Answers for Calculus A Midterm Examination

1. Use implicit differentiation to find the tangent line at $(2,2)$ of the graph of the function $2x^3 - 3y^2 = 4$. Also find the second derivative d^2y/dx^2 at $(2,2)$.

Sol : Using implicit differentiation, we obtain

$$2x^3 - 3y^2 = 4, \quad 6x^2 - 6y\frac{dy}{dx} = 0, \quad 12x - 6(\frac{dy}{dx})^2 - 6y\frac{d^2y}{dx^2} = 0.$$

At $(2,2)$, $24 - 12\frac{dy}{dx} = 0$, and $24 - 6(\frac{dy}{dx})^2 - 12\frac{d^2y}{dx^2} = 0$. Then

$$\frac{dy}{dx} = 2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 0 \quad \text{at } (2,2)$$

The tangent line at $(2,2)$ is $y - 2 = 2(x-2)$, i.e. $y = 2x - 2$.

2. For a ball with radius r meter, its surface area and volume are $A = 4\pi r^2$ meter^2 and $v = \frac{4}{3}\pi r^3$ meter^3, respectively.

(a) When $r = 2$ meter, what is the instantaneous rate of change of colume with respect to the radius? You should indicate the unit of your answer.

(b) Let h be a very small positive number. When the radius increases from 2 meter to $(2+h)$ meter, use differential to give an approximation on the increase of volume.

(c) Use geometric point of view to explain the relationship between the increase of volume derived in (b) to the surface area of the ball with radius 2 meter.

Sol : (a) The instantaneous rate of change of volume at $r = 2$ equals

$$\frac{dV}{dr} = 4\pi r^2 = 16\pi \text{ (meter}^2\text{)}$$

(b) Since $\frac{dV}{dr} = 4\pi r^2$, for a small h, $V(r+h) \approx V(r) + \frac{dV}{dr} \cdot h = \frac{4}{3}\pi r^3 + 4\pi h r^2$. For $r = 2$, $\Delta V = V(2+h) - V(2) \approx 4\pi h \cdot 2^2 = 16\pi h$.

(c) Since $\Delta V \approx 16\pi h$ meter^3, and the surface area of the ball with radius 2 meter is 16π meter^2, the increase of volume ΔV is approximately the same as the surface area multiplies the increment of radius h.

3. Set $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a) Find $f'(0)$.

(b) When $x \neq 0$, find $f'(x) =$?

(c) Dos $f''(0)$ exist? If it exists, please find its value. If not, give reason to support your argument.

Sol : (a) $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0.$

(b) When $x \neq 0$, since x^2 and $\sin\frac{1}{x}$ are differentiable, $f(x) = x^2\sin\frac{1}{x}$ is also differentiable, and

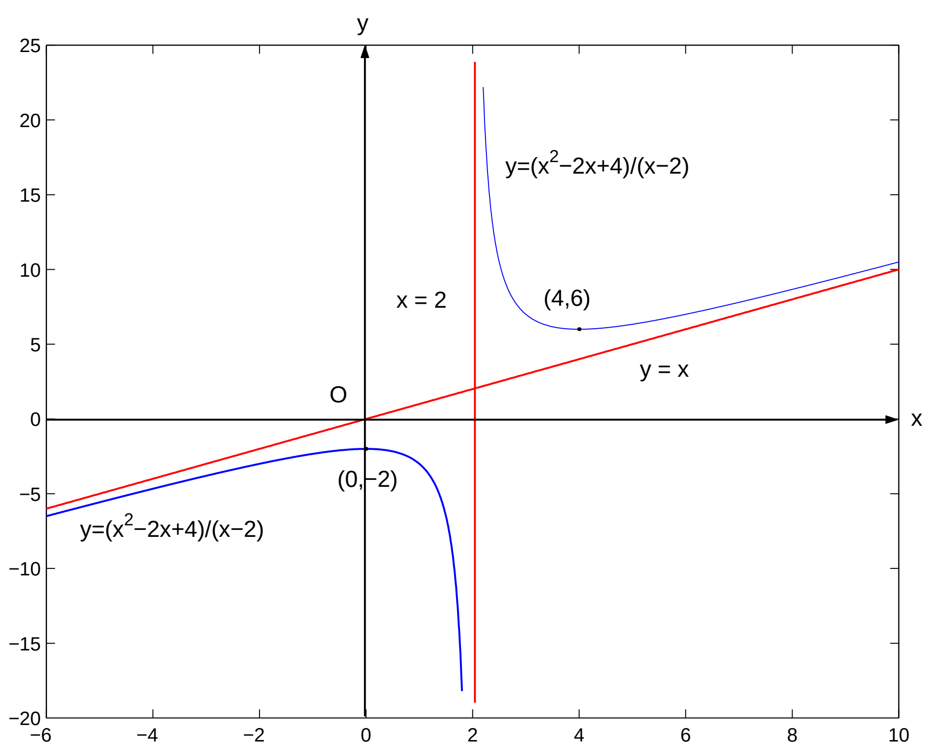
$$f'(x) = \frac{d}{dx}(x^2 \cdot \sin\frac{1}{x}) = (\frac{d}{dx}x^2) \cdot \sin\frac{1}{x} + x^2 \cdot (\frac{d}{dx}\sin\frac{1}{x}) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

(c) $f''(0) = \lim_{h\to 0}\frac{f'(h) - f'(0)}{h} = \lim_{h\to 0}(2\sin\frac{1}{h} - \frac{1}{h}\cos\frac{1}{h})$.

If we make h tend to 0 by $h = \frac{1}{2n\pi}$, the absoulute value of the limit tends to infinity. So the limit does not exist.

4. Sketch the graph of a rational function $y = f(x) = \frac{x^2-2x+4}{x-2}$. In the graph, you should discuss the concavity of f, describe the rise and fall of f, and find asymptotes, local minimum and maximum if they exist.

Sol : $y = f(x) = \frac{x^2-2x+4}{x-2} = x + \frac{4}{x-2}$, $f'(x) = 1 - \frac{4}{(x-2)^2}$, $f''(x) = \frac{8}{(x-2)^3}$.
$f'(x) = 0$ iff $x = 0$ or 4 $f(4) = 6$, $f(0) = -2$.
$f' > 0$ if $x > 4$ or $x < 0$, the graph rises in $x > 4$ or $x < 0$.
$f' < 0$ if $0 < x < 2$ or $2 < x < 4$, the graph falls in $0 < x < 2$, $2 < x < 4$.
$f'' > 0$ if $x > 2$, the graph is always concave up in $x > 2$.
$f'' < 0$ if $x < 2$, the graph is always concave down in $x < 2$.
The graph of $y = f(x) = \frac{x^2-2x+4}{x-2}$ and its asymptotes:



5. Assume $0 \le a \le 1$. Find the value of a such that $\int_0^1 |x^2 - ax|dx$ achieves its maximum.

Sol : $0 \le a \le 1$, $|x^2 - ax| = \begin{cases} -x^2 + ax, & 0 \le x \le a \\ x^2 - ax, & a \le x \le 1 \end{cases}$. Then

$$\int_0^1 |x^2 - ax|dx = \int_0^a (-x^2 + ax)dx + \int_a^1 (x^2 - ax)dx$$

$= (-\frac{1}{3}x^3 + \frac{1}{2}ax^2)|_0^a + (\frac{1}{3}x^3 - \frac{1}{2}ax^2)|_a^1 = \frac{1}{3}a^3 - \frac{1}{2}a + \frac{1}{3} = f(a).$

The integral can achieve its maximum only at the end points 0, 1 or the point $c \in (0,1)$ for which $f'(c) = 0$.

$f(0) = \frac{1}{3}; f(1) = \frac{1}{6}.$

Since $f'(c) = c^2 - \frac{1}{2}$, $f'(c) = 0$ iff $c^2 = \frac{1}{2}$, or $c = \frac{1}{\sqrt{2}}$, $f(\frac{1}{\sqrt{2}}) = \frac{1}{3} - \frac{1}{3\sqrt{2}}$.

So the integral achieves its maximum at $a = 0$.

6. Find $\frac{d}{dx}\int_{2x}^{x^2} \cos\sqrt{t}dt$ when $x > 0$.

Sol : Let $F(x) = \int_0^x \cos\sqrt{t}dt$, since $\cos\sqrt{t}$ is continuous, by fundamental theorem of calculus, we have $F'(x) = \cos\sqrt{x}$, and for all $a, b \in \mathbb{R}$, $F(b) - F(a) = \int_a^b \cos\sqrt{t}dt$. So when $x > 0$,

$$\frac{d}{dx}\int_{2x}^{x^2} \cos\sqrt{t}dt = \frac{d}{dx}[F(x^2) - F(2x)] = F'(x^2)\cdot 2x - F'(2x)\cdot 2 = 2x\cos x - 2\cos\sqrt{2x}.$$

7. Evaluate the following integral:

(a) $\int \frac{\sin(3t+2)}{\cos^5(3t+2)}dt$; (b) $\int \frac{1}{x^2}\sqrt{1 - \frac{1}{x}}dx$.

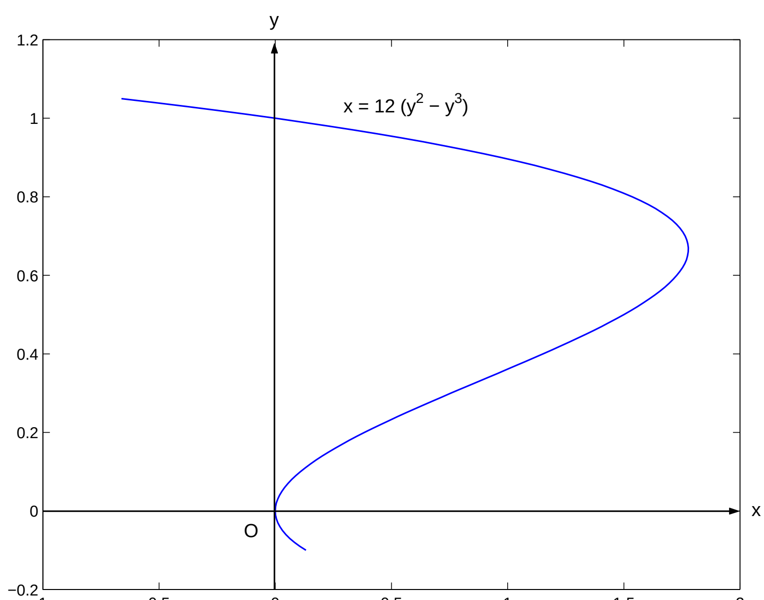
Sol : (a) Let $u = \cos(3t+2)$, then $du = -3\sin(3t+2)dt$,

$$\int \frac{\sin(3t+2)}{\cos^5(3t+2)}dt = -\frac{1}{3}\int \frac{du}{u^5} = \frac{1}{12}u^{-4} + C = \frac{1}{12}\cdot\frac{1}{\cos^4(3t+2)} + C$$

(b) Let $u = 1 - \frac{1}{x}$, then $du = \frac{1}{x^2}dx$,

$$\int \frac{1}{x^2}\sqrt{1 - \frac{1}{x}}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1 - \frac{1}{x})^{\frac{3}{2}} + C$$

8. Find the volume of the solid generated by revolving the shaded region (which is enclosed by $x = 0$ and $x = 12(y^2 - y^3)$ as indicated in the following figure) about the x-axis.



Sol : By shell method:

$$V = \int_0^1 2\pi y \cdot 12(y^2 - y^3)dy = 24\pi \int_0^1 (y^3 - y^4)dy$$

$$= 24\pi(\frac{y^4}{4} - \frac{y^5}{5})|_0^1 = 24\pi(\frac{1}{4} - \frac{1}{5}) = \frac{24\pi}{20} = \frac{6\pi}{5}$$

9. Suppose that $y = \frac{1}{x^2+1}$, $x \in \mathbb{R}$. Form a tangent line $L = \overline{PQ}$ by picking two points P and Q in the graph of this funcrtion. Let m denote the slope of L. Show that $|m| \leq 3\sqrt{4}/8$.

Proof : Give $P = (a, \frac{1}{a^2+1})$ and $Q = (b, \frac{1}{b^2+1})$, the slope of the line L is $\frac{f(b)-f(a)}{b-a}$
By mean value theorem, there is a c between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = -\frac{2c}{(c^2 + 1)^2}$$

Since $\frac{d}{dx}(-\frac{2x}{(x^2+1)^2}) = -2\frac{1-3x^2}{(x^2+1)^3}$, $f''(x)$ attains its maximum and minimum at the point $x = \pm\frac{1}{\sqrt{3}}$. As x tends to infinity, $f'(x)$ tends to zero.
So the absolute maximun and minimum of the function $f'(x)$ are

$$f'(-\frac{1}{\sqrt{3}}) = \frac{3\sqrt{3}}{8} \quad , \quad f'(\frac{1}{\sqrt{3}}) = -\frac{3\sqrt{3}}{8} \quad \text{respectively.}$$

So $|m| \leq \frac{3\sqrt{3}}{8}$.