微甲 I 組 (01-06班)-考題部分

1. 設 $f(x)=\ln(x+\sqrt{x^2+1})$，試求 $f^{(n)}(0)$．(10分)

解. 因$f'(x)=\frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}=\frac{\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}=(1+x^2)^{-\frac{1}{2}}$

$=\sum_{m=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2m-1}{2})}{m!}\cdot x^{2m},\ x\in(-1,1).$

故$f(x)=\sum_{m=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2m-1}{2})}{m!(2m+1)}\cdot x^{2m+1}$

因此, 若$n=2m$時,$\frac{f^{(n)}(0)}{n!}=\frac{f^{(2m)}(0)}{(2m)!}=0$, 故$f^{(2m)}(0)=0.$

若$n=2m+1$時,$\frac{f^{(n)}(0)}{n!}=\frac{f^{(2m+1)}(0)}{(2m+1)!}=\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2m-1}{2})}{m!}(2m+1)$. 故$f^{(2m+1)}(0)=$
$\frac{(2m)!}{m!}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2m-1}{2})=\frac{(2m)!}{m!}(-1)^m\cdot\frac{1}{2^m}\cdot 1\cdot 3\cdots(2m-1)=\frac{(-1)^m[(2m)!]^2}{2^{2m}(m!)^2}$

2. 設 $x\neq 1$，試求 $\tan^{-1}x+\tan^{-1}\frac{x+1}{x-1}$ 之值. (10分)

解1. 設$\alpha=\tan^{-1}x$,$\beta=\tan^{-1}\frac{x+1}{x-1}$, 則$\tan\alpha=x$,$\tan\beta=\frac{x+1}{x-1}$, 且$-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$,$-\frac{\pi}{2}<\beta<\frac{\pi}{2}$. 又$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}=\frac{x+\frac{x+1}{x-1}}{1-x\cdot\frac{x+1}{x-1}}=\frac{x^2-x+x+1}{x-1-x^2-x}=-1$,
當$x>1$時,$\frac{\pi}{4}<\alpha<\frac{\pi}{2}$,$\frac{\pi}{4}<\beta<\frac{\pi}{2}$, 故$\frac{\pi}{2}<\alpha+\beta<\pi$, 因此$\alpha+\beta=\frac{3\pi}{4}$;
當$x<1$時,$-\frac{\pi}{2}<\alpha<\frac{\pi}{4}$,$-\frac{\pi}{2}<\beta<\frac{\pi}{4}$,$-\pi<\alpha+\beta<\frac{\pi}{2}$, 因此,$\alpha+\beta=-\frac{\pi}{4}$.

解2. 考慮$f(x)=\tan^{-1}x+\tan^{-1}\frac{x+1}{x-1}$, $x\neq 1$, 因$f'(x)=\frac{1}{1+x^2}+\frac{-\frac{2}{(x-1)^2}}{1+(\frac{x+1}{x-1})^2}=\frac{1}{1+x^2}-\frac{2}{(x-1)^2+(x+1)^2}=\frac{1}{1+x^2}-\frac{2}{2(1+x^2)}=0$. 故$f(x)$在$(-\infty,1)$及$(1,\infty)$分別爲常數, 設$f(x)=\begin{cases}c_+, & 若x>1,\\ c_-, & 若x<1.\end{cases}$ 因$\lim_{x\to\infty}f(x)=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$, 故得$c_+=\frac{3\pi}{4}$, 又$f(0)=0-\frac{\pi}{4}=-\frac{\pi}{4}$, 故得$c_-=-\frac{\pi}{4}$.

3. 設D 爲一由曲線 $y=\frac{1}{x^2}$, $x=1$, $x=3$ 和 $y=0$ 等所圍成的區域．
試求 D 的形心. (10分)

解． $\bar{x}=\frac{\int_1^3 x\cdot\frac{1}{x^2}dx}{\int_1^3\frac{1}{x^2}dx}=\frac{\ln 3}{(\frac{2}{3})}=\frac{3}{2}\ln 3.$

$\bar{y}=\frac{\int_1^3\frac{1}{2}\cdot(\frac{1}{x^2})^2dx}{\int_1^3\frac{1}{x}dx}=\frac{\frac{13}{54}}{\frac{2}{3}}=\frac{13}{54}$

4. 求積分$\int\frac{e^x-1}{e^{2x}+e^{-x}}dx$.(10分)

解 令$e^x=t$, 則$e^x dx=dt$, $dx=\frac{1}{t}dt$

$\int\frac{e^x-1}{e^{2x}+e^{-x}}dx=e\int\frac{t-1}{t^2+t^{-1}}\frac{1}{t}dt=\int\frac{t-1}{t^3+1}dt$.

其中$\frac{t-1}{t^3+1}=\frac{t-1}{(t+1)(t^2-t+1)}=\frac{-2}{3(t+1)}+\frac{2t-1}{3(t^2-t+1)}$,

所以$\int\frac{e^x-1}{e^{2x}+e^{-x}}dx=\int\frac{-2}{3(t+1)}dt+\int\frac{2t-1}{3(t^2-t+1)}dt=-\frac{2}{3}\int\frac{1}{t+1}dt+\frac{1}{3}\int\frac{2t-1}{t^2-t+1}dt=$
$-\frac{2}{3}\ln|t+1|+\frac{1}{3}\ln|t^2-t+1|+c$

5. 設 $y=f(x)$ 在 $0\le x\le t$ 上的弧長 $s(t)=\frac{1}{2}(e^t-e^{-t})$ 並且 $f(x)$ 在$x=0$處有最小值1，試求 $f(x)$．(10分)

解 $\int_0^t\sqrt{1+(f'(x))^2}dx=\frac{1}{2}(e^t+e^{-t})$, 對$t$微分, 得$\sqrt{1+(f'(t))^2}=\frac{1}{2}(e^t+e^{-t})$,
解得$f'(x)=\pm\frac{1}{2}(e^x-e^{-x})$ (t換成x), $f(x)=\int f'(x)dx=\pm\frac{1}{2}(e^x+e^{-x})+C$,
已知 $f(0)=\pm1+C=1$, 故$C=0$或2, $C=0$時,$f(x)=-\frac{1}{2}(e^x+e^{-x})+2$(不合);$C=0$時,$f(x)=\frac{1}{2}(e^x+e^{-x})=\cosh x$(合)

6. 試求 $\lim_{x\to0}(\frac{1}{2-2\cos x}-\frac{1}{x^2})$．(10分)

解1. 因爲 $\lim_{x\to0}(\frac{1}{2-2\cos x}-\frac{1}{x^2})=\lim_{x\to0}\frac{x^2-(2-2\cos x)}{x^2(2-2\cos x)}=\lim_{x\to0}\frac{2x-2\sin x}{2x(2-2\cos x)+x^2\cdot2\sin x}=$
$\lim_{x\to0}\frac{x-\sin x}{2x-2x\cos x+x^2\sin x}=\lim_{x\to0}\frac{1-\cos x}{2-2\cos x+2x\sin x+2x\sin x+x^2\cos x}$
$=\lim_{x\to0}\frac{\sin x}{2\sin x+4\sin x+4x\cos x+2x\cos x-x^2\sin x}=\lim_{x\to0}\frac{\sin x}{6\sin x+6x\cos x-x^2\sin x}=$
$\lim_{x\to0}\frac{1}{6+\frac{x}{\sin x}\cos x-x^2}=\frac{1}{12}$

解2. 又 $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...+\frac{(-1)^nx^{2n}}{(2n)!}+...$

$\frac{1}{2-2\cos x}-\frac{1}{x^2}=\frac{x^2-2(1-\cos x)}{2(1-\cos x)x^2}=\frac{\frac{x^2}{2}-(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-...\frac{(-1)^{n+1}x^{2n}}{(2n)!}+...)}{x^2(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-...-\frac{(-1)^{n+1}x^{2n}}{(2n)!}+...)}=\frac{x^4(\frac{1}{24}-\frac{x^2}{6!}+...+\frac{(-1)^{n+1}x^{2n-4}}{(2n)!}-...)}{x^4(\frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-...+\frac{(-1)^{n+1}x^{2n-2}}{(2n)!}+...)}$
$\Rightarrow\frac{1}{12}$ as $x\to0$

7. (i) 若 $y=\frac{\ln x}{x}$，描繪其圖形 .(5分)

(ii) 試說明 $\pi^e<e^\pi$.(5分)

解

(i) $f(x)=\frac{\ln x}{x}$, $x>0$, $f'(x)=\frac{1-\ln x}{x^2}$.
On $x>e$ $\ln x>1$, $f'(x)<0$, $f(x)$ decreasing on $(0,\infty)$.
On $0<x<e$, $\ln x<1$, $f'(x)>0$, $f(x)$ increasing on $(0,e)$.
Hence $f(e)=\frac{\ln e}{e}=\frac{1}{e}$ is the absolutly maximum value of f.
$f''(x)=\frac{(-\frac{1}{x})x^2-(1-\ln x)2x}{x^4}=\frac{2(\ln x-\frac{3}{2})}{x^3}$.
On $x>e^{\frac{3}{2}}$, $\ln x>\frac{3}{2}$, $f''(x)>0$, f concave upward.

On $0 < x < e^{\frac{3}{2}}$, $\ln x < \frac{3}{2}$, $f''(x) < 0$, f concave downward.
Hence $(e^{\frac{3}{2}}, f(e^{\frac{3}{2}})) = (e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}})$ is the point of inflection.
$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$, $y = 0$ is a horizontal asymptote.
$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$, $x = 0$ is a vertical asymptote

(ii) $f(e) > f(\pi)$, $\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$, $\pi \ln e > e \ln \pi$, $e^\pi > \pi^e$.

8. 求積分$\int_0^{\frac{\pi}{2}} \sin^6 x dx$.(10分)

解 $\int_0^{\frac{\pi}{2}} \sin^6 x dx = -\int_0^{\frac{\pi}{2}} \sin^5 x d\cos x = -\sin^5 x \cos x|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot 5 \sin^4 x \cos x dx =$
$5\int_0^{\frac{\pi}{2}} \sin^4 x dx - 5\int_0^{\frac{\pi}{2}} \sin^6 x dx$, 所以
$6\int_0^{\frac{\pi}{2}} \sin^6 x dx = 5\int_0^{\frac{\pi}{2}} \sin^4 x dx$, $\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x dx$. 同理,
$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx = \frac{3}{4} \cdot \frac{1}{2}(x - \frac{1}{2}\sin 2x)|_0^{\frac{\pi}{2}} =$
$\frac{3}{8} \cdot \frac{\pi}{2}$. 所以
$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6} \times \frac{3}{8} \times \frac{\pi}{2} = \frac{5}{32}\pi$.

9. 試討論 p 值的範圍與下列無窮級數在收斂、發散上的關係
$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$. (10分)

解

(i) $\int \frac{dx}{x(\ln x)^p} = \int (\ln x)^{-p} d\ln x = \begin{cases} \ln\ln x & \text{, if } p = 1; \\ (\ln x)^{1-p} & \text{,otherwise} \end{cases}$. Hence $\int_2^\infty \frac{dx}{x(\ln x)^p} =$
$\lim_{a\to\infty} \int_2^a \frac{dx}{x(\ln x)^p} = \begin{cases} \lim_{a\to\infty}(\ln\ln a - \ln\ln 2) = \infty & \text{, if } p = 1; \\ \lim_{a\to\infty}((\ln a)^{1-p} - (\ln 2)^{1-p}) = \infty & \text{, if } p < 1; \\ \lim_{a\to\infty}((\ln a)^{1-p} - (\ln 2)^{1-p}) = -(\ln 2)^{1-p} & \text{, if } p > 1; \end{cases}$
Therefore $\int_2^\infty \frac{dx}{x(\ln x)^p}$ converges iff $p > 1$.

(ii) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges iff $p > 1$, by part(ii) and the integral test.

10. (i) 試求無窮級數 $\sum_{n=1}^{\infty} n(-x)^{n+1} = x^2 - 2x^3 + 3x^4 + ... + n(-x)^{n+1} + ...$ 的收斂半徑 .(5分)

(ii) 令 $f(x) = \sum_{n=1}^{\infty} n(-x)^{n+1}$, 對所有 $|x| < r$. 試求 $f(x)$ 與 $f(0.5)$, $f'(0.5)$兩者之值 . (5分)

(i) By the ratio test, $\lim_{x\to\infty} |\frac{(n+1)(-x)^{n+2}}{|n(-x)^{n+1}|}| = |x|$, so, if $|x| < 1$ then it's abs. conv ; if $|x| > 1$, then it is div. so the radius of conv. $r = 1$

(ii) Let $f(x) = \sum_{n=1}^{\infty} n(-x)^{n+1} = x^2 \sum_{n=1}^{\infty} n(-x)^{n-1}$
$= x^2(1 - 2x + 3x^2 + ... + n(-x)^{n-1} + ...)$.
Let $\sum_{n=1}^{\infty} (-x)^n = 1 + (-x) + (-x)^2 + ... + (-x)^n + ...$,

so $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$ for $|x| < 1$,
$(\frac{1}{1+x})' = \sum_{n=1}^{\infty} n(-x)^{n-1} \cdot (-1)$ for $|x| < 1$,
i.e. $(-1)(\frac{1}{(1+x)^2}) = \sum_{n=1}^{\infty} n(-x)^{n-1} \cdot (-1)$ for $|x| < 1$,
and $(\frac{1}{(1+x)^2}) = \sum_{n=1}^{\infty} n(-x)^{n-1}$ for $|x| < 1$.
Then $f(x) = \frac{x^2}{(1+x)^2}$, $f'(x) = (-2)\frac{-1}{(1+x)^2} + \frac{-2}{(1+x)^3}$.
Hence$f(0.5) = \frac{\frac{1}{4}}{(\frac{3}{2})^2} = \frac{1}{4} \times \frac{4}{9} = \frac{1}{9}$, $f'(0.5) = \frac{8}{27}$.