+$. We enlarge M so that $\pi_{\underline{Q}} \otimes \varrho_{\mathbb{A}}^{-1}$ has tame conductor M^3 and $(\pi_{\underline{Q},l} \otimes \varrho_l^{-1})_u$ is trivial for all prime factors l of N and $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+$.

Fix $(\underline{Q}_0, \underline{Q}'_0) \in \mathfrak{X}_{\mathbb{V}}^{\mathrm{crit}}$. Let

$$f_{\varrho} \in e_{\mathrm{ord}} \mathbf{S}_{k_{\underline{Q}_0}}^G(p^{\ell_{\underline{Q}}} \mathfrak{M}^3, \mathcal{R}, \chi \varrho^3), \quad g_{\varrho} \in e_{\mathrm{ord}} \mathbf{S}_{k_{\underline{Q}'_0}}^H(p^{\ell_{\underline{Q}'}} \mathfrak{M}^2, \mathcal{R}', \chi' \varrho^{-2})$$

be p -ordinary newforms associated to $\pi_{\underline{Q}_0} \otimes \varrho_{\mathbb{A}}^{-1}$ and $\sigma_{\underline{Q}'_0} \otimes \varrho_{\mathbb{A}}$. Theorem 2.13 allows us to lift f_{ϱ} and g_{ϱ} to Hida families

$$\mathbf{f}_{\varrho} \in e_{\mathrm{ord}} \mathbf{S}^G(\mathfrak{M}^3, \mathcal{R}, \chi \varrho^3), \quad \mathbf{g}_{\varrho} \in e_{\mathrm{ord}} \mathbf{S}^H(\mathfrak{M}^2, \mathcal{R}', \chi' \varrho^{-2}).$$

For our choice of ϱ we see that \mathbf{g}_{ϱ} satisfies (H_3) , and

$$\begin{aligned} \mathfrak{Y}_{\mathbf{f}_{\varrho}} &= \mathfrak{X}_{\mathbf{f}_{\varrho}}, & \Omega_{\underline{Q}(\mathbf{f}_{\varrho})}^{\dagger} &= [\mathcal{K}_0(\mathfrak{N}) : \mathcal{K}_0(\mathfrak{M}^3)] \Omega^{(NN')} (V_{\underline{Q}(\mathbf{f})}), \\ \mathfrak{Y}_{\mathbf{g}_{\varrho}} &= \mathfrak{X}_{\mathbf{g}_{\varrho}}, & \Omega_{\underline{Q}'(\mathbf{g}_{\varrho})}^{\dagger} &= [\mathcal{K}'_0(\mathfrak{N}') : \mathcal{K}'_0(\mathfrak{M}^2)] \Omega^{(NN')} (V_{\underline{Q}'(\mathbf{g})}). \end{aligned}$$

Theorem 4.14 applied to \mathbf{f}_{ϱ} and \mathbf{g}_{ϱ} shows that

$$\mathcal{Q}(\mathcal{L}_{\mathbf{f}_{\varrho}, \mathbf{g}_{\varrho}})^2 = \frac{\Gamma(0, \mathbf{V}_{\underline{Q}}) L(0, \mathbf{V}_{\underline{Q}})}{\Omega_{\underline{Q}(\mathbf{f}_{\varrho})} \Omega_{\underline{Q}'(\mathbf{g}_{\varrho})}} \mathcal{E}(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}})$$

for $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbb{V}}^{\mathrm{crit}} \cap (\mathfrak{X}_{\mathbf{f}_{\varrho}} \times \mathfrak{X}_{\mathbf{g}_{\varrho}})$. Therefore

$$L_p(\mathbf{V}) = [\mathcal{K}_0(\mathfrak{N}) : \mathcal{K}_0(\mathfrak{M}^3)] [\mathcal{K}'_0(\mathfrak{N}') : \mathcal{K}'_0(\mathfrak{M}^2)] \frac{\mathcal{L}_{\mathbf{f}_{\varrho}, \mathbf{g}_{\varrho}}^2}{\eta_{\mathbf{f}_{\varrho}} \eta_{\mathbf{g}_{\varrho}}} \in \mathrm{Frac}(\mathcal{R} \widehat{\otimes}_{\mathcal{O}} \mathcal{R}').$$

satisfies the interpolation formula in Theorem 1.1 for $\underline{Q}_0 \in \mathfrak{X}_{\mathbb{V}}^{\mathrm{crit}} \cap (\mathfrak{X}_{\mathbf{f}_{\varrho}} \times \mathfrak{X}_{\mathbf{g}_{\varrho}})$. Since this interpolation formula determines $L_p(\mathbf{V})$, it holds for all $\underline{Q} \in \mathfrak{X}_{\mathbb{V}}^{\mathrm{crit}}$.

4.13. Proof of Theorem 4.14. Recall the theta element $\Theta_{\mathbf{F}}$ defined in §1.7.

Proposition 4.16. *Notations and assumptions being as in Theorem 4.14, we have*

$$\zeta_p(1)^4 \mathcal{Q}(\Theta_{\mathbf{F}})^2 = \frac{L(\frac{1}{2}, \pi_{\underline{Q}} \times \sigma_{\underline{Q}'})}{\Omega_{\underline{Q}(\mathbf{f})} \Omega_{\underline{Q}'(\mathbf{g})}} \mathcal{E}(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}}) \chi'(-1) \frac{2^{3+t_E} D_T^2}{D_E^3} \prod_{l|NN'} \mathfrak{f}_{\pi_{\underline{Q},l}, \sigma_{\underline{Q}',l}}$$

for $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{f}, \mathbf{g}}^{\mathrm{crit}}$, where $D_T = \prod_{q \in \Sigma_T^- \setminus \Sigma_E^r} q$.

Proof. Let $\pi = \pi_{\underline{Q}}$ and $\sigma = \sigma_{\underline{Q}'}$. Hypotheses (H_0) and (H_1) hold by (splt), (H_2) holds by Remark 4.3, and (H_4) holds by [Ger19, Lemma 5.4]. Thanks to Remark 1.2(2), we can apply Proposition 4.12 to $\underline{Q}(\mathbf{f})$ and $\underline{Q}'(\mathbf{g})$. If ℓ is sufficiently large, then Proposition 3.4 yields

$$\zeta_p(1)^4 \mathcal{Q}(\Theta_{\mathbf{F}})^2 = \left(\frac{\zeta_p(1)}{\zeta_p(2)} [\mathcal{K}' : \mathcal{K}'_0(\mathfrak{M}^2)] \frac{\mathcal{P}_{\mathcal{K}'}(\pi(\mathfrak{t}_{\ell} \zeta^{(p)}) \Phi_{\mathbf{F}})}{(p^{-5} \alpha_{\pi_p} \alpha_{\sigma_p})^{\ell}} \right)^2 \prod_{q \in \Sigma_T^- \setminus \Sigma_E^r} (q+1)$$

$$= \eta_{\varphi_\pi} \eta_{\varphi_\sigma} \frac{L\left(\frac{1}{2}, \pi \times \sigma\right)}{\text{Pet}(\pi)\text{Pet}(\sigma)} (-1)^{k'_1 - k'_2} (\rho_p \nu'_p) (-1) \mathcal{E}(\pi_p, \sigma_p) \frac{2^{3+t_E} I_- D_T^2 \prod_{l|NN'} \mathfrak{f}_{\pi_l, \sigma_l}}{D_E^3 \prod_{q \in \Sigma_T^- \setminus \Sigma_E^r} L(1, \epsilon_{E_q/\mathbb{Q}_q})^2}.$$

We can let the stabilization of φ_π be $f_{\mathbf{v}_{k_Q}}$. Then Proposition 2.22, Remark 2.6 and Corollary 6.6 below give

$$\begin{aligned} \frac{\underline{Q}(\eta_{\mathbf{f}})}{\eta_{\varphi_\pi}} &= \frac{[\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N})] \alpha_{\pi_p}^{-\ell} (\tau_{p^\ell \mathfrak{N}} f_{\mathbf{v}_{k_Q}}, f_{\mathbf{v}_{k_Q}}) \mathcal{K}}{(\varphi_\pi, \varphi_\pi^\vee) \mathcal{K}} \\ &= \frac{[\mathcal{K} : \mathcal{K}_0(p^\ell \mathfrak{N})] \alpha_{\pi_p}^{-\ell} \mathcal{B}_{\pi_p^{\text{ord}}}^{[\ell]}}{\mathcal{B}_{\pi_p}} \\ &= [\mathcal{K} : \mathcal{K}_0(\mathfrak{N}_\pi)] \mathcal{E}(\pi_p, \text{Ad}) \rho_p(-1) \end{aligned}$$

Since $1 = \omega_\sigma(-1) = (-1)^{k_{Q'_1} - k_{Q'_2}} (\mu'_p \nu'_p) (-1) \chi'(-1)$, we have

$$\begin{aligned} \frac{\underline{Q}'(\eta_{\mathbf{g}})}{\eta_{\varphi_\sigma}} &= [\mathcal{K}' : \mathcal{K}'_0(\mathfrak{N}_\sigma)] \mathcal{E}(\sigma_p, \text{Ad}) \mu'_p(-1) \\ &= [\mathcal{K}' : \mathcal{K}'_0(\mathfrak{N}_\sigma)] \mathcal{E}(\sigma_p, \text{Ad}) \nu'_p(-1) \chi'(-1) (-1)^{k_{Q'_1} - k_{Q'_2}}. \end{aligned}$$

The stated interpolation formula follows from (4.2). \square

Lemma 4.17. *For each prime factor l of NN' there is an element $\sqrt{\mathfrak{f}_{\pi_l, \sigma_l}} \in \mathcal{R} \widehat{\otimes} \mathcal{R}'$ such that for $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$*

$$\underline{Q}(\sqrt{\mathfrak{f}_{\pi_l, \sigma_l}})^2 = \mathfrak{f}_{\pi_{\underline{Q}, l}, \sigma_{\underline{Q}', l}}.$$

Proof. We denote the l -primary part of χ' by χ'_l . Recall that if $l|N'$, then

$$\mathfrak{f}_{\pi_{\underline{Q}, l}, \sigma_{\underline{Q}', l}} = \frac{l^{6c(\chi'_l)} \chi'_l(l)^{-4c(\chi'_l)} \varepsilon\left(\frac{1}{2}, \sigma_{\underline{Q}', l}, \psi_l\right)^2}{\varepsilon\left(\frac{1}{2}, \chi_l^{-1}, \psi_l\right)^4 L^{\text{GL}}\left(\frac{1}{2}, \pi_{\underline{Q}, l} \times \sigma_{\underline{Q}', l}\right)^2} \varepsilon^{\text{GL}}\left(\frac{1}{2}, \pi_{\underline{Q}, l} \times \sigma_{\underline{Q}', l}, \psi_l\right).$$

If $l \nmid N'$, then $\mathfrak{f}_{\pi_{\underline{Q}, l}, \sigma_{\underline{Q}', l}} = \varepsilon^{\text{GL}}\left(\frac{1}{2}, \pi_{\underline{Q}, l} \times \sigma_{\underline{Q}', l}, \psi_l\right)$. One can construct elements $\varepsilon_{\sigma_l}, \varepsilon_{\pi_l, \sigma_l} \in (\mathcal{R} \widehat{\otimes} \mathcal{R}')^\times$ and $P_{\pi_l, \sigma_l} \in \mathcal{R} \widehat{\otimes} \mathcal{R}'$ such that

$$\begin{aligned} \underline{Q}(\varepsilon_{\sigma_l}) &= \varepsilon\left(\frac{1}{2}, \sigma_{\underline{Q}', l}, \psi_l\right), & \underline{Q}(\varepsilon_{\pi_l, \sigma_l}) &= \varepsilon^{\text{GL}}\left(\frac{1}{2}, \pi_{\underline{Q}, l} \times \sigma_{\underline{Q}', l}, \psi_l\right), \\ \underline{Q}(P_{\pi_l, \sigma_l}) &= \frac{1}{L^{\text{GL}}\left(\frac{1}{2}, \pi_{\underline{Q}, l} \times \sigma_{\underline{Q}', l}\right)} \end{aligned}$$

for $\underline{Q} = (\underline{Q}, \underline{Q}') \in \mathfrak{X}_{\mathbf{V}}^{\text{crit}}$ as in the proof of Proposition 6.11 of [Hsi21]. \square

Define the fudge factor $\sqrt{\mathfrak{f}_{\pi, \sigma}} \in \mathcal{R} \widehat{\otimes} \mathcal{R}'$ by

$$\sqrt{\mathfrak{f}_{\pi, \sigma}} = \prod_{l|NN'} \sqrt{\mathfrak{f}_{\pi_l, \sigma_l}}.$$

We can prove Theorem 1.1 by defining the theta element $\mathcal{L}_{\mathbf{f}, \mathbf{g}} \in \text{Frac}(\mathcal{R} \widehat{\otimes} \mathcal{R}')$ attached to \mathbf{f} and \mathbf{g} by

$$(4.6) \quad \mathcal{L}_{\mathbf{f}, \mathbf{g}} = \frac{\zeta_p(1)^2 D_E \sqrt{\chi'(-1)} D_E}{2^{\frac{3+t_E}{2}} D_T \sqrt{\mathfrak{f}_{\pi, \sigma}}} \Theta_{\mathbf{F}}$$

in view of Proposition 4.16.

5. THE CALCULATION OF THE LOCAL ZETA INTEGRAL AT THE p -ADIC CASE

5.1. The JPSS integrals. Let F be a finite extension of \mathbb{Q}_p which contains the integer ring \mathfrak{o} having a single prime ideal \mathfrak{p} . We denote the order of the residue field $\mathfrak{o}/\mathfrak{p}$ by q . The absolute value $\alpha_F = |\cdot|$ on F is normalized via $|\varpi| = q^{-1}$ for any generator ϖ of \mathfrak{p} , where q denotes the order of the residue field $\mathfrak{o}/\mathfrak{p}$.

Let $B_n = T_n N_n$ be the Borel subgroup of GL_n , where T_n is the group of diagonal matrices in $\mathrm{GL}_n(F)$ and N_n is the group of upper triangular unipotent matrices in $\mathrm{GL}_n(F)$. Let w_n be the longest Weyl element in $\mathrm{GL}_n(F)$.

Fix an additive character $\psi : F \rightarrow \mathbb{C}^\times$ which is trivial on \mathfrak{o} and non-trivial on \mathfrak{p}^{-1} . We write $\mathcal{S}(F)$ for the space of locally constant compactly supported functions on F . The Fourier transform of $\phi \in \mathcal{S}(F)$ is defined by

$$\widehat{\phi}(y) = \int_F \phi(x) \psi(-xy) dx.$$

The measure dx is chosen so that $\widehat{\widehat{\phi}}(x) = \phi(-x)$.

Let π be an irreducible admissible generic representation of $\mathrm{GL}_{m+1}(F)$. We write $\mathcal{W}_\psi(\pi)$ for the Whittaker model of π with respect to an arbitrarily fixed additive character ψ of F . One can define an invariant perfect pairing

$$\langle \cdot, \cdot \rangle : \mathcal{W}_\psi(\pi) \otimes \mathcal{W}_{\psi^{-1}}(\pi^\vee) \rightarrow \mathbb{C}$$

by

$$(5.1) \quad \langle W_1, W_2 \rangle = \int_{N_m \backslash \mathrm{GL}_m(F)} W_1 \left(\begin{bmatrix} g & \\ & 1 \end{bmatrix} \right) W_2 \left(\begin{bmatrix} g & \\ & 1 \end{bmatrix} \right) dg.$$

where dg be the Haar measure on $\mathrm{GL}_m(F)$ giving $\mathrm{GL}_m(\mathfrak{o})$ volume 1. Given $W \in \mathcal{W}_\psi(\pi)$, we define $\widetilde{W} \in \mathcal{W}_{\psi^{-1}}(\pi^\vee)$ by

$$\widetilde{W}(g) = W(w_{m+1} {}^t g^{-1}).$$

Let n be a positive integer which is equal or less than m . Put $l = m - n$. Let σ be an irreducible admissible generic representation of $\mathrm{GL}_n(F)$ whose central character is ω_σ . We associate to Whittaker functions $W \in \mathcal{W}_\psi(\pi)$ and $W' \in \mathcal{W}_{\psi^{-1}}(\sigma)$ the local zeta integrals

$$\begin{aligned} Z(s, W, W') &= \int_{N_n \backslash \mathrm{GL}_n(F)} W \left(\begin{bmatrix} h & & \\ & \mathbf{1}_{l+1} & \\ & & 1 \end{bmatrix} \right) W'(h) |\det h|^{s - \frac{l+1}{2}} dh, \\ \widetilde{Z}(s, \widetilde{W}, \widetilde{W}') &= \int_{N_n \backslash \mathrm{GL}_n(F)} \int_{M_{l,n}(F)} \widetilde{W} \left(\begin{bmatrix} h & & \\ x & \mathbf{1}_l & \\ & & 1 \end{bmatrix} \right) \widetilde{W}'(h) |\det h|^{s - \frac{l+1}{2}} dx dh, \end{aligned}$$

which converge absolutely for $\Re s \gg 0$, where dh is the Haar measure on $\mathrm{GL}_n(F)$ giving $\mathrm{GL}_n(\mathfrak{o})$ volume 1.

We write $L^{\mathrm{GL}}(s, \pi \times \sigma)$, $\varepsilon^{\mathrm{GL}}(s, \pi \times \sigma, \psi)$ and $\gamma^{\mathrm{GL}}(s, \pi \times \sigma, \psi)$ for the L , epsilon and gamma factors associated to π and σ . These local factors are studied extensively in [JPSS83]. The gamma factor is defined as the proportionality constant of the functional equation

$$(5.2) \quad Z(1 - s, \pi^\vee(w_{m+1,n}) \widetilde{W}, \widetilde{W}') = \omega_\sigma(-1)^m \gamma^{\mathrm{GL}}(s, \pi \times \sigma, \psi) \widetilde{Z}(s, W, W')$$

(see Theorem in §2.7 of [JPSS83]), where

$$w_{m+1,n} = \begin{bmatrix} \mathbf{1}_n & \\ & w_{m-n+1} \end{bmatrix}.$$

Remark 5.1. When we view π and σ are representations of unitary groups over the split quadratic algebra $F \oplus F$,

$$L(s, \pi \times \sigma) = L^{\text{GL}}(s, \pi \times \sigma) L^{\text{GL}}(s, \pi^\vee \times \sigma^\vee).$$

When $n = 1$ and χ is a character of F^\times , we will write

$$\begin{aligned} L(s, \pi \otimes \chi) &= L^{\text{GL}}(s, \pi \times \chi), \\ \varepsilon(s, \pi \otimes \chi, \psi) &= \varepsilon^{\text{GL}}(s, \pi \times \chi, \psi), \\ \gamma(s, \pi \otimes \chi, \psi) &= \gamma^{\text{GL}}(s, \pi \times \chi, \psi). \end{aligned}$$

These local factors are studied extensively also in [GJ72].

5.2. The splitting lemma. Let π and σ be irreducible admissible tempered representations of $\text{GL}_{m+1}(F)$ and $\text{GL}_m(F)$ respectively. We consider the integral

$$J(W_1, W_2, W'_1, W'_2) = \int_{\text{GL}_m(F)} \left\langle \pi \left(\begin{bmatrix} h & \\ & 1 \end{bmatrix} \right) W_1, W_2 \right\rangle \langle \sigma(h) W'_1, W'_2 \rangle' dh,$$

which is convergent for

$$W_1 \in \mathscr{W}_\psi(\pi), \quad W_2 \in \mathscr{W}_{\psi^{-1}}(\pi^\vee), \quad W'_1 \in \mathscr{W}_{\psi^{-1}}(\sigma), \quad W'_2 \in \mathscr{W}_\psi(\sigma^\vee).$$

The following result is called a splitting lemma and was proved by Wei Zhang in Proposition 4.10 of [Zha14] by using the work of Lapid and Mao [LM14]. It is worth noting that Proposition 4.10 of [Zha14] uses unnormalized local Haar measures (cf. §2.1 of [Zha14]) while we here use normalized ones.

Lemma 5.2. *Notation being as above, we have*

$$J(W_1, W_2, W'_1, W'_2) = Z\left(\frac{1}{2}, W_1, W'_1\right) Z\left(\frac{1}{2}, W_2, W'_2\right) \prod_{i=1}^{m-1} \zeta_F(i).$$

Remark 5.3. (1) Assume that π and σ are unramified and that ψ is trivial on \mathfrak{o} but non-trivial on \mathfrak{p}^{-1} . Let $W_\pi \in \mathscr{W}_\psi(\pi)$ be the normalized essential vector defined by $W_\pi(k) = 1$ for $k \in \text{GL}_{m+1}(\mathfrak{o})$. Define W_{π^\vee} , W_σ and W_{σ^\vee} similarly. Then

$$(5.3) \quad \begin{aligned} Z(s, W_\pi, W_\sigma) &= L^{\text{GL}}(s, \pi \times \sigma), \\ \langle W_\pi, W_{\pi^\vee} \rangle &= \frac{L^{\text{GL}}(1, \pi \times \pi^\vee)}{\zeta_F(m+1)}. \end{aligned}$$

by [Zha14, (3.3)]. Lemma 5.2 reproves the formula

$$\frac{J(W_\pi, W_{\pi^\vee}, W_\sigma, W_{\sigma^\vee})}{\langle W_\pi, W_{\pi^\vee} \rangle \langle W_\sigma, W_{\sigma^\vee} \rangle'} = \frac{L\left(\frac{1}{2}, \pi \times \sigma\right) \prod_{i=1}^{m+1} \zeta_F(i)}{L^{\text{GL}}(1, \pi \times \pi^\vee) L^{\text{GL}}(1, \sigma \times \sigma^\vee)}$$

(cf. Remark 5.1), which was proved in Theorem 2.12 of [Har14].

(2) Proposition 5.1 of [Hsi21] is a triple product analogue of this lemma (cf. Proposition 3.8 of [CH18]).

5.3. Ordinary vectors of representations of $\mathrm{GL}_n(F)$. We write \mathbb{I}_S for the characteristic function of a subset S . For a compact subgroup Γ of $\mathrm{GL}_n(F)$ and a representation (π, V) of $\mathrm{GL}_n(F)$ we let

$$V^\Gamma = \{v \in V \mid \pi(\gamma)v = v \text{ for } \gamma \in \Gamma\}$$

be the space of Γ -invariant vectors in V .

Let $\mu_1, \mu_2, \dots, \mu_n$ be characters of F^\times . The space V of $\pi = I(\mu_1, \mu_2, \dots, \mu_n)$ consists of functions $h : \mathrm{GL}_n(F) \rightarrow \mathbb{C}$ which satisfy

$$h(tug) = h(g)\varrho_n(t)^{1/2} \prod_{i=1}^n \mu_i(t_i), \quad \varrho_n(t) = \prod_{i=1}^n |t_i|^{n+1-2i}$$

for $t = \mathrm{diag}[t_1, t_2, \dots, t_n] \in T_n$, $u \in N_n$ and $g \in \mathrm{GL}_n(F)$.

We define a function h^{ord} on $B_n w_n N_n$ by

$$h_\pi^{\mathrm{ord}}(tuw_nv) = \mathbb{I}_{\mathcal{N}_n}(v)\varrho_n(t)^{1/2} \prod_{i=1}^n \mu_i(t_i)$$

for $t = \mathrm{diag}[t_1, t_2, \dots, t_n] \in T_n$ and $u, v \in N_n$, where $\mathbb{I}_{\mathcal{N}_n}$ denotes the characteristic function of $\mathcal{N}_n = N_n \cap \mathrm{GL}_n(\mathfrak{o})$. Since $B_n w_n N_n$ is the cell of the longest Weyl element w_n in the Bruhat decomposition of $\mathrm{GL}_n(F)$, we can extend h_π^{ord} by zero to an element of V (cf. [Car79, (B), p. 138]), which we call an ordinary vector. Define the operator $U_{\mathfrak{p}}$ on $V^{\mathcal{N}_n}$ by

$$U_{\mathfrak{p}}h = \sum_{\substack{u=(u_{i,j}) \in \mathcal{N}_n \\ u_{i,j} \in \mathfrak{o}/\mathfrak{p}^{j-i} \text{ for } i < j}} \pi(uD_{n,\varpi})h,$$

where

$$D_{n,\varpi} = \begin{bmatrix} \varpi^{n-1} & 0 & \dots & 0 & 0 \\ 0 & \varpi^{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varpi & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Proposition 5.4.

$$U_{\mathfrak{p}}h_\pi^{\mathrm{ord}} = h_\pi^{\mathrm{ord}} \prod_{i=1}^n \left(\mu_i(\varpi)q^{i-\frac{n+1}{2}} \right)^{i-1}.$$

Proof. Let $g \in \mathrm{GL}_n(F)$ be such that $[U_{\mathfrak{p}}h_\pi^{\mathrm{ord}}](g) \neq 0$. There exists $u \in \mathcal{N}_n$ such that $h_\pi^{\mathrm{ord}}(guD_{n,\varpi}) \neq 0$. We have $guD_{n,\varpi} \in B_n w_n \mathcal{N}_n$. Since $D_{n,\varpi} \mathcal{N}_n D_{n,\varpi}^{-1} \subset \mathcal{N}_n$, we get $g \in B_n w_n \mathcal{N}_n$. By the characterization of h_π^{ord} we see that h_π^{ord} is an eigenvector of $U_{\mathfrak{p}}$ with eigenvalue $[U_{\mathfrak{p}}h_\pi^{\mathrm{ord}}](w_n)$. By definition we know that $[U_{\mathfrak{p}}h_\pi^{\mathrm{ord}}](w_n)$ equals

$$\sum_{u_{i,j} \in \mathfrak{o}/\mathfrak{p}^{j-i}} h_\pi^{\mathrm{ord}}(w_n u D_{n,\varpi}) = h_\pi^{\mathrm{ord}}(w_n D_{n,\varpi}),$$

from which we obtain the formula for the eigenvalue. \square

5.4. An inductive property. Fix an additive character ψ of F which is trivial on \mathfrak{o} but non-trivial on \mathfrak{p}^{-1} . Define the additive character of N_n by

$$\psi(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$$

for $u = (u_{i,j}) \in N_n$. For an irreducible admissible generic representation σ of $GL_n(F)$ we write $\mathscr{W}_\psi(\sigma)$ for the Whittaker model of σ with respect to ψ .

Let π be the irreducible generic constituent of $I(\mu_1, \mu_2, \dots, \mu_n)$. For $h \in V$ we define $W_\psi(h) \in \mathscr{W}_\psi(\pi)$ by

$$(5.4) \quad W_\psi(g, h) = \int_{N_n} h(w_n u g) \overline{\psi(u)} du$$

for $g \in GL_n(F)$. Put

$$(5.5) \quad W_\pi^{\text{ord}} = W_\psi(h_\pi^{\text{ord}}), \quad e_{n-1} = (0, 0, \dots, 0, 1) \in F^{n-1}.$$

Define $\mathbb{J}_n(y) = \mathbb{I}_{\mathfrak{o}^n}({}^t y)$ for column vectors y . Let π' be the irreducible generic constituent of $I(\mu_2, \mu_3, \dots, \mu_n)$ and π'' that of $I(\mu_1, \mu_2, \dots, \mu_{n-1})$.

Lemma 5.5. *If $n \geq 2$, then for $g \in GL_{n-1}(F)$*

$$\begin{aligned} W_\pi^{\text{ord}} \left(\begin{bmatrix} g & \\ & 1 \end{bmatrix} \right) &= |\det g|^{\frac{1}{2}} W_{\pi'}^{\text{ord}}(g) \mathbb{J}_{n-1}(e_{n-1} g), \\ W_\pi^{\text{ord}} \left(\begin{bmatrix} 1 & \\ & g \end{bmatrix} \right) &= |\det g|^{-\frac{1}{2}} W_{\pi''}^{\text{ord}}(g) \mathbb{J}_{n-1}(e_{n-1} w_{n-1} {}^t g^{-1}). \end{aligned}$$

Proof. Put

$$w_{(n-1,1)} = \begin{bmatrix} & 1 \\ \mathbf{1}_{n-1} & \end{bmatrix}, \quad \mathbf{u}(x) = \begin{bmatrix} \mathbf{1}_{n-1} & x \\ & 1 \end{bmatrix}$$

for $x \in F^{n-1}$. Since

$$\begin{bmatrix} 1 & \\ & b w_{n-1} v \end{bmatrix} w_{(n-1,1)} \mathbf{u}(y) = \begin{bmatrix} 1 & \\ & b \end{bmatrix} w_n \begin{bmatrix} v & v y \\ & 1 \end{bmatrix}$$

for $b \in B_{n-1}$ and $v \in N_{n-1}$, the section h_π^{ord} satisfies

$$h_\pi^{\text{ord}} \left(\begin{bmatrix} 1 & \\ & g \end{bmatrix} w_{(n-1,1)} \mathbf{u}(y) \right) = |\det g|^{-\frac{1}{2}} h_{\pi'}^{\text{ord}}(g) \mathbb{I}_{\mathfrak{o}^{n-1}}(y)$$

for $g \in GL_{n-1}(F)$ and $y \in F^{n-1}$. Since

$$w_n \begin{bmatrix} u & \\ & 1 \end{bmatrix} \mathbf{u}(y) \begin{bmatrix} g & \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & w_{n-1} u g \end{bmatrix} w_{(n-1,1)} \mathbf{u}(g^{-1} y)$$

for $u \in N_{n-1}$ and $y \in F^{n-1}$ and since $d(gy) = |\det g| dy$, we get

$$\begin{aligned} & W_\psi \left(\begin{bmatrix} g & \\ & 1 \end{bmatrix}, h_\pi^{\text{ord}} \right) \\ &= \int_{F^{n-1}} \int_{N_{n-1}} h_\pi^{\text{ord}} \left(w_n \begin{bmatrix} u & \\ & 1 \end{bmatrix} \mathbf{u}(y) \begin{bmatrix} g & \\ & 1 \end{bmatrix} \right) \overline{\psi(u) \psi(y_{n-1})} dy du \\ &= \int_{F^{n-1}} \int_{N_{n-1}} \frac{h_{\pi'}^{\text{ord}}(w_{n-1} u g)}{|\det g|^{\frac{1}{2}}} \mathbb{I}_{\mathfrak{o}^{n-1}}(g^{-1} y) \overline{\psi(u) \psi(e_{n-1} y)} dy du \\ &= |\det g|^{\frac{1}{2}} \int_{N_{n-1}} h_{\pi'}^{\text{ord}}(w_{n-1} u g) \overline{\psi(u)} du \int_{F^{n-1}} \mathbb{I}_{\mathfrak{o}^{n-1}}(y) \overline{\psi(e_{n-1} g y)} dy \\ &= |\det g|^{\frac{1}{2}} W_\psi(g, h_{\pi'}^{\text{ord}}) \mathbb{I}_{\mathfrak{o}^{n-1}}({}^t g e_{n-1}) \end{aligned}$$

as claimed. The second formula can be proved in the same way. \square

5.5. The pairing. Let σ be an irreducible admissible generic unitary representation of $\mathrm{GL}_n(F)$. Given $W \in \mathcal{W}_\psi(\sigma)$, we define $\widetilde{W} \in \mathcal{W}_{\psi^{-1}}(\sigma^\vee)$ by $\widetilde{W}(g) = W(w_n {}^t g^{-1})$. Let π be an irreducible generic unitary constituent of $I(\mu_1, \mu_2, \dots, \mu_n)$. Put

$$\alpha_\pi = q^{\frac{n(n^2-1)}{12}} \prod_{i=1}^n \mu_i(p)^{i-1},$$

$$\mathcal{I}_0^{(n)}(\mathfrak{p}^\ell) = \{(g_{ij}) \in \mathrm{GL}_n(\mathfrak{o}) \mid g_{ij} \in \mathfrak{p}^{\ell(i-j)} \text{ for } i > j\},$$

$$\mathcal{B}_{\pi^{\mathrm{ord}}}^{[\ell]} = \frac{\zeta_F(n)}{L^{\mathrm{GL}}(1, \pi \times \pi^\vee)} \langle W_\pi^{\mathrm{ord}}, \pi^\vee(D_{n, \varpi}^{-\ell}) \widetilde{W}_\pi^{\mathrm{ord}} \rangle.$$

The following formula generalizes Lemma 2.8 of [Hsi21].

Proposition 5.6. *If π is unitary and ℓ is sufficiently large, then*

$$\mathcal{B}_{\pi^{\mathrm{ord}}}^{[\ell]} = \frac{\zeta_F(1)^n}{L^{\mathrm{GL}}(1, \pi \times \pi^\vee)} \cdot \frac{\alpha_\pi^\ell}{[\mathrm{GL}_n(\mathfrak{o}) : \mathcal{I}_0^{(n)}(\mathfrak{p}^\ell)]} \prod_{i < j} \frac{\mu_i(-1)}{\gamma(1, \mu_i^{-1} \mu_j, \psi)}.$$

The proof of Proposition 5.6 consists of several steps. Let $B_n^- = w_n B_n w_n^{-1}$ be the group of lower triangular matrices in $\mathrm{GL}_n(F)$. We denote the unipotent radical of B_n^- by N_n^- . Since there is nothing to prove if $n = 1$, we suppose that $n \geq 2$. Put

$$m = n - 1, \quad \Delta_m = \varpi^{-\ell} D_{m, \varpi}^{-\ell}, \quad \gamma_m = \prod_{i=1}^m \frac{\zeta_F(i)}{\zeta_F(1)}.$$

Lemma 5.7. *Let $W \in \mathcal{W}_\psi(\sigma)$ and $b \in B_m^-$. If ℓ is sufficiently large, then $W\left(\begin{bmatrix} b\Delta_m & \\ & 1 \end{bmatrix}\right) = 0$ unless ${}^t(e_m b) \in \mathfrak{o}^m$.*

Proof. We write

$$b = \begin{bmatrix} g & \\ {}^t y & t \end{bmatrix} = \begin{bmatrix} g & \\ & t \end{bmatrix} \begin{bmatrix} \mathbf{1}_{m-1} & \\ t^{-1} {}^t y & 1 \end{bmatrix},$$

where g has size $m - 1$. Notice that $e_m b = ({}^t y, t)$. Since

$$\left[\begin{array}{c|c} \mathbf{1}_{m-1} & \\ \hline t^{-1} {}^t y & 1 \\ \hline & 1 \end{array} \right] \left[\begin{array}{c|c} \mathbf{1}_m & x \\ \hline & z \\ \hline & 1 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{1}_m & x \\ \hline & z + \frac{{}^t y x}{t} \\ \hline & 1 \end{array} \right] \left[\begin{array}{c|c} \mathbf{1}_{m-1} & \\ \hline t^{-1} {}^t y & 1 \\ \hline & 1 \end{array} \right]$$

for $x, y \in F^{m-1}$ and $z \in F$, we have

$$W\left(\begin{bmatrix} b\Delta_m & \\ & 1 \end{bmatrix} \left[\begin{array}{c|c} \mathbf{1}_m & \Delta_m^{-1} \begin{pmatrix} x \\ z \end{pmatrix} \\ \hline & 1 \end{array} \right]\right) = \psi(tz + {}^t y x) W\left(\begin{bmatrix} b\Delta_m & \\ & 1 \end{bmatrix}\right).$$

If ℓ is sufficiently large, then the left hand side is $W\left(\begin{bmatrix} b\Delta_m & \\ & 1 \end{bmatrix}\right)$ for all $x \in \mathfrak{o}^{m-1}$ and $z \in \mathfrak{o}$, which implies that ${}^t y \in \mathfrak{o}^{m-1}$ and $t \in \mathfrak{o}$. \square

Lemma 5.8. *If ℓ is sufficiently large, then $B_{\pi^{\mathrm{ord}}}^{[\ell]} = \gamma_m \lim_{s \rightarrow 0} \mathcal{F}_\pi(s)$, where*

$$\mathcal{F}_\pi(s) = \int_{N_m^-} \int_{T_m} \widetilde{W}_\pi^{\mathrm{ord}}\left(\begin{bmatrix} vt\Delta_m & \\ & 1 \end{bmatrix}\right) \prod_{i=1}^m \mu_{n+1-i}(t_i) |t_i|^{s+1+\frac{m}{2}-i} dt dv.$$

Proof. Put $W = \widetilde{W}_\pi^{\text{ord}}$. We define the function $\mathcal{B}_\pi(s)$ by the integral

$$\mathcal{B}_\pi(s) = \int_{N_m \backslash \text{GL}_m(F)} |\det g|^s W_\pi^{\text{ord}} \left(\begin{bmatrix} g & \\ & 1 \end{bmatrix} \right) W \left(\begin{bmatrix} g\Delta_m & \\ & 1 \end{bmatrix} \right) dg$$

for $\Re s \gg 0$. Then $B_{\pi^{\text{ord}}}^{[\ell]} = \lim_{s \rightarrow 0} \mathcal{B}_\pi(s)$. We write

$$\begin{aligned} \mathcal{B}_\pi(s) &= \int_{N_m \backslash \text{GL}_m(F)} |\det g|^{s+\frac{1}{2}} W_{\pi'}^{\text{ord}}(g) \mathbb{J}_m(e_m g) W \left(\begin{bmatrix} g\Delta_m & \\ & 1 \end{bmatrix} \right) dg \\ &= \int_{\text{GL}_m(F)} |\det g|^{s+\frac{1}{2}} h_{\pi'}^{\text{ord}}(w_m g) \mathbb{J}_m(e_m g) W \left(\begin{bmatrix} g\Delta_m & \\ & 1 \end{bmatrix} \right) dg, \end{aligned}$$

using Lemma 5.5 and substituting the integral expression (5.4) of $W_{\pi'}^{\text{ord}}$.

Since $[\text{GL}_m(\mathfrak{o}) : \mathcal{I}_m] = q^{m(m-1)/2} \gamma_m^{-1}$, where \mathcal{I}_m is the Iwahori subgroup of $\text{GL}_m(\mathfrak{o})$, we see the following integral formula from (2) on p. 240 of [Wal03]:

$$(5.6) \quad \int_{\text{GL}_m(F)} \mathcal{F}(g) dg = \gamma_m \int_{N_m^-} \int_{T_m} \int_{N_m} \mathcal{F}(vtu) \varphi_m(t) dudtdv$$

for an integrable function \mathcal{F} on $\text{GL}_m(F)$. It follows that

$$\frac{\mathcal{B}_\pi(s)}{\gamma_m} = \int_{N_m^-} \int_{T_m} |\det t|^{s+\frac{1}{2}} h_{\pi'}^{\text{ord}}(w_m t) \mathbb{J}_m(e_m vt) W \left(\begin{bmatrix} vt\Delta_m & \\ & 1 \end{bmatrix} \right) \varphi_m(t) dt dv.$$

The right hand side coincides with $\mathcal{F}(s)$ by Lemma 5.7. \square

Lemma 5.9. *If $n \geq 2$, then*

$$\mathcal{F}_\pi(s) = q^{-(n-1)\ell} \left(s + \frac{n}{4} \right) \mu_n(\varpi^\ell)^{n-1} \zeta_F(s+1) \mathcal{F}_{\pi''}(s) \prod_{i=1}^{n-1} \frac{\mu_i(-1)}{\gamma(s+1, \mu_i^{-1} \mu_n, \psi)}.$$

Here we understand that $\mathcal{F}_{\pi''}(s) = 1$ when $n = 2$.

Proof. By the definition of the JPSS integral we have

$$\mathcal{F}_\pi(s) = \int_{N_{m-1}^-} \int_{T_{m-1}} \tilde{Z}(s+1, W_{v't'}, \mu_n) \prod_{i=2}^m \mu_{n+1-i}(t_i) |t_i|^{s+1+\frac{m}{2}-i} dt' dv',$$

where $t' = \text{diag}[t_2, \dots, t_n]$, and $W_{b'} \in \mathcal{W}_{\psi^{-1}}(\pi^\vee)$ is defined by

$$W_{b'}(g) = \widetilde{W}_\pi^{\text{ord}} \left(g \left[\begin{array}{c|c} 1 & \\ \hline & b' \\ \hline & & 1 \end{array} \right] D_{n,\varpi}^{-\ell} \right)$$

for $b' \in B_{m-1}^-$. The functional equation (5.2) gives

$$\tilde{Z}(s+1, W_{b'}, \mu_n) = \mu_n(-1)^m \frac{Z(-s, \pi(w_{n,1}) \widetilde{W}_{b'}, \mu_n^{-1})}{\gamma(s+1, \pi^\vee \otimes \mu_n, \psi^{-1})}.$$

Since

$$\begin{aligned} &\pi(w_{n,1}) \widetilde{W}_{b'} \left(\begin{bmatrix} a & \\ & \mathbf{1}_m \end{bmatrix} \right) \\ &= W_{b'} \left(w_n \begin{bmatrix} a^{-1} & \\ & w_m \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned}
&= \widetilde{W}_\pi^{\text{ord}} \left(w_n \begin{bmatrix} a^{-1} & & \\ & w_m & \\ & & 1 \end{bmatrix} \left[\begin{array}{c|c} 1 & \\ \hline & b' \\ & & 1 \end{array} \right] D_{n,\varpi}^{-\ell} \right) \\
&= W_\pi^{\text{ord}} \left(\begin{bmatrix} a & & \\ & w_m & \\ & & 1 \end{bmatrix} \left[\begin{array}{c|c} 1 & \\ \hline & b'^{-1} \\ & & 1 \end{array} \right] D_{n,\varpi}^\ell \right) \\
&= W_\pi^{\text{ord}} \left(\left[\begin{array}{c|c} a\varpi^{m\ell} & \\ \hline & w_m \begin{bmatrix} b' & \\ & 1 \end{bmatrix}^{-1} \\ & & 1 \end{array} \right] D_{m,\varpi}^\ell \right) \\
&= \mu_n(a\varpi^{m\ell}) \left| \frac{(a\varpi^{m\ell})^m \det b'}{\det D_{m,\varpi}^\ell} \right|^{\frac{1}{2}} \widetilde{W}_{\pi''}^{\text{ord}} \left(\begin{bmatrix} b' \Delta_{m-1} & \\ & 1 \end{bmatrix} \right) \mathbb{I}_\circ(a\varpi^{m\ell})
\end{aligned}$$

by Lemma 5.5, we have

$$\begin{aligned}
&Z(-s, \pi(w_{n,1}) \widetilde{W}_{b', \mu_n^{-1}}) \\
&= \int_{F^\times} \mu_n(\varpi^{m\ell}) \left| \frac{(a\varpi^{m\ell})^m \det b'}{\det D_{m,\varpi}^\ell} \right|^{\frac{1}{2}} \widetilde{W}_{\pi''}^{\text{ord}} \left(\begin{bmatrix} b' \Delta_{m-1} & \\ & 1 \end{bmatrix} \right) \mathbb{I}_\circ(a\varpi^{m\ell}) |a|^{-s-\frac{m}{2}} da \\
&= \mu_n(\varpi^{m\ell}) q^{-m\ell(s+\frac{m+1}{4})} |\det b'|^{\frac{1}{2}} \widetilde{W}_{\pi''}^{\text{ord}} \left(\begin{bmatrix} b' \Delta_{m-1} & \\ & 1 \end{bmatrix} \right) \zeta_F(-s).
\end{aligned}$$

We therefore get

$$\widetilde{Z}(s+1, W_{b', \mu_n}) = \frac{\mu_n(-\omega^\ell)^m}{q^{m\ell(s+\frac{m+1}{4})}} |\det b'|^{\frac{1}{2}} \frac{\widetilde{W}_{\pi''}^{\text{ord}} \left(\begin{bmatrix} b' \Delta_{m-1} & \\ & 1 \end{bmatrix} \right)}{\prod_{i=1}^m \gamma(s+1, \mu_i^{-1} \mu_n, \psi^{-1})} \zeta_F(s+1)$$

by the multiplicativity of the gamma factor. Substituting this expression, we get the inductive formula for $\mathcal{F}_\pi(s)$. \square

We are now ready to prove Proposition 5.6. We have

$$\mathcal{F}_\pi(s) = \frac{\zeta_F(s+1)^{n-1}}{q^{\frac{n(n-1)\ell}{2}(s+\frac{n+1}{3})}} \alpha_\pi^\ell \prod_{i < j} \frac{\mu_i(-1)}{\gamma(s+1, \mu_i^{-1} \mu_j, \psi)},$$

applying Lemma 5.9 inductively. Since

$$[\text{GL}_n(\mathfrak{o}) : I_0^{(n)}(\mathfrak{p}^\ell)] = q^{\frac{n(n^2-1)\ell}{6}} \gamma_n^{-1},$$

we immediately deduce the declared formula from Lemma 5.8. \square

5.6. The JPSS zeta integral. Let π be an irreducible unitary generic constituent of the principal series $I(\nu, \rho, \mu)$ and σ that of $I(\mu', \nu')$. The matrix ς is defined in §3.1.

Proposition 5.10. *Put*

$$W_\pi^{\text{ord}} = W_\psi(h_\pi^{\text{ord}}), \quad W_\sigma^{\text{ord}} = W_{\psi^{-1}}(h_\sigma^{\text{ord}}), \quad \mathbf{t}_\ell = \begin{bmatrix} 0 & 0 & -\varpi^{-\ell} \\ 0 & \varpi^\ell & 1 \\ \varpi^{2\ell} & \varpi^\ell & 0 \end{bmatrix}.$$

If ℓ is sufficiently large, then

$$Z\left(\frac{1}{2}, \pi(\varsigma \mathbf{t}_\ell) W_\pi^{\text{ord}}, W_\sigma^{\text{ord}}\right) = \frac{\pm \zeta_F(2) (q^{-5/2} \rho(\varpi) \mu(\varpi)^2 \nu'(\varpi))^\ell}{\zeta_F(1) \gamma(\frac{1}{2}, \mu \nu', \psi) \gamma(\frac{1}{2}, \rho \nu', \psi) \gamma(\frac{1}{2}, \mu \mu', \psi)}.$$

The proof of Proposition 5.10 relies on the following lemmas.

Lemma 5.11. *Put*

$$W_b = \pi \left(\left[\begin{array}{c|c} 1 & \\ \hline & 1 \\ & b \end{array} \right] \mathbf{t}_\ell \right) W_\pi^{\text{ord}}.$$

If ℓ is sufficiently large, then

$$Z(s, \pi(\varsigma \mathbf{t}_\ell) W_\pi^{\text{ord}}, W_\sigma^{\text{ord}}) = \frac{\zeta_F(2)}{\zeta_F(1)} \int_{F^\times} Z(1-s, \widetilde{W}_b, \nu'^{-1}) \mu'(b) |b|^{s-1} db.$$

Proof. Put $W = \pi(\mathbf{t}_\ell) W_\pi^{\text{ord}}$. We get

$$Z(s, \pi(\varsigma) W, W_\sigma^{\text{ord}}) = \int_{\text{GL}_2(F)} W \left(\left[\begin{array}{c|c} g & \\ \hline & 1 \end{array} \right] \varsigma \right) h_\sigma^{\text{ord}} \left(\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] g \right) |\det g|^{s-\frac{1}{2}} dg,$$

substituting the integral expression (5.4) of W_σ^{ord} . Put $\mathbf{u}(y) = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$ for $y \in F$.

Then $\frac{\zeta_F(1)}{\zeta_F(2)} Z(s, \pi(\varsigma) W, W_\sigma^{\text{ord}})$ equals

$$\int_{F^{\times 2} \times F^2} W \left(\left[\begin{array}{cc|c} a & & \mathbf{u}(y) \\ x & b & \\ \hline & & 1 \end{array} \right] \varsigma \right) \left| \frac{b}{a} \right|^{1/2} \mu'(b) \nu'(a) \mathbb{I}_\sigma(y) |ab|^{s-\frac{1}{2}} dy \frac{dx da db}{|b|}$$

by the integration formula (5.6). Since

$$\varsigma^{-1} \left[\begin{array}{cc|c} \mathbf{u}(y) & & \\ & & \\ \hline & & 1 \end{array} \right] \varsigma \mathbf{t}_\ell = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{t}_\ell = \mathbf{t}_\ell \begin{bmatrix} 1 - \varpi^\ell y & -y & 0 \\ \varpi^{2\ell} y & 1 + \varpi^\ell y & 0 \\ -\varpi^{3\ell} y & -\varpi^{2\ell} y & 1 \end{bmatrix},$$

if ℓ is sufficiently large, then $\pi \left(\varsigma^{-1} \left[\begin{array}{cc|c} \mathbf{u}(y) & & \\ & & \\ \hline & & 1 \end{array} \right] \varsigma \right) W = W$ for $y \in \mathfrak{o}$. We get

$$\begin{aligned} & \frac{\zeta_F(1)}{\zeta_F(2)} Z(s, \pi(\varsigma) W, W_\sigma^{\text{ord}}) \\ &= \int_{F^{\times 2} \times F} W \left(\left[\begin{array}{c|c} a & \\ \hline x & b \\ & 1 \end{array} \right] \varsigma \right) \mu'(b) \nu'(a) |ab|^{s-1} dx da db \\ &= \int_{F^\times} \widetilde{Z}(s, \pi(\varsigma) W_b, \nu') \mu'(b) |b|^{s-1} db, \end{aligned}$$

where

$$W_b = \pi \left(\varsigma \left[\begin{array}{c|c} 1 & \\ \hline & b \\ & 1 \end{array} \right] \varsigma \right) W = \pi \left(\left[\begin{array}{c|c} 1 & \\ \hline & 1 \\ & b \end{array} \right] \mathbf{t}_\ell \right) W_\pi^{\text{ord}}.$$

The stated formula follows from the functional equation (5.2)

$$\widetilde{Z}(s, \pi(\varsigma) W_b, \nu') = \gamma(s, \pi \otimes \nu', \boldsymbol{\psi})^{-1} Z(1-s, \widetilde{W}_b, \nu'^{-1}).$$

We here follow the convention in Remark 5.1. \square

Now we need the following formula:

Lemma 5.12. *Put $\Phi = \mathbb{I}_{1+\mathfrak{p}^\ell}$. Then for $a, b \in F^\times$*

$$\widetilde{W}_b \left(\left[\begin{array}{c|c} a & \\ \hline & \mathbf{1}_2 \end{array} \right] \right) = \rho(\varpi^\ell) \mu(b\varpi^{2\ell}) |b| \widehat{\Phi}(-b) \frac{|a\varpi^\ell| \widehat{\Phi}(a\varpi^\ell)}{\nu(-a\varpi^\ell)}.$$

Proof. Observe that

$$\widetilde{W}_b(g) = W_b(w_3 {}^t g^{-1}) = W_\pi^{\text{ord}} \left(w_3 {}^t g^{-1} \left[\begin{array}{c|c} 1 & \\ \hline & 1 \quad b \end{array} \right] \mathbf{t}_\ell \right).$$

It follows that

$$\begin{aligned} \widetilde{W}_b \left(\left[\begin{array}{c|c} a & \\ \hline & \mathbf{1}_2 \end{array} \right] \right) &= W_\pi^{\text{ord}} \left(w_3 \left[\begin{array}{c|c} a^{-1} & \\ \hline & 1 \quad b \end{array} \right] \left[\begin{array}{ccc} 0 & 0 & -\varpi^{-\ell} \\ 0 & \varpi^\ell & 1 \\ \varpi^{2\ell} & \varpi^\ell & 0 \end{array} \right] \right) \\ &= W_\pi^{\text{ord}} \left(\left[\begin{array}{ccc} b\varpi^{2\ell} & 0 & 0 \\ 0 & \varpi^\ell & 0 \\ 0 & 0 & -(a\varpi^\ell)^{-1} \end{array} \right] \left[\begin{array}{ccc} 1 & \varpi^{-\ell} & 0 \\ 0 & 1 & \varpi^{-\ell} \\ 0 & 0 & 1 \end{array} \right] \right). \end{aligned}$$

Since

$$\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b\varpi^{2\ell} & 0 & 0 \\ 0 & \varpi^\ell & 0 \\ 0 & 0 & \frac{-1}{a\varpi^\ell} \end{bmatrix} = \begin{bmatrix} b\varpi^{2\ell} & 0 & 0 \\ 0 & \varpi^\ell & 0 \\ 0 & 0 & \frac{-1}{a\varpi^\ell} \end{bmatrix} \begin{bmatrix} 1 & \frac{y}{b\varpi^\ell} & -\frac{z}{ab\varpi^{3\ell}} \\ 0 & 1 & -\frac{x}{a\varpi^{2\ell}} \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$\begin{aligned} &\widetilde{W}_b \left(\left[\begin{array}{c|c} a & \\ \hline & \mathbf{1}_2 \end{array} \right] \right) \\ &= \int_{F^3} h_\pi^{\text{ord}} \left(w_3 \left[\begin{array}{ccc} b\varpi^{2\ell} & 0 & 0 \\ 0 & \varpi^\ell & 0 \\ 0 & 0 & \frac{-1}{a\varpi^\ell} \end{array} \right] \left[\begin{array}{ccc} 1 & \frac{y+b}{b\varpi^\ell} & \frac{a\varpi^\ell y-z}{ab\varpi^{3\ell}} \\ 0 & 1 & \frac{a\varpi^\ell x}{a\varpi^{2\ell}} \\ 0 & 0 & 1 \end{array} \right] \right) \overline{\psi(x+y)} \, dx dy dz \\ &= \frac{\rho(\varpi^\ell) \mu(b\varpi^{2\ell})}{\nu(-a\varpi^\ell) |ab\varpi^{3\ell}|} \int_{F^3} h_\pi^{\text{ord}} \left(w_3 \left[\begin{array}{ccc} 1 & \frac{y+1}{\varpi^\ell} & z \\ 0 & 1 & \frac{1-x}{\varpi^\ell} \\ 0 & 0 & 1 \end{array} \right] \right) \frac{|a^2 b^2 \varpi^{4\ell}|}{\psi(a\varpi^\ell x + by)} \, dx dy dz \\ &= \frac{\rho(\varpi^\ell) \mu(b\varpi^{2\ell})}{\nu(-a\varpi^\ell)} |ab\varpi^\ell| \int_{F^2} \Phi(y) \Phi(x) \overline{\psi(a\varpi^\ell x - by)} \, dx dy \end{aligned}$$

from which one can complete the proof of Lemma 5.12. \square

We are now ready to prove Proposition 5.10. For $\phi \in \mathcal{S}(F)$ and a character χ of F^\times we define Tate's local integral by

$$Z(s, \phi, \chi) = \int_{F^\times} \phi(a) \chi(a) |a|^s \, da.$$

Substituting this expression in Lemma 5.12, we get

$$\begin{aligned} Z(1-s, \widetilde{W}_b, \nu'^{-1}) &= \rho(\varpi^\ell) \mu(b\varpi^{2\ell}) |b| \widehat{\Phi}(-b) \int_{F^\times} \frac{|a\varpi^\ell| \widehat{\Phi}(a\varpi^\ell)}{\nu(-a\varpi^\ell)} \nu'(a)^{-1} \frac{da}{|a|^s} \\ &= \rho(\varpi^\ell) \mu(b\varpi^{2\ell}) |b| \widehat{\Phi}(-b) \frac{\nu'(\varpi^\ell)}{\nu(-1) q^{\ell s}} Z(1-s, \widehat{\Phi}, (\nu\nu')^{-1}). \end{aligned}$$

We conclude that $\frac{\zeta_F(1)}{\zeta_F(2)}\gamma(s, \pi \otimes \nu', \boldsymbol{\psi})Z(s, \pi(\varsigma)W, W_\sigma^{\text{ord}})$ equals

$$\begin{aligned} & Z(1-s, \widehat{\Phi}, (\nu\nu')^{-1})\rho(\varpi^\ell) \frac{\nu'(\varpi^\ell)}{\nu(-1)q^{\ell s}} \int_{F^\times} \mu(b\varpi^{2\ell})|b|\widehat{\Phi}(-b)\mu'(b)|b|^{s-1} db \\ & = Z(1-s, \widehat{\Phi}, (\nu\nu')^{-1})\rho(\varpi^\ell)(\nu\mu\mu')(-1)\nu'(\varpi^\ell)q^{-\ell s}\mu(\varpi^{2\ell})Z(s, \widehat{\Phi}, \mu\mu'). \end{aligned}$$

If ℓ is sufficiently large, then since $\Phi = \mathbb{I}_{1+\mathfrak{p}^\ell}$,

$$\begin{aligned} Z(1-s, \widehat{\Phi}, (\nu\nu')^{-1}) &= \gamma(1-s, (\nu\nu')^{-1}, \boldsymbol{\psi})^{-1}(\nu\nu')(-1)q^{-\ell}, \\ Z(s, \widehat{\Phi}, (\mu\mu')^{-1}) &= \gamma(s, (\mu\mu')^{-1}, \boldsymbol{\psi})^{-1}(\mu\mu')(-1)q^{-\ell} \end{aligned}$$

by the functional equation (see Theorem 3.3 of [GJ72]). We conclude that

$$Z\left(\frac{1}{2}, \pi(\varsigma\mathbf{t}_\ell)W_\pi^{\text{ord}}, W_\sigma^{\text{ord}}\right) = \frac{\nu'(-1) \frac{\zeta_F(2)}{q^{5\ell/2}\zeta_F(1)}\rho(\varpi)^\ell \mu(\varpi)^{2\ell} \nu'(\varpi)^\ell}{\gamma(\frac{1}{2}, \pi \otimes \nu', \boldsymbol{\psi})\gamma(\frac{1}{2}, (\nu\nu')^{-1}, \boldsymbol{\psi})\gamma(\frac{1}{2}, \mu\mu', \boldsymbol{\psi})}.$$

One can deduce the stated formula from the multiplicativity of the gamma factor and the functional equation $\gamma(\frac{1}{2}, \nu\nu', \boldsymbol{\psi})\gamma(\frac{1}{2}, (\nu\nu')^{-1}, \boldsymbol{\psi}) = (\nu\nu')(-1)$. \square

6. RAMIFIED COMPUTATIONS: THE SPLIT CASE

6.1. Essential vectors. We choose a non-trivial additive character $\boldsymbol{\psi}$ of F so that the maximal fractional ideal on which it is trivial is \mathfrak{o} . Let π be an irreducible admissible generic representation of $\text{GL}_{m+1}(F)$. Given an open compact subgroup Γ of $\text{GL}_{m+1}(F)$ and its character $\mathcal{X} : \Gamma \rightarrow \mathbb{C}^\times$, we put

$$\mathscr{W}_\boldsymbol{\psi}(\pi, \Gamma, \mathcal{X}) = \{W \in \mathscr{W}_\boldsymbol{\psi}(\pi) \mid \pi(\gamma)W = \mathcal{X}(\gamma)W \text{ for } \gamma \in \Gamma\}.$$

Assume that $m \geq 1$. For a positive integer ℓ the subgroup $\mathcal{K}_0^{(m+1)}(\mathfrak{p}^\ell)$ consists of matrices of the form

$$\begin{bmatrix} A & B \\ C & d \end{bmatrix} \quad (A \in \text{GL}_m(\mathfrak{o}), B \in \mathfrak{o}^m, {}^tC \in (\mathfrak{p}^\ell)^m, d \in \mathfrak{o}^\times).$$

When $\ell = 0$, we set $\mathcal{K}_0^{(m+1)}(\mathfrak{p}^\ell) = \text{GL}_{m+1}(\mathfrak{o})$. Given a character ω of $(\mathfrak{o}/\mathfrak{p}^\ell)^\times$, we define the characters $\omega^\downarrow : \mathcal{K}_0^{(m+1)}(\mathfrak{p}^\ell) \rightarrow \mathbb{C}^\times$ and $\omega^\uparrow : \mathcal{K}_0^{(2)}(\mathfrak{p}^\ell) \rightarrow \mathbb{C}^\times$ by

$$\omega^\downarrow \left(\begin{bmatrix} A & B \\ C & d \end{bmatrix} \right) = \omega(d), \quad \omega^\uparrow \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \omega(a).$$

We write ω_π for the central character of π . Let $c(\pi)$ denote the exponent of the conductor of π , i.e., the epsilon factor of π is of the form

$$(6.1) \quad \varepsilon \left(s + \frac{1}{2}, \pi, \boldsymbol{\psi} \right) = q^{-c(\pi)s} \varepsilon \left(\frac{1}{2}, \pi, \boldsymbol{\psi} \right).$$

Théorème on p. 211 of [JPSS81] says that

$$\dim \mathscr{W}_\boldsymbol{\psi}(\pi, \mathcal{K}_0^{(m+1)}(\mathfrak{p}^{c(\pi)}), \omega_\pi^\downarrow) = 1.$$

Theorem 3.1 of [Mat13] enables us to normalize a basis vector of this one-dimensional space in the following way:

Definition 6.1 (essential vectors). There exists a unique vector

$$W_\pi \in \mathscr{W}_\boldsymbol{\psi}(\pi, \mathcal{K}_0^{(m+1)}(\mathfrak{p}^{c(\pi)}), \omega_\pi^\downarrow)$$

which satisfies $W_\pi(\mathbf{1}_{m+1}) = 1$. This vector W_π is called a normalized essential Whittaker vector of π with respect to $\boldsymbol{\psi}$.

6.2. The Atkin-Lehner operator.

Proposition 6.2. *Let π be an irreducible admissible generic representation of $\mathrm{GL}_{m+1}(F)$. Put $\ell = c(\pi)$ and $\xi_{m,\ell} = \begin{bmatrix} \varpi^{-\ell} \mathbf{1}_m & 0 \\ 0 & 1 \end{bmatrix}$. Let W_{π^\vee} be the essential vector of π^\vee with respect to ψ^{-1} . Then $\pi^\vee(\xi_{m,\ell})\widetilde{W}_\pi = \varepsilon\left(\frac{1}{2}, \pi, \psi\right)^m W_{\pi^\vee}$.*

Proof. One can immediately see that

$$\pi^\vee(\xi_{m,\ell})\widetilde{W}_\pi \in \mathscr{W}_{\psi^{-1}}(\pi^\vee, \mathcal{K}_0^{(m+1)}(\mathfrak{p}^\ell), \omega_{\pi^\vee}^\downarrow)$$

from

$${}^t \left(\xi_{m,\ell}^{-1} \begin{bmatrix} A & B \\ C & d \end{bmatrix} \xi_{m,\ell} \right) = \begin{bmatrix} {}^t A & \frac{{}^t C}{\varpi^\ell} \\ {}^t B \varpi^\ell & d \end{bmatrix}.$$

Thus $\pi^\vee(\xi_{m,\ell})\widetilde{W}_\pi = cW_{\pi^\vee}$ with $c \in \mathbb{C}^\times$.

To determine c , we take an irreducible unramified principal series $\sigma = \alpha_F^{s_1} \times \cdots \times \alpha_F^{s_m}$ of $\mathrm{GL}_m(F)$. Recall the functional equation

$$\begin{aligned} & \gamma^{\mathrm{GL}}(s, \pi \times \sigma, \psi) \int_{N_m \backslash \mathrm{GL}_m(F)} W \left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \right) W_\sigma(g) |\det g|^{s-\frac{1}{2}} dg \\ &= \int_{N_m \backslash \mathrm{GL}_m(F)} \widetilde{W} \left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \right) \widetilde{W}_\sigma(g) |\det g|^{\frac{1}{2}-s} dg \\ &= \omega_\sigma(\varpi)^\ell q^{\ell m(1-2s)/2} \int_{N_m \backslash \mathrm{GL}_m(F)} \pi^\vee(\xi_{m,\ell})\widetilde{W} \left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \right) \widetilde{W}_\sigma(g) |\det g|^{\frac{1}{2}-s} dg \end{aligned}$$

for every $W \in \mathscr{W}_\psi(\pi)$. Letting $W = W_\pi$, we get

$$\gamma^{\mathrm{GL}}(s, \pi \times \sigma, \psi) L(s, \pi \times \sigma) = \omega_\sigma(\varpi)^\ell q^{\ell m(1-2s)/2} c L(1-s, \pi^\vee \times \sigma^\vee)$$

by (4.3). Since

$$\varepsilon^{\mathrm{GL}}\left(\frac{1}{2}, \pi \times \sigma, \psi\right) = \omega_\sigma(\varpi)^\ell \varepsilon\left(\frac{1}{2}, \pi, \psi\right)^m,$$

we obtain the relation by (6.1). \square

Let $m = 2$. Thus π is an irreducible admissible generic representation of $\mathrm{GL}_3(F)$ and σ that of $\mathrm{GL}_2(F)$. Put

$$\ell = c(\pi), \quad n = c(\sigma), \quad \tau_\ell = w_3 \xi_{2,\ell}^{-1}, \quad \tau'_n = w_2 \xi_{1,n}^{-1}.$$

Let W_σ (resp. W_{σ^\vee}) be the essential vector of σ^\vee (resp. σ) with respect to ψ (resp. ψ^{-1}) defined in Definition 6.1. We rewrite Proposition 6.2 in the following manner.

Corollary 6.3. Notation being as above, we have

$$W_\pi(\tau_\ell) = \varepsilon\left(\frac{1}{2}, \pi, \psi\right)^2, \quad W_\sigma(\tau'_n) = \varepsilon\left(\frac{1}{2}, \sigma, \psi\right).$$

6.3. Computation of the pairing. Let π be an irreducible admissible generic unitary representation of $\mathrm{GL}_n(F)$. We write π_u for the unramified component of the first nonzero spherical Bernstein-Zelevinsky derivative $\pi^{(n-r)}$ of π (see Definition 1.3 of [Mat13] for the precise definition).

Proposition 6.4. *Let W_π be the essential vector of π with respect to ψ and W_{π^\vee} the essential vector of π^\vee with respect to ψ^{-1} . If π is unitary, then*

$$\mathcal{B}_\pi := \frac{\zeta_F(n)}{L^{\text{GL}}(1, \pi \times \pi^\vee)} \langle W_\pi, W_{\pi^\vee} \rangle = \begin{cases} 1 & \text{if } \pi \text{ is unramified,} \\ \frac{\zeta_F(n) L^{\text{GL}}(1, \pi_u \times \pi_u^\vee)}{L^{\text{GL}}(1, \pi \times \pi^\vee)} & \text{if } \pi \text{ is ramified.} \end{cases}$$

Proof. We may assume π to be ramified in view of Remark 5.1. Put $m = n - 1$. Let $t = \text{diag}[t_1, t_2, \dots, t_m] \in T_m$. Put $t' = \text{diag}[t_1, t_2, \dots, t_r] \in T_r$. Corollary 3.2 of [Mat13] gives

$$W_\pi \left(\begin{bmatrix} t & \\ & 1 \end{bmatrix} \right) = W_{\pi_u}(t') |\det t'|^{\frac{m+1-r}{2}} \mathbb{I}_\mathfrak{o}(t_r) \prod_{i=r+1}^m \mathbb{I}_\mathfrak{o} \times (t_i)$$

if $r \geq 1$, and $W_\pi \left(\begin{bmatrix} t & \\ & 1 \end{bmatrix} \right) = \prod_{i=1}^m \mathbb{I}_\mathfrak{o} \times (t_i)$ if $r = 0$. If $r \geq 1$, then

$$\begin{aligned} \langle W_\pi, W_{\pi^\vee} \rangle &= \int_{T_m} W_\pi \left(\begin{bmatrix} t & \\ & 1 \end{bmatrix} \right) W_{\pi^\vee} \left(\begin{bmatrix} t & \\ & 1 \end{bmatrix} \right) \prod_{i=1}^m |t_i|^{2i-m-1} dt \\ &= \int_{T_r} W_{\pi_u}(t') W_{\pi_u^\vee}(t') \mathbb{I}_\mathfrak{o}(t_r) \prod_{i=1}^r |t_i|^{2i-r} dt \end{aligned}$$

by the Iwasawa decomposition. The last integral equals

$$\int_{N_r \backslash \text{GL}_r(F)} W_{\pi_u}(g) W_{\pi_u^\vee}(g) \mathbb{I}_\mathfrak{o}(e_r g) |\det g| dg = L^{\text{GL}}(1, \pi_u \times \pi_u^\vee)$$

by Proposition 2.3 of [JS81]. The case $r = 0$ is trivial. \square

Definition 6.5. Suppose that π is a generic constituent of $I(\mu_1, \mu_2, \dots, \mu_n)$, where we order $\mu_1, \mu_2, \dots, \mu_n$ as in Definition 2.3. Put

$$\frac{1}{\mathcal{E}(\pi, \text{Ad}, \psi)} = L^{\text{GL}}(1, \pi_u \times \pi_u^\vee) \prod_{i < j} \gamma(1, \mu_i^{-1} \mu_j, \psi) \times \begin{cases} \frac{1}{\zeta_F(1)^n} & \text{if } c(\pi) = 0, \\ \frac{q^{(n-1)c(\pi)}}{\zeta_F(1)^{n-1}} & \text{if } c(\pi) > 0. \end{cases}$$

Corollary 6.6. Notations and assumptions being as in Proposition 5.6, if ℓ is sufficiently large, then

$$\frac{\mathcal{B}_{\pi^{\text{ord}}}^{[\ell]}}{\mathcal{B}_\pi} = \frac{\alpha_\pi^\ell}{[\mathcal{K}_0^{(n)}(\mathfrak{p}^{c(\pi)}) : I_0^{(n)}(\mathfrak{p}^\ell)]} \mathcal{E}(\pi, \text{Ad}, \psi) \prod_{i < j} \mu_i(-1).$$

Proof. Since

$$[\text{GL}_n(\mathfrak{o}) : \mathcal{K}_0^{(n)}(\mathfrak{p}^{c(\pi)})] = \begin{cases} 1 & \text{if } c(\pi) = 0, \\ q^{(n-1)c(\pi)} \frac{\zeta_F(1)}{\zeta_F(n)} & \text{if } c(\pi) > 0, \end{cases}$$

the stated formula follows from Propositions 5.6 and 6.4. \square

6.4. A depletion à la Schmidt. We consider the embedding

$$t' : \text{GL}_2(F) \hookrightarrow \text{GL}_3(F), \quad t'(g) = \begin{bmatrix} g & \\ & 1 \end{bmatrix}.$$

Let χ be a character of F^\times . When χ is ramified, the conductor $c(\chi)$ of χ is defined as the smallest positive integer n such that χ is trivial on $1 + \mathfrak{p}^n$. When χ is unramified, we set $c(\chi) = 0$. If $c(\chi) \geq 1$, then the Gauss sum is defined by

$$\mathfrak{g}(\chi, \psi) = \sum_{a \in (\mathfrak{o}/\mathfrak{p}^{c(\chi)})^\times} \chi(a)^{-1} \psi\left(-\frac{a}{\varpi^{c(\chi)}}\right).$$

When $c(\chi) = 0$, we formally set $\mathfrak{g}(\chi, \psi) = 1$. The Gauss sum is related to the epsilon factor in the following way:

$$\mathfrak{g}(\chi, \psi) = q^{c(\chi)/2} \chi(\varpi)^{-c(\chi)} \varepsilon\left(\frac{1}{2}, \chi, \psi^{-1}\right).$$

Proposition 6.7 ([Sch93]). *Let χ be a character of \mathfrak{o}^\times . Put $f = c(\chi)$. Assume that $f > 0$. Given $W \in \mathscr{W}_\psi(\pi, \mathcal{K}_0^{(3)}(\mathfrak{p}^\ell), \omega_\pi^\perp)$, we define $\mathbf{U}^\times W \in \mathscr{W}_\psi(\pi)$ by*

$$\mathbf{U}^\times W = \sum_{i,j \in (\mathfrak{o}/\mathfrak{p}^f)^\times} \sum_{y \in \mathfrak{o}/\mathfrak{p}^{2f}} \chi(ij) \pi \left(\begin{bmatrix} 1 & \frac{i}{\varpi^f} & \frac{y}{\varpi^{2f}} \\ 0 & 1 & \frac{j}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} \right) W.$$

Then $\mathbf{U}^\times W$ possesses the following properties:

- (i) $\pi(\iota'(\gamma))\mathbf{U}^\times W = \chi^\uparrow(\gamma)^{-1}\mathbf{U}^\times W$ for $\gamma \in \mathcal{K}_0^{(2)}(\mathfrak{p}^{2f})$;
- (ii) $\mathbf{U}^\times W(\iota'(h)) = 0$ unless $h \in N_2\mathcal{K}_0^{(2)}(\mathfrak{p}^{2f})$;
- (iii) $\mathbf{U}^\times W(\mathbf{1}_3) = q^{3f} \chi(\varpi)^{2f} \varepsilon\left(\frac{1}{2}, \chi, \psi\right)^{-2} W(\mathbf{1}_3)$.

Proof. The proof is the same as that of Lemma 2.3 of [Sch93]. However, we reproduce the proof here for the reader's convenience. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{K}_0^{(2)}(\mathfrak{p}^{2f})$.

Observe that

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{i}{\varpi^f} & \frac{y}{\varpi^{2f}} \\ 0 & 1 & \frac{j}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{ai}{d\varpi^f} & \frac{y'}{\varpi^{2f}} \\ 0 & 1 & \frac{dj}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a - \frac{aci}{d\varpi^f} & b - \frac{aci^2}{d\varpi^{2f}} & 0 \\ c & d + \frac{ci}{\varpi^f} & \frac{cy}{\varpi^{2f}} \\ 0 & 0 & 1 \end{bmatrix},$$

where $y' = ya\left(1 - \frac{ci}{d\varpi^f}\right) + bj\varpi^f$. Since $a\left(1 - \frac{ci}{d\varpi^f}\right) \in \mathfrak{o}^\times$, y' runs over a full system of representatives mod \mathfrak{p}^{2f} as y does (for fixed i, j). We get

$$\pi(\iota'(\gamma))\mathbf{U}^\times W = \sum_{i,j \in (\mathfrak{o}/\mathfrak{p}^f)^\times} \sum_{y \in \mathfrak{o}/\mathfrak{p}^{2f}} \chi(ij) \pi \left(\begin{bmatrix} 1 & \frac{ai}{d\varpi^f} & \frac{y}{\varpi^{2f}} \\ 0 & 1 & \frac{dj}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} \right) W.$$

The right hand side is $\chi(a)^{-1}\mathbf{U}^\times W$, which proves (i).

To prove (ii), we have only to prove $\mathbf{U}^\times W\left(\iota'\left(\begin{bmatrix} 0 & \varpi^n \\ \varpi^m & 0 \end{bmatrix}\right)\right) = 0$ and

$$\mathbf{U}^\times W\left(\iota'\left(\begin{bmatrix} \varpi^n & 0 \\ \varpi^m c & \varpi^m \end{bmatrix}\right)\right) \neq 0 \Rightarrow m = n = 0, c \in \mathfrak{p}^{2f}$$

by the Iwasawa decomposition. Since

$$\begin{bmatrix} 0 & \varpi^n & 0 \\ \varpi^m & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{i}{\varpi^f} & \frac{y}{\varpi^{2f}} \\ 0 & 1 & \frac{j}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\varpi^{f+n}}{i\varpi^m} & \frac{j\varpi^n}{\varpi^f} \\ 0 & 1 & \frac{y\varpi^m}{\varpi^{2f}} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varpi^{n+f} & 0 & 0 \\ 0 & \varpi^{m-f} & 0 \\ 0 & 0 & 1 \end{bmatrix} k,$$

where $k = \iota' \left(\begin{bmatrix} -i^{-1} & 0 \\ \varpi^f & i \end{bmatrix} \right)$, we get

$$\begin{aligned} & \mathbf{U}^\times W \left(\iota' \left(\begin{bmatrix} 0 & \varpi^n \\ \varpi^m & 0 \end{bmatrix} \right) \right) \\ &= \sum_{i,j \in (\mathfrak{o}/\mathfrak{p}^f)^\times} \sum_{y \in \mathfrak{o}/\mathfrak{p}^{2f}} \chi(ij) \psi \left(\frac{\varpi^{f+n}}{i\varpi^m} + \frac{y\varpi^m}{\varpi^{2f}} \right) W \left(\iota' \left(\begin{bmatrix} \varpi^{n+f} & 0 \\ 0 & \varpi^{m-f} \end{bmatrix} \right) \right). \end{aligned}$$

The right hand side is zero as $\sum_{j \in (\mathfrak{o}/\mathfrak{p}^f)^\times} \chi(j) = 0$.

Finally we let $c \in \mathfrak{o}$ and assume that

$$(6.2) \quad \mathbf{U}^\times W \left(\iota' \left(\begin{bmatrix} \varpi^n & 0 \\ \varpi^m c & \varpi^m \end{bmatrix} \right) \right) \neq 0.$$

Since we can write

$$\begin{bmatrix} \varpi^n & 0 & 0 \\ \varpi^m c & \varpi^m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{i}{\varpi^f} & \frac{y}{\varpi^{2f}} \\ 0 & 1 & \frac{j}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{i\varpi^n}{\varpi^{f+m}} & \frac{y\varpi^n}{\varpi^{2f}} \\ 0 & 1 & \frac{\varpi^m c y}{\varpi^{2f}} + \frac{\varpi^m j}{\varpi^f} \\ 0 & 0 & 1 \end{bmatrix} g_{n,m,i},$$

where

$$g_{n,m,i} = \iota' \left(\begin{bmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{bmatrix} \begin{bmatrix} 1 - \frac{ci}{\varpi^f} & -\frac{ci^2}{\varpi^{2f}} \\ c & 1 + \frac{ci}{\varpi^f} \end{bmatrix} \right),$$

we get

$$\begin{aligned} & \mathbf{U}^\times W \left(\iota' \left(\begin{bmatrix} \varpi^n & 0 \\ \varpi^m c & \varpi^m \end{bmatrix} \right) \right) \\ &= \sum_{i,j \in (\mathfrak{o}/\mathfrak{p}^f)^\times} \sum_{y \in \mathfrak{o}/\mathfrak{p}^{2f}} \chi(ij) \psi \left(\frac{i\varpi^n}{\varpi^{f+m}} + \frac{\varpi^m c y}{\varpi^{2f}} + j \frac{\varpi^m}{\varpi^f} \right) W(g_{n,m,i}). \end{aligned}$$

Replacing j by $j + \varpi^f$, we find that $m \geq 0$ by (6.2). Since

$$(6.3) \quad \sum_{j \in (\mathfrak{o}/\mathfrak{p}^f)^\times} \chi(j) \psi \left(j \frac{\varpi^m}{\varpi^f} \right) \neq 0 \Rightarrow m = 0,$$

we see that $m = 0$. Since $\sum_{y \in \mathfrak{o}/\mathfrak{p}^{2f}} \psi \left(\frac{cy}{\varpi^{2f}} \right) = 0$ unless $c \in \mathfrak{p}^{2f}$, we find that $c \in \mathfrak{p}^{2f}$.

We conclude that $n = 0$ again by (6.3). We see that

$$\mathbf{U}^\times W(\mathbf{1}_3) = q^{2f} \mathfrak{g}(\chi^{-1}, \psi)^2 W(\mathbf{1}_3) = \frac{q^{3f} \chi(\varpi)^{2f}}{\varepsilon(\frac{1}{2}, \chi, \psi)^2} W(\mathbf{1}_3)$$

by letting $n = m = c = 0$ in the formula above. \square

6.5. The zeta integral of $U^{\omega_\sigma} W_\pi$. Let π be an irreducible admissible generic representation of $GL_3(F)$ and σ an irreducible admissible *ramified* generic representation of $GL_2(F)$.

Proposition 6.8. *Notation being as above, if $c(\sigma) \leq 2c(\omega_\sigma)$, then*

$$Z(s, \pi(\iota'(\xi_{1,c(\sigma)} w_2)) \mathbf{U}^{\omega_\sigma} W_\pi, W_\sigma) = \frac{q^{3c(\omega_\sigma)} \omega_\sigma(\varpi)^{2c(\omega_\sigma)} \varepsilon(\frac{1}{2}, \sigma, \psi)}{\varepsilon(\frac{1}{2}, \omega_\sigma, \psi)^2 [\mathcal{K}_0^{(2)}(\mathfrak{p}^{2c(\omega_\sigma)})]}.$$

Proof. Put $f = c(\omega_\sigma)$, $n = c(\sigma)$ and $W' = \sigma(\tau'_n)W_\sigma$. The left hand side is $Z(s, \mathbf{U}^{\omega_\sigma}W_\pi, W')$ by the invariance of the JPSS integral. Since $\sigma(u)W' = \omega_\sigma^\uparrow(u)W'$ for $u \in \mathcal{K}_0^{(2)}(\mathfrak{p}^n)$. Proposition 6.7(ii) gives

$$Z(s, \mathbf{U}^{\omega_\sigma}W_\pi, W') = \int_{\mathcal{K}_0^{(2)}(\mathfrak{p}^{2f})} \mathbf{U}^{\omega_\sigma}W_\pi(\iota'(h))W'(h) |\det h|^{s-\frac{1}{2}} dh.$$

The right hand side is $\mathbf{U}^{\omega_\sigma}W_\pi(\mathbf{1}_3) \cdot W_\sigma(\tau'_n)[\mathrm{GL}_2(\mathfrak{o}) : \mathcal{K}_0^{(2)}(\mathfrak{p}^{2f})]^{-1}$ by Proposition 6.7(i). Since $W'(\mathbf{1}_2) = W_\sigma(\tau'_n) = \varepsilon(\frac{1}{2}, \sigma, \psi)$ by Corollary 6.3, the proof is complete by Proposition 6.7(iii). \square

Remark 6.9. If $f = c(\chi)$ is large enough, then

$$c(\sigma \otimes \chi) = 2f, \quad \varepsilon(s, \sigma \otimes \chi, \psi) = \varepsilon(s, \chi, \psi)\varepsilon(s, \omega_\sigma\chi, \psi)$$

by stability of the epsilon factor (see [JS85, Proposition 2.2]).

APPENDIX A. ARCHIMEDEAN COMPUTATIONS

A.1. Local factors. For a positive integer n let

$$\mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t g^c g = \mathbf{1}_n\}$$

be the compact unitary group. Let \mathbb{C}^1 denote the group of complex numbers of absolute value 1. Define the character $\varepsilon : \mathbb{C}^\times \rightarrow \mathbb{C}^1$ by $\varepsilon(x) = \frac{x}{|x|}$. We view it as a character of any unitary group via composition with the determinant character.

Let $W_\mathbb{R}$ and $W_\mathbb{C}$ be the Weil groups of \mathbb{R} and \mathbb{C} . Recall that $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. Fix tuples $\lambda_1 > \cdots > \lambda_n$ and $\mu_1 > \cdots > \mu_{n-1}$ of half integers such that $\lambda_i - \frac{n+1}{2} \in \mathbb{Z}$ and $\mu_j - \frac{n}{2} \in \mathbb{Z}$. Let π be an irreducible representation of $\mathrm{U}(n)$ with Harish-Chandra parameter $(\lambda_1, \dots, \lambda_n)$ and σ an irreducible representation of $\mathrm{U}(n-1)$ with Harish-Chandra parameter $(\mu_1, \dots, \mu_{n-1})$. The L -parameters of π and σ restricted to $W_\mathbb{C} = \mathbb{C}^\times \subset W_\mathbb{R}$ are given by

$$\phi_\pi|_{W_\mathbb{C}} = \varepsilon^{2\lambda_1} \oplus \cdots \oplus \varepsilon^{2\lambda_n}, \quad \phi_\sigma|_{W_\mathbb{C}} = \varepsilon^{2\mu_1} \oplus \cdots \oplus \varepsilon^{2\mu_{n-1}}.$$

The L -factors are given by

$$\begin{aligned} L(s, \pi \times \sigma) &= \prod_{i=1}^n \prod_{j=1}^{n-1} \Gamma_\mathbb{C}(s + |\lambda_i + \mu_j|), \\ L(s, \pi, \mathrm{Ad}) &= \Gamma_\mathbb{R}(s+1)^n \prod_{i < j} \Gamma_\mathbb{C}(s + \lambda_i - \lambda_j), \\ L(s, \sigma, \mathrm{Ad}) &= \Gamma_\mathbb{R}(s+1)^{n-1} \prod_{i < j} \Gamma_\mathbb{C}(s + \mu_i - \mu_j). \end{aligned}$$

One can easily compute the adjoint L -factors, combining Remark 4.1 with [Pra92, Lemma 7.1]. Theorem 1.1 of [He17] says that σ^\vee appears as a subrepresentation of $\pi|_{\mathrm{U}(n-1)}$ if and only if

$$\lambda_1 > -\mu_{n-1} > \lambda_2 > \cdots > \lambda_{n-1} > -\mu_1 > \lambda_n$$

In this case it is easy to check that

$$(A.1) \quad \frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \mathrm{Ad})L(1, \sigma, \mathrm{Ad})} \in \pi^{\frac{n(n+1)}{2}} \mathbb{Q}^\times.$$

A.2. Representations of $U(2)$. For a commutative ring A of characteristic 0 and a non-negative integer n we write $L_n(A)$ for the module of two variable homogeneous polynomials of degree n over A . The group $GL_2(A)$ acts on this module $L_n(A)$ by

$$\varrho_n(\alpha)P(X, Y) = P((X, Y)\alpha).$$

Define the pairing

$$\ell_n : L_n(\mathbb{Q}) \otimes L_n(\mathbb{Q}) \rightarrow \mathbb{Q}$$

by

$$(A.2) \quad \ell_n(X^i Y^{n-i} \otimes X'^j Y'^{n-j}) = \begin{cases} (-1)^i \binom{n}{i}^{-1} & \text{if } i+j = n, \\ 0 & \text{if } i+j \neq n. \end{cases}$$

Put $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. It is well-known that for $\alpha \in M_2(A)$ and $P, Q \in L_n(A)$

$$\ell_n(\varrho_n(\alpha)P \otimes Q) = \ell_n(P \otimes \varrho_n(J^t \alpha J^{-1})Q).$$

We view $U(2)$ as a subgroup of $GL_2(\mathbb{C})$ and regard ϱ_n as an irreducible representation of $U(2)$ of dimension $n+1$. Note that ϱ_n is irreducible, has central character ε^n , highest weight $(n, 0)$ and Harish-Chandra parameter $(n + \frac{1}{2}, -\frac{1}{2})$. For $\lambda \in \mathbb{Z}$ we will write $\varrho_{(n+\lambda, \lambda)} = \varrho_n \otimes \varepsilon^\lambda$. Let σ be an irreducible representation of $U(2)$ of dimension $n+1$. Then there is an integer λ such that $\sigma \simeq \varrho_{(n+\lambda, \lambda)}$. We define the perfect pairing $\ell_\sigma : \sigma \otimes \sigma^\vee \rightarrow \mathbb{C}$ by

$$\ell_\sigma(\sigma(h)P \otimes Q) = \varepsilon(\det h)^\lambda \cdot \ell_n(\varrho_n(h)P \otimes Q)$$

for $h \in U(2)$ and $P, Q \in L_n(\mathbb{C})$.

We identify the contragredient representation $\varrho_{(a,b)}^\vee$ of $\varrho_{(a,b)}$ with $\varrho_{(-b, -a)}$. Define the representation $\varrho_{(a,b)}^\vartheta$ of $GL_2(A)$ by $\varrho_{(a,b)}^\vartheta(\alpha) = \varrho_{(a,b)}({}^t \alpha^{-1})$ and an equivariant isomorphism $\vartheta : \varrho_{(a,b)}^\vartheta \simeq \varrho_{(a,b)}^\vee$ by

$$P^\vartheta(X, Y) = \varrho_{(a,b)}(J)P(X, Y) = P(-Y, X).$$

Define $\mathbf{P}_n, \mathbf{P}'_n \in L_n(A) \otimes L_n(A)$ by

$$\mathbf{P}_n = (X_1 Y_2 - Y_1 X_2)^n, \quad \mathbf{P}'_n = (X'_1 Y'_2 - Y'_1 X'_2)^n.$$

Since $\varrho_n(\alpha) \otimes \varrho_n(\alpha) \mathbf{P}_n = (\det \alpha)^n \mathbf{P}_n$ for $\alpha \in GL_n(\mathbb{C})$, this vector \mathbf{P}_n spans the line of $\sigma \otimes \sigma^\vee$ fixed by the diagonal action of $U(2)$. Set $\ell_{\sigma \otimes \sigma^\vee} = \ell_\sigma \otimes \ell_{\sigma^\vee}$, where polynomials of X_i, Y_i are paired with those of X'_i, Y'_i for $i = 1, 2$.

Lemma A.1.

$$\ell_{\sigma \otimes \sigma^\vee}(\mathbf{P}_n \otimes \mathbf{P}'_n) = n + 1.$$

Proof. Since $\mathbf{P}_n = \sum_{i=0}^n \binom{n}{i} (X_1 Y_2)^i (-Y_1 X_2)^{n-i}$, we have

$$\mathbf{P}_n \otimes \mathbf{P}'_n = \sum_{i=0}^n \sum_{j=0}^n \binom{n}{i} \binom{n}{j} (-1)^{i+j} X_1^i Y_1^{n-i} X_2^{n-i} Y_2^i \otimes X_1'^j Y_1'^{n-j} X_2'^{n-j} Y_2'^j$$

and hence

$$\ell_{\sigma \otimes \sigma^\vee}(\mathbf{P}_n \otimes \mathbf{P}'_n) = (-1)^n \sum_{i=0}^n \binom{n}{i}^2 (-1)^i \binom{n}{i}^{-1} (-1)^{n-i} \binom{n}{n-i}^{-1} = n + 1$$

by (A.2). □

A.3. Representations of $U(3)$. Fix a commutative integral domain A of characteristic zero. Let $\mathcal{P}(A)$ be the set of polynomials in $z = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ with coefficients in A . Fix non-negative integers a, b . The submodule $\mathcal{P}_{a,b}(A)$ of $\mathcal{P}(A)$ consists of homogeneous polynomials of degree a in x_1, x_2, x_3 and of degree b in y_1, y_2, y_3 . The group $GL_3(A)$ acts on the module $\mathcal{P}_{a,b}(A)$ by

$$\rho(g)P \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} xg \\ ytg^{-1} \end{pmatrix},$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.

We define a submodule $\mathcal{T}_{a,b}(A)$ of $\mathcal{P}_{a,b}(A)$ by

$$\mathcal{T}_{a,b}(A) = (x_1y_1 + x_2y_2 + x_3y_3)\mathcal{P}_{a-1,b-1}(A).$$

Since $\mathcal{T}_{a,b}(A)$ is stable under the action of $GL_3(A)$, the group $GL_3(A)$ acts on the quotient module $\mathfrak{H}_{a,b}(A) = \mathcal{P}_{a,b}(A)/\mathcal{T}_{a,b}(A)$. If A is a field, then $\mathfrak{H}_{a,b}(A)$ is an irreducible representation of $GL_3(A)$ with highest weight $(a, 0, -b)$ (cf. Chapter 7 of [HIM22]).

A.4. Contragredient representations. We define a bilinear form

$$l_{b,a} : \mathcal{P}_{b,a}(A) \otimes \mathcal{P}_{a,b}(A) \rightarrow A$$

by

$$l_{b,a}(Q \otimes P) = \frac{1}{a!b!} Q \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix} P \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Note that

$$l_{b,a} \left(\prod_{i=1}^3 x_i^{n_i} y_i^{m_i} \otimes x_i^{n'_i} y_i^{m'_i} \right) = \begin{cases} \frac{\prod_{i=1}^3 n_i! m_i!}{a!b!} & \text{if } n_i = m'_i \text{ and } m_i = n'_i, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma A.2. For $P \in \mathcal{P}_{a,b}(A)$, $Q \in \mathcal{P}_{b,a}(A)$ and $g \in GL_3(A)$ we have

$$l_{b,a}(\rho(g)Q \otimes \rho(g)P) = l_{b,a}(Q \otimes P).$$

Proof. Let $h = (h_{ij})$ be the inverse matrix of $g = (g_{ij})$. Put $s_j = \sum_{i=1}^3 x_i g_{ij}$ and $t_j = \sum_{i=1}^3 y_i h_{ji}$. Then $\frac{\partial}{\partial s_j} = \sum_{i=1}^3 h_{ji} \frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial t_j} = \sum_{i=1}^3 g_{ij} \frac{\partial}{\partial y_i}$. We therefore see that

$$\rho(g)P \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} s \\ t \end{pmatrix}, \quad \rho(g)Q \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} = Q \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial s} \end{pmatrix}$$

from which Lemma A.2 follows. \square

Put $\mathfrak{H}_{b,a}^\vee(A) = \{P \in \mathcal{P}_{a,b}(A) \mid \Delta P = 0\}$, where

$$\Delta = \frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial x_2 \partial y_2} + \frac{\partial^2}{\partial x_3 \partial y_3}.$$

If A is a field, then since

$$\mathfrak{H}_{b,a}^\vee(A) = \{P \in \mathcal{P}_{a,b}(A) \mid l_{b,a}(Q \otimes P) = 0 \text{ for } Q \in \mathcal{T}_{b,a}(A)\},$$

the restriction of ρ to $\mathfrak{H}_{b,a}^\vee(A)$ is the contragredient representation of $\mathfrak{H}_{b,a}(A)$. In particular, $\mathfrak{H}_{b,a}^\vee(A)$ is an irreducible representation of $GL_3(A)$ with highest weight $(a, 0, -b)$. The linear form $l_{b,a}$ induces a perfect pairing

$$l_{b,a} : \mathfrak{H}_{b,a}(A) \otimes \mathfrak{H}_{b,a}^\vee(A) \rightarrow A.$$

Remark A.3. When we view $\mathfrak{H}_{b,a}^\vee(\mathbb{C})$ as a representation of $U(3)$, it is a theta lift of the discrete series of $U(1, 1)$ with Harish-Chandra parameter $(\frac{a+3}{2}, \frac{b-1}{2})$ (cf. [KV78]). Any irreducible representation of $U(3)$ is of the form $\mathfrak{H}_{b,a}^\vee \otimes \varepsilon^\lambda$ with $0 \leq a, b \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}$.

We define the isomorphism $\vartheta : \mathcal{P}_{b,a}(A) \rightarrow \mathcal{P}_{a,b}(A)$ by

$$P^\vartheta \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} y \\ x \end{pmatrix}.$$

Lemma A.4. $\mathcal{P}_{a,b}(\mathbb{Q})$ is a direct sum of $\mathfrak{H}_{b,a}^\vee(\mathbb{Q})$ and $\mathcal{T}_{a,b}(\mathbb{Q})$.

Proof. It suffices to show that

$$\mathfrak{H}_{b,a}^\vee(\mathbb{Q}) \cap \mathcal{T}_{a,b}(\mathbb{Q}) = \{0\}.$$

Let $f = \sum a_{lmn}^{ijk} \cdot x_1^i x_2^j x_3^k y_1^l y_2^m y_3^n \in \mathfrak{H}_{b,a}^\vee(\mathbb{Q}) \cap \mathcal{T}_{a,b}(\mathbb{Q})$. Since $f^\vartheta \in \mathcal{T}_{b,a}(\mathbb{Q})$, we have $l_{b,a}(f^\vartheta \otimes f) = 0$ by the definition of $\mathfrak{H}_{b,a}^\vee(\mathbb{Q})$. Observe that

$$l_{b,a}(f^\vartheta \otimes f) = (a!b!)^{-1} \sum (a_{lmn}^{ijk})^2 i!j!k!l!m!n!,$$

which shows that $f = 0$. □

We endow the space $\mathfrak{H}_{b,a}^\vartheta := \mathfrak{H}_{b,a}$ with the action $\rho^\vartheta(g) = \rho({}^t g^{-1})$ of $GL_3(A)$. Since $(\rho^\vartheta(g)P)^\vartheta = \rho(g)P^\vartheta$, we can define the equivariant isomorphism

$$\vartheta : \mathfrak{H}_{b,a}^\vartheta(\mathbb{Q}) \simeq \mathfrak{H}_{b,a}^\vee(\mathbb{Q}),$$

letting P^ϑ be the unique polynomial $Q \in \mathfrak{H}_{b,a}^\vee(\mathbb{Q})$ such that $Q - \tilde{P}^\vartheta \in \mathcal{T}_{a,b}(\mathbb{Q})$ for any lift $\tilde{P} \in \mathcal{P}_{b,a}(\mathbb{Q})$ of $P \in \mathfrak{H}_{b,a}$.

A.5. The setting. Let V be a three dimensional positive definite Hermitian space and $V' \subset V$ a two dimensional subspace. We fix a basis of V so that the embedding $\iota : GL(V') \hookrightarrow GL(V)$ is given by

$$\iota : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}.$$

Fix a triplet of integers $\lambda_1 > \lambda_2 > \lambda_3$ and a pair $\mu_1 > \mu_2$ of half integers. Let π be an irreducible representation of $U(V)$ with Harish-Chandra parameter $(\lambda_1, \lambda_2, \lambda_3)$ and σ an irreducible representation of $U(V')$ with Harish-Chandra parameter (μ_1, μ_2) . The highest weight of π is $(-k_1, -k_2, -k_3) := (\lambda_1 - 1, \lambda_2, \lambda_3 + 1)$ while the highest weight of σ is $(-k'_1, -k'_2) := (\mu_1 - \frac{1}{2}, \mu_2 + \frac{1}{2})$. Recall that

$$\mathcal{L}(\pi \times \sigma) = \frac{L(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \text{Ad})L(1, \sigma, \text{Ad})} \prod_{i=1}^n L(i, \epsilon_{\mathbb{C}/\mathbb{R}}^i).$$

Recall that σ^\vee appears as a subrepresentation of $\pi|_{U(V')}$ if and only if

$$(\star) \quad \lambda_1 > -\mu_2 > \lambda_2 > -\mu_1 > \lambda_3 \Leftrightarrow k_1 \leq -k'_2 \leq k_2 \leq -k'_1 \leq k_3.$$

We consider the additive character $\psi_{-2}^{\mathbb{C}}(x) = e^{2\pi(\bar{x}-x)}$ for $x \in \mathbb{C}$. Since

$$\varepsilon\left(\frac{1}{2}, \varepsilon^{2\kappa}, \psi_{-2}^{\mathbb{C}}\right) = \begin{cases} -1 & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa < 0 \end{cases}$$

for $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, if (\star) holds, then

$$(A.3) \quad \varepsilon\left(\frac{1}{2}, \phi_\pi \otimes \phi_\sigma, \psi_{-2}^{\mathbb{C}}\right) = \prod_{i,j} \varepsilon\left(\frac{1}{2}, \varepsilon^{2(\lambda_i + \mu_j)}, \psi_{-2}^{\mathbb{C}}\right) = -1.$$

Put

$$a = k_3 - k_2, \quad b = k_2 - k_1, \quad n = k_3 + k'_1, \quad l = -k_1 - k'_2.$$

If (\star) holds, then $\sigma \simeq \varrho_{(a-n, l-b)} \otimes \varepsilon^{k_2}$ and

$$\frac{\mathcal{L}(\pi \times \sigma)}{(2\pi)^2 n! l!} = \frac{\Gamma(a+b-l+2)\Gamma(a-n+b+2)\Gamma(a-n+1)\Gamma(b-l+1)}{2^{-1}\Gamma(a+b+3)\Gamma(a-n+b-l+2)\Gamma(a+2)\Gamma(b+2)}.$$

A.6. The $U(V')$ -invariant vector. We hereafter assume (\star) . Then the $\mathrm{GL}_2(\mathbb{C})$ -invariant subspace of $\pi \otimes \sigma$ is one-dimensional. We will construct a basis vector of this $U(V')$ -invariant line. Replacing π by $\pi \otimes \varepsilon^{k_2}$ and σ by $\sigma \otimes \varepsilon^{-k_2}$, we may assume that $k_2 = \lambda_2 = 0$. Then

$$\pi \simeq \mathfrak{H}_{b,a}, \quad \pi^\vee \simeq \mathfrak{H}_{b,a}^\vee, \quad \sigma \simeq \varrho_{(a-n, l-b)}, \quad \sigma^\vee \simeq \varrho_{(b-l, n-a)}.$$

Put $\Pi = \pi \otimes \sigma$ and $\Pi^\vartheta = \pi^\vartheta \otimes \sigma^\vartheta$. The maps constructed in §A.2 and §A.4 give the equivariant isomorphism $\vartheta \otimes \vartheta : \Pi^\vartheta \simeq \Pi^\vee$.

We define $\mathbf{W}_\Pi^H \in \Pi$ by

$$\mathbf{W}_\Pi^H \equiv \det \begin{bmatrix} x_1 & x_3 \\ X_2 & Y_2 \end{bmatrix}^{b-l} \det \begin{bmatrix} X_2 & Y_2 \\ y_3 & -y_1 \end{bmatrix}^{a-n} x_2^l y_2^n \pmod{\mathcal{T}_{b,a}(\mathbb{Q})}$$

and define a polynomial \mathbf{P}_{Π^\vee} in $x_1, x_2, x_3, y_1, y_2, y_3, X_2, Y_2$ by

$$\mathbf{P}_{\Pi^\vee} = \sum_{m=0}^{\min\{n, l\}} (-1)^m a_{\Pi^\vee}^{(m)} \mathbf{P}_{\Pi^\vee}^{(m)} \cdot x_2^{n-m} y_2^{l-m}$$

for $0 \leq n \leq a$ and $0 \leq l \leq b$, where

$$\begin{aligned} \mathbf{P}_{\Pi^\vee}^{(m)} &= (x_1 y_1 + x_3 y_3)^m (x_1 Y_2 - x_3 X_2)^{a-n} (y_1 X_2 + y_3 Y_2)^{b-l}, \\ a_{\Pi^\vee}^{(m)} &= \binom{n}{m} \binom{l}{m} \frac{\Gamma(m+1)\Gamma(a-n+b+2)\Gamma(a+b-l+2)}{\Gamma(a+b+2)\Gamma(a-n+b-l+m+2)}. \end{aligned}$$

Lemma A.5.

$$\sum_{m=0}^{\min\{n, l\}} a_{\Pi^\vee}^{(m)} = 1.$$

Proof. If $n \geq l$, then Vandermonde's convolution gives

$$\sum_{m=0}^{\min\{n, l\}} a_{\Pi^\vee}^{(m)} = \binom{a+b+1}{l}^{-1} \sum_{m=0}^l \binom{n}{m} \binom{a-n+b+1}{l-m} = 1$$

as claimed. The case $n \leq l$ can be proved in the same way. \square

- Proposition A.6.**
- (1) $\mathbf{P}_{\Pi^\vee} \in \mathfrak{H}_{b,a}^\vee(\mathbb{Q}) \otimes L_{a-n+b-l}(\mathbb{Q})$.
 - (2) $\Pi^\vee(\iota(h), h)\mathbf{P}_{\Pi^\vee} = \mathbf{P}_{\Pi^\vee}$ for $h \in \mathrm{GL}(V')$.
 - (3) $\mathbf{W}_{\Pi}^{H^\vartheta \otimes \vartheta} = \mathbf{P}_{\Pi^\vee}$.
 - (4) $\Pi(\iota(h), h)\mathbf{W}_{\Pi}^H = \mathbf{W}_{\Pi}^H$ for $h \in \mathrm{GL}(V')$.

Proof. The polynomial $P_{n,l}$ is homogeneous of degree a in x_1, x_2, x_3 and of degree b in y_1, y_2, y_3 and of degree $a - n + b - l$ in X_2, Y_2 . Since

$$\left(\frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial x_3 \partial y_3} \right) \mathbf{P}_{\Pi^\vee}^{(m)} = m(m+1+a-n+b-l) \mathbf{P}_{\Pi^\vee}^{(m-1)}$$

and since

$$m(m+1+a-n+b-l)a_{\Pi^\vee}^{(m)} = (n-m+1)(l-m+1)a_{\Pi^\vee}^{(m-1)}$$

for $m = 1, 2, \dots, \min\{l, n\}$, we prove (1). Recall that $\mathrm{GL}(V')$ acts on

$$P = \sum_{i,j} P_{i,j} \cdot x_2^{n-i} y_2^{l-j} \in \mathfrak{H}_{b,a}^\vee(\mathbb{Q}) \otimes L_{c-d}(\mathbb{Q})$$

by

$$\pi^\vee(\iota(h)) \otimes \varrho_{(c,d)}(h) P_{i,j} \left(\begin{bmatrix} x_1 & x_3 \\ y_3 & -y_1 \\ X_2 & Y_2 \end{bmatrix} \right) = (\det h)^{l-b-j+d} P_{i,j} \left(\begin{bmatrix} x_1 & x_3 \\ y_3 & -y_1 \\ X_2 & Y_2 \end{bmatrix} h \right)$$

for $h \in \mathrm{GL}(V')$. It follows that

$$\pi^\vee(\iota(h)) \otimes \sigma^\vee(h) \mathbf{P}_{\Pi^\vee}^{(m)} = \mathbf{P}_{\Pi^\vee}^{(m)},$$

which proves (2). Since

$$\mathbf{P}_{\Pi^\vee}^{(m)} = (-x_2 y_2)^m (x_1 Y_2 - x_3 X_2)^{a-n} (y_1 X_2 + y_3 Y_2)^{b-l} \pmod{\mathcal{T}_{a,b}(\mathbb{Q})},$$

we see that

$$\begin{aligned} \mathbf{P}_{\Pi^\vee} &\equiv \sum_m a_{\Pi^\vee}^{(m)} (x_1 Y_2 - x_3 X_2)^{a-n} (y_1 X_2 + y_3 Y_2)^{b-l} x_2^n y_2^l \\ &\equiv \left(\sum_m a_{\Pi^\vee}^{(m)} \right) \mathbf{W}_{\Pi}^{H^\vartheta \otimes \vartheta} \pmod{\mathcal{T}_{a,b}(\mathbb{Q})}. \end{aligned}$$

Now (3) is a consequence of Lemma A.5. One sees (4) from (3). \square

A.7. The restriction to $U(V')$. The branching law for the restriction of π^\vee to $U(V')$ is well-known:

$$\pi^\vee|_{U(V')} \simeq \bigoplus_{n=0}^a \bigoplus_{l=0}^b \varrho_{(a-n, -b+l)}.$$

Hara and Namikawa [HN21] explicitly give a $\mathrm{GL}(V')$ -equivariant map

$$\nabla_{n,l} : \mathfrak{H}_{b,a}^\vee \rightarrow \varrho_{(a-n, -b+l)}$$

by

$$(\nabla_{n,l} P)(X_1, Y_1) = \frac{1}{n!l!} \cdot \frac{\partial^{n+l} P}{\partial x_2^n \partial y_2^l} \left(\begin{bmatrix} X_1 & 0 & Y_1 \\ -Y_1 & 0 & X_1 \end{bmatrix} \right).$$

Define a $\mathrm{GL}(V')$ -equivariant map $\nabla_{n,l} : \mathfrak{H}_{a,b} \rightarrow \varrho_{(b-l,n-a)}$ so that the diagram

$$\begin{array}{ccc} \mathfrak{H}_{b,a} & \xrightarrow{\nabla_{n,l}} & \varrho_{(b-l,n-a)} \\ \vartheta \downarrow & & \vartheta \downarrow \\ \mathfrak{H}_{b,a}^\vee & \xrightarrow{\nabla_{n,l}} & \varrho_{(a-n,-b+l)} \end{array}$$

is commutative. Since the $\mathrm{GL}(V)$ -invariant pairing $l_{b,a}$ is the sum of appropriate $\mathrm{GL}(V')$ -invariant pairings on $\mathrm{GL}(V')$ -irreducible summands, we can define the constants $c_{n,l}$ so that for all $Q \in \mathfrak{H}_{b,a} \simeq \pi$ and $P \in \mathfrak{H}_{b,a}^\vee \simeq \pi^\vee$

$$l_{b,a}(Q \otimes P) = \sum_{n=0}^a \sum_{l=0}^b c_{n,l} \ell_{a-n+b-l}(\nabla_{n,l}Q \otimes \nabla_{n,l}P).$$

A.8. The Ichino-Ikeda integral. We will consider the integral

$$J(\mathbf{P} \otimes \mathbf{Q}) = \int_{\mathbf{U}(V')} \ell_{\Pi}((\Pi(\iota(h), h)\mathbf{P}) \otimes \mathbf{Q}) dh$$

for $\mathbf{P} \in \Pi$ and $\mathbf{Q} \in \Pi^\vee$, where we set $\ell_{\Pi} = \ell_{\pi} \otimes \ell_{\sigma}$.

Since $\nabla_{n,l}\mathbf{P}_{\Pi^\vee} = a_{\Pi^\vee}^{(0)}\mathbf{P}_{a-n+b-l}$ and since $\nabla_{n',l'}\mathbf{P}_{\Pi^\vee} = 0$ unless $n' = n$ and $l' = l$, Proposition A.6 and Lemma A.1 show that

$$\begin{aligned} J(\mathbf{W}_{\Pi}^H \otimes \mathbf{W}_{\Pi}^{H\vartheta \otimes \vartheta}) &= J(\mathbf{P}_{\Pi^\vee}^{\vartheta^{-1} \otimes \vartheta^{-1}} \otimes \mathbf{P}_{\Pi^\vee}) \\ &= \ell_{\Pi}(\mathbf{P}_{\Pi^\vee}^{\vartheta^{-1} \otimes \vartheta^{-1}} \otimes \mathbf{P}_{\Pi^\vee}) \\ &= c_{n,l} \ell_{\sigma \otimes \sigma^\vee} (a_{\Pi^\vee}^{(0)} \mathbf{P}_{a-n+b-l}^{\vartheta^{-1} \otimes \vartheta^{-1}} \otimes a_{\Pi^\vee}^{(0)} \mathbf{P}_{a-n+b-l}) \\ &= a_{\Pi^\vee}^{(0)} c_{n,l} a_{\Pi^\vee}^{(0)} (a - n + b - l + 1). \end{aligned}$$

We denote the dimensions of π and σ by $d(\pi)$ and $d(\sigma)$. Since

$$d(\pi) = \frac{1}{2}(a+b+2)(a+1)(b+1), \quad d(\sigma) = a - n + b - l + 1,$$

we have

$$\begin{aligned} & \frac{J(\mathbf{W}_{\Pi}^H \otimes \mathbf{W}_{\Pi}^{H\vartheta \otimes \vartheta})}{\mathcal{L}(\pi \times \sigma)} \\ &= \frac{d(\pi)d(\sigma)a_{\Pi^\vee}^{(0)}c_{n,l}a_{\Pi^\vee}^{(0)}\Gamma(a+b+2)\Gamma(a-n+b-l+2)\Gamma(a+1)\Gamma(b+1)}{n!l!(2\pi)^2\Gamma(a+b-l+2)\Gamma(a-n+b+2)\Gamma(a-n+1)\Gamma(b-l+1)} \\ &= \frac{d(\pi)d(\sigma)}{(2\pi)^2} \binom{a}{n} \binom{b}{l} a_{\Pi^\vee}^{(0)}c_{n,l}a_{\Pi^\vee}^{(0)} \frac{\Gamma(a+b+2)\Gamma(a-n+b-l+2)}{\Gamma(a+b-l+2)\Gamma(a-n+b+2)} \\ &= \frac{d(\pi)d(\sigma)}{(2\pi)^2} \binom{a}{n} \binom{b}{l} a_{\Pi^\vee}^{(0)}c_{n,l}. \end{aligned}$$

Let $W_\sigma = X_1^{a-n+b-l}$ and $W_\pi = x_1^b y_3^a$ be highest weight vectors. Since

$$W_{\sigma^\vee} := W_\sigma^\vartheta = (-Y_1')^{a-n+b-l}, \quad W_{\pi^\vee} := W_\pi^\vartheta = y_1^b x_3^{l_a}$$

(cf. (3.1) and (3.2)), we have

$$\ell_{\Pi}((W_\pi \otimes W_\sigma) \otimes (W_{\pi^\vee} \otimes W_{\sigma^\vee})) = 1.$$

Now the following formula is a consequence of Proposition A.8 below.

Proposition A.7.

$$\frac{\mathcal{L}(\pi \times \sigma)^{-1} J(\mathbf{W}_{\Pi}^H \otimes \mathbf{W}_{\Pi}^{H\vartheta \otimes \vartheta})}{\ell_{\Pi}((W_{\pi} \otimes W_{\sigma}) \otimes (W_{\pi^{\vee}} \otimes W_{\sigma^{\vee}}))} = (-1)^{k'_1 - k'_2} \frac{d(\pi)d(\sigma)}{(2\pi)^2}.$$

A.9. Computation of $c_{n,l}$. We will prove the following formula:

Proposition A.8.

$$\binom{a}{n} \binom{b}{l} a_{\Pi^{\vee}}^{(0)} c_{n,l} = (-1)^{a-n+b-l}.$$

Proof. Assume that $n \geq l$. Let $\mathfrak{gl}(V)$ denote the Lie algebra of $\mathrm{GL}(V)$. Put

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\pi(-{}^t E_1) = y'_2 \frac{\partial}{\partial y'_1} - x'_1 \frac{\partial}{\partial x'_2}, \quad \pi(-{}^t E_2) = x'_2 \frac{\partial}{\partial x'_3} - y'_3 \frac{\partial}{\partial y'_2}.$$

Let $\bar{w}_{0,0} = y_1^a x_3^b \in \mathfrak{H}_{a,b}^{\vee} \subset \mathcal{P}_{b,a}$ be the lowest weight vector. Put

$$\bar{w}_{n,0} = \frac{(a-n)!}{a!} \pi(-{}^t E_1)^n \bar{w}_{0,0} = y_1^{a-n} y_2^n x_3^b.$$

We define a vector $\bar{w}_{n,l} \in \mathfrak{H}_{a,b}^{\vee} \subset \mathcal{P}_{b,a}$ by $\bar{w}_{n,l} = \pi(-{}^t E_2)^l \bar{w}_{n,0}$. Observe that

$$\begin{aligned} \bar{w}_{n,l} &= \left(x'_2 \frac{\partial}{\partial x'_3} - y'_3 \frac{\partial}{\partial y'_2} \right)^l y_1^{a-n} y_2^n x_3^b \\ &= \sum_{m=0}^{\min\{n,l\}} \binom{l}{m} \frac{n!b!}{(n-m)!(b-l+m)!} y_1^{a-n} y_2^{n-m} x_2^{l-m} (-y'_3)^m x_3^{b-l+m}. \end{aligned}$$

By the definition of $c_{n,l}$ we have

$$\begin{aligned} &\ell_{\Pi}(\bar{w}_{n,l} X_2^{a-n+b-l} \otimes \mathbf{P}_{\Pi^{\vee}}) \\ &= c_{n,l} \ell_{\sigma \otimes \sigma^{\vee}}(\nabla_{n,l} \bar{w}_{n,l} X_2^{a-n+b-l} \otimes \nabla_{n,l} \mathbf{P}_{\Pi^{\vee}}) \\ &= c_{n,l} \ell_{\sigma \otimes \sigma^{\vee}} \left(\frac{b! Y_1^{a-n+b-l} X_2^{a-n+b-l}}{(b-l)!} \otimes a_{\Pi^{\vee}}^{(0)} \mathbf{P}_{a-n+b-l} \right) = (-1)^{a-n+b-l} \frac{c_{n,l} a_{\Pi^{\vee}}^{(0)} b!}{(b-l)!}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\ell_{\Pi}(\bar{w}_{n,l} X_2^{a-n+b-l} \otimes \mathbf{P}_{\Pi^{\vee}}) \\ &= \sum_{m=0}^{\min\{n,l\}} (-1)^m a_{\Pi^{\vee}}^{(m)} \ell_{\Pi}(\bar{w}_{n,l} X_2^{a-n+b-l} \otimes \mathbf{P}_{\Pi^{\vee}}^{(m)}) \cdot x_2^{n-m} y_2^{l-m} \end{aligned}$$

is the sum of the product of $(-1)^m a_{\Pi^{\vee}}^{(m)} \binom{l}{m} \frac{n!b!}{(n-m)!(b-l+m)!}$ and

$$\begin{aligned} &\ell_{\Pi}(y_1^{a-n} y_2^{n-m} x_2^{l-m} (-y'_3)^m x_3^{b-l+m} X_2^{a-n+b-l} \otimes \mathbf{P}_{\Pi^{\vee}}^{(m)}) \cdot x_2^{n-m} y_2^{l-m} \\ &= (-1)^m (n-m)!(l-m)! \frac{(a-n)!}{a!} m! \frac{(b-l+m)!}{b!} \end{aligned}$$

over $m = 0, 1, 2, \dots, \min\{n, l\}$. It follows that

$$\begin{aligned} (-1)^{a-n+b-l} c_{n,l} a_{\Pi^\vee}^{(0)} &= \frac{(b-l)!}{b!} \sum_{m=0}^{\min\{n,l\}} (-1)^m a_{\Pi^\vee}^{(m)} \binom{l}{m} n! (-1)^m (l-m)! \frac{(a-n)!}{a!} m! \\ &= \binom{a}{n}^{-1} \binom{b}{l}^{-1} \sum_{m=0}^{\min\{n,l\}} a_{\Pi^\vee}^{(m)} = \binom{a}{n}^{-1} \binom{b}{l}^{-1} \end{aligned}$$

by Lemma A.5. \square

APPENDIX B. LOCAL INTEGRALS AT NON-SPLIT PRIMES

Let E/F be a quadratic extension of non-archimedean local fields. Fix a prime element ϖ of the integer ring \mathfrak{o} of F . The order of the residue field of \mathfrak{o} is denoted by q . Put $E^- = \{x \in E \mid x^c = -x\}$. We write $|\cdot|_E$ for the normalized absolute value of E . We denote by e the ramification index of E/F .

In this section we assume that H is quasi-split. Then we may view G and H as unitary groups of Hermitian matrices

$$T_0 = w_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T'_0 = w_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Fix unramified characters χ, ρ of E^\times . Put $\beta = \chi(\varpi_E)$, $\gamma = \rho(\varpi_E)$ and

$$\begin{aligned} \mathbf{m}'(\alpha) &= \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha}^{-1} \end{bmatrix}, & \mathbf{u}(b) &= \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \\ \mathbf{m}'(\alpha, g_0) &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & g_0 & 0 \\ 0 & 0 & \bar{\alpha}^{-1} \end{bmatrix}, & \mathbf{u}(\beta, b) &= \begin{bmatrix} 1 & \beta & -\frac{\beta\beta^c}{2} + b \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

for $\alpha \in E^\times$, $b \in E^-$, $g_0 \in E^1$ and $\beta \in E$, where ϖ_E is a generator of the maximal ideal of \mathfrak{t} . The space V'_ρ of the principal series $\sigma = I'(\rho)$ consists of functions $f' : H \rightarrow \mathbb{C}$ which satisfy

$$f'(\mathbf{m}'(\alpha)\mathbf{u}(b)h) = \rho(\alpha)|\alpha|_E^{1/2} f'(h)$$

for $\alpha \in E^\times$, $b \in E^-$ and $h \in H$. The space V_χ of the principal series $\pi = I(\chi)$ consists of functions $f : G \rightarrow \mathbb{C}$ which satisfy

$$f(\mathbf{m}'(\alpha, g_0)\mathbf{u}(\beta, b)g) = \chi(\alpha)|\alpha|_E f(g)$$

for $\alpha \in E^\times$, $g_0 \in G_0$, $\beta \in E$, $b \in E^-$ and $g \in G$.

Let \mathcal{K}' and \mathcal{K} be special good maximal bounded subgroups of H and G in the sense of §1.1 of [Cas80]. There exist $f'_{\mathcal{K}'} \in V'$ and $f_{\mathcal{K}} \in V$ such that $f'_{\mathcal{K}'}(k') = f_{\mathcal{K}}(k) = 1$ for $k' \in \mathcal{K}'$ and $k \in \mathcal{K}$. We consider the zonal spherical functions

$$L'(h) = \langle \sigma(h) f'_\rho, f'_{\rho^{-1}} \rangle', \quad L(g) = \langle \pi(g) f_\chi, f_{\chi^{-1}} \rangle$$

for $h \in H$ and $g \in G$. We here normalize the local perfect pairings $\langle \cdot, \cdot \rangle'$ and $\langle \cdot, \cdot \rangle$ so that $L'(k') = L(k) = 1$ for $k' \in \mathcal{K}'$ and $k \in \mathcal{K}$. Put

$$C'(\rho) = \frac{1 - q^{-1}\gamma^{-e}}{1 - \gamma^{-e}}, \quad C(\chi) = \frac{(1 - q^{-2/e}\beta^{-1})(1 + q^{e-2}\beta^{-1})}{1 - \beta^{-2}}.$$

Proposition B.1.

$$\begin{aligned} L'(\mathbf{m}'(\varpi_E^m)) &= (1 + q^{-1})^{-1} q^{-m/e} (C'(\rho)\gamma^m + C'(\rho^{-1})\gamma^{-m}), \\ L(\mathbf{m}'(\varpi_E^m, 1)) &= (1 + q^{2e-5})^{-1} q^{-2m/e} (C(\chi)\beta^m + C(\chi^{-1})\beta^{-m}). \end{aligned}$$

Proof. Let α' be the positive root of H and α the positive root of G . In view of (15) of [Cas80] we have

$$\begin{aligned} q_{\alpha'} &= q_{\alpha} = q, & q_{\alpha'/2} &= 1, & q_{\alpha/2} &= q^{4-2e}, \\ a'_{\alpha} &= \begin{bmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{bmatrix}, & a_{\alpha} &= \begin{bmatrix} \varpi_E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\varpi_E^{-1})^c \end{bmatrix}. \end{aligned}$$

Then the c -function for V'_ρ (resp. V_χ) is given by $C'(\rho)$ (resp. $C(\chi)$). The stated formulas are special cases of the Macdonald formula (see Theorem 4.2 of [Cas80]). \square

We will compute the following integral

$$J(f_\chi, f'_\rho) = \int_H L(u(h))L'(h) dh.$$

Proposition B.2.

$$J(f_\chi, f'_\rho) = \mathcal{L}(\pi \times \sigma).$$

Proof. When $e = 1$, the formula is proved in [Har14, Theorem 2.12]. Recall that

$$\begin{aligned} L(s, \pi \times \sigma) &= L(s, \chi\rho)L(s, \chi\rho^{-1})L(s, \chi^{-1}\rho)L(s, \chi^{-1}\rho^{-1})L(s, \rho)L(s, \rho^{-1}), \\ L(s, \sigma, \text{Ad}) &= \zeta_F(s)L(s, \epsilon_{E/F})L(s, \rho|_{F^\times})L(s, \rho^{-1}|_{F^\times}), \\ L(s, \pi, \text{Ad}) &= \zeta_E(s)L(s, \epsilon_{E/F})L(s, \chi)L(s, \chi^{-1})L(s, \epsilon_{E/F}\chi_0)L(s, \epsilon_{E/F}\chi_0^{-1}) \end{aligned}$$

(cf. Remark 4.1), where χ_0 stands for the restriction of χ to F^\times . By the Cartan decomposition we have

$$J(f_\chi, f'_\rho) = \sum_{m=0}^{\infty} L(\mathbf{m}'(\varpi_E^m, 1))L'(\mathbf{m}'(\varpi_E^m))[\mathcal{K}'\mathbf{m}'(\varpi_E^m)\mathcal{K}' : \mathcal{K}'].$$

Since $[\mathcal{K}'\mathbf{m}'(\varpi_E^m)\mathcal{K}' : \mathcal{K}'] = q^{2m/e}(1 + q^{-1})$ if $m > 0$, Proposition B.1 gives

$$J(f_\chi, f'_\rho) = 1 + \frac{1}{1 + q^{2e-5}} \sum_{m=1}^{\infty} \frac{(C(\chi)\beta^m + \frac{C(\chi^{-1})}{\beta^m})(C'(\rho)\gamma^m + \frac{C'(\rho^{-1})}{\gamma^m})}{q^{m/e}}.$$

Put $x = q^{-1/2}$. If $e = 2$, then

$$\mathcal{L}(\pi \times \sigma) = \frac{(1 - x^2)(1 + \gamma x)(1 + \gamma^{-1}x)(1 - \beta x^2)(1 - \beta^{-1}x^2)}{(1 + x^2)(1 - \beta\gamma x)(1 - \beta\gamma^{-1}x)(1 - \beta^{-1}\gamma x)(1 - \beta^{-1}\gamma^{-1}x)}$$

and $(1 + x^2)(J(f_\chi, f'_\rho) - 1)$ equals

$$\begin{aligned} & \frac{1 - x^2\gamma^{-2}}{1 - \gamma^{-2}} \cdot \frac{1 - x^2\beta^{-1}}{1 - \beta^{-1}} \cdot \frac{\beta\gamma x}{1 - \beta\gamma x} + \frac{1 - x^2\gamma^2}{1 - \gamma^2} \cdot \frac{1 - x^2\beta^{-1}}{1 - \beta^{-1}} \cdot \frac{\beta\gamma^{-1}x}{1 - \beta\gamma^{-1}x} \\ & + \frac{1 - x^2\gamma^{-2}}{1 - \gamma^{-2}} \cdot \frac{1 - x^2\beta}{1 - \beta} \cdot \frac{\beta^{-1}\gamma x}{1 - \beta^{-1}\gamma x} + \frac{1 - x^2\gamma^2}{1 - \gamma^2} \cdot \frac{1 - x^2\beta}{1 - \beta} \cdot \frac{\beta^{-1}\gamma^{-1}x}{1 - \beta^{-1}\gamma^{-1}x}. \end{aligned}$$

We can prove the wanted identity by a brute force calculation. \square

APPENDIX C. RAMIFIED COMPUTATIONS: THE MINUS SIGN CASE

C.1. Maximal compact subgroups. In this section E is a quadratic extension of a local field F of odd residual characteristic. We write \mathfrak{o} (resp. \mathfrak{r}) for the maximal compact subring of F (resp. E) and denote by \mathfrak{p} (resp. \mathfrak{q}) its maximal ideal. Given a Hermitian matrix T , we define the Hermitian form on $W = E^3$ by $(u, v)_T = {}^t u^c T v$, where

$$T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t_2 \end{bmatrix}.$$

We denote its unitary group by G . Put

$$e_1 = {}^t(1, 0, 0), \quad e_2 = {}^t(0, 1, 0), \quad e_3 = {}^t(0, 0, 1), \quad W' = Ee_1 \oplus Ee_3.$$

Let $H = \{h \in G \mid he_2 = e_2\}$ be the unitary group of W' .

Fix a generator ϖ of \mathfrak{p} . We choose t_1, t_2 in the following way:

- (i) if E/F is unramified, then $t_1 = -\frac{1}{2\varpi}$ and $t_2 = -1$;
- (ii) if E/F is ramified, then $t_1, t_2 \in \mathfrak{o}^\times$ and $-t_1 t_2 \notin N_{E/F}(E^\times)$.

Note that $T' = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$ is not split. The group H is compact.

We call an \mathfrak{r} -lattice L in W integral if $(x, x) \in \mathfrak{o}$ for every $x \in L$ and call L maximal if it is maximal among the integral \mathfrak{r} -lattices. Take the maximal \mathfrak{r} -integral lattice $\mathcal{L} = \mathfrak{r}e_1 \oplus \mathfrak{r}e_2 \oplus \mathfrak{r}e_3$ in Case (ii). Clearly,

$$(e_2, e_2)_T = 1, \quad e_2 \in \mathcal{L}, \quad \{(x, e_2)_T \mid x \in \mathcal{L}\} = \mathfrak{r}.$$

Put

$$\eta = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -\varpi & \varpi \end{bmatrix}, \quad \mathcal{L} = \left\{ \eta^{-1}x \mid x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1, x_2, x_3 \in \mathfrak{r} \right\}$$

in Case (i). Then

$${}^t \eta^c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \eta = -2\varpi T =: \mathcal{T}.$$

Thus \mathcal{L} is a maximal \mathfrak{r} -integral lattice with respect to \mathcal{T} . Note that

$$(e_2, e_2)_{\mathcal{T}} = -2\varpi, \quad e_2 \in \mathcal{L}, \quad \{(x, e_2)_{\mathcal{T}} \mid x \in \mathcal{L}\} = \mathfrak{r}.$$

Let $\mathcal{K} = \{g \in G \mid g\mathcal{L} = \mathcal{L}\}$ be a maximal compact subgroup of G .

C.2. The Ichino-Ikeda integral. Let χ be an unramified character of E^\times . We retain the notation in Appendix B. We will compute the integral

$$J(f_\chi, 1_H) = \int_H L(\iota(h)) dh.$$

Proposition C.1.

$$\mathcal{L}(\pi \times \sigma^\vee)^{-1} J(f_\chi, 1_H) = L(1, \epsilon_{E/F})^2.$$

Remark C.2. Let St_E denote the Steinberg representation of $\text{GL}_2(E)$. We write $\mathbb{1}_H$ for the trivial representation of H . Fix a non-trivial additive character Ψ on E . If χ is unramified and unitary, then

$$L(s, I(\chi) \times \mathbb{1}_H) = L(s, \text{St}_E \otimes \chi) L(s, \text{St}_E \otimes \chi^{-1}) L(s, \text{St}_E),$$

$$\varepsilon(s, I(\chi) \times \mathbf{1}_H, \Psi) = \varepsilon(s, \text{St}_E \otimes \chi, \Psi) \varepsilon(s, \text{St}_E \otimes \chi^{-1}, \Psi) \varepsilon(s, \text{St}_E, \Psi).$$

In particular,

$$\varepsilon(1/2, I(\chi) \times \mathbf{1}_H, \Psi) = -1.$$

Proof. Since $L(s, \mathbf{1}_H, \text{Ad}) = L(s, \epsilon_{E/F}) \zeta_F(s+1)$, we have

$$\mathcal{L}(\pi \times \mathbf{1}_H) = \frac{L(3, \epsilon_{E/F})}{L(1, \epsilon_{E/F}) L(1, \epsilon_{E/F} \chi_0) L(1, \epsilon_{E/F} \chi_0^{-1})}$$

(see Remark C.2 and the proof of Proposition B.2). In particular, we have $\mathcal{L}(\pi \times \mathbf{1}_H) = 1$ in Case (ii). Since $H \subset \mathcal{K}$ by Lemma 3.14 of [Shi08] applied with $q = 1$, $\varphi_0 = T$, $L = \mathcal{L}$, $\mathfrak{b} = \mathfrak{r}$ and $h = e_2$, there is nothing to compute in Case (ii).

Finally, we consider Case (i). Lemma 3.14 of [Shi08] applied with $q = -2\varpi$, $\varphi_0 = \mathcal{T}$, $L = \mathcal{L}$, $\mathfrak{b} = \mathfrak{r}$ and $h = e_2$ shows that $[H : H \cap \mathcal{K}] = 1 + q$. Moreover, its proof shows that

$$J(f_\chi, 1_H) = \frac{1}{q+1} \sum_{h \in H/H \cap \mathcal{K}} L(\iota(h)) = \frac{1}{q+1} (1 + qL(\mathbf{m}'(\varpi, 1))).$$

Put $\beta = \chi(\varpi)$. Proposition B.1 gives

$$L(\mathbf{m}'(\varpi, 1)) = \frac{C(\chi)\beta + C(\chi^{-1})\beta^{-1}}{q^2(1+q^{-3})} = \frac{L(3, \epsilon_{E/F})}{q^2} (\beta + \beta^{-1} + q^{-1} - q^{-2}),$$

which completes our proof. \square

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