Answers for Calculus A Final Examination

1. Consider a segment of the curve described by the equation $x^{2/3} + y^{2/3} = 1$ in the first quadrant (*i.e.*, when $0 \le x \le 1$ and $0 \le y \le 1$).

(a) Find the length of the curve.

(b) Find the area of the surface generated by revolving the curve about the x-axis.

(c) Find the centroid of the curve. From that, what can you say about the area of the surface generated by revolving the curve about the $x = -1$ line.

Sol : (a) View y as a function of x: $y = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$, for $0 \le x \le 1$.

$$dy = \frac{3}{2}(1 - x^{\frac{2}{3}})^{\frac{1}{2}} \cdot (-\frac{2}{3}x^{-\frac{1}{3}})dx = -(x^{-\frac{2}{3}} - 1)^{\frac{1}{2}}dx$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + x^{-\frac{2}{3}} - 1}dx = x^{-\frac{1}{3}}dx$$

So the length of the curve $= \displaystyle\int_0^1 x^{-\frac{1}{3}}dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1 = \frac{3}{2}$

(b) The area of the surface generated by revolving the cureve about the x-axis is

$$\begin{aligned}
\int_0^1 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}dx &= \int_0^1 2\pi(1 - x^{\frac{2}{3}})^{\frac{3}{2}}x^{-\frac{1}{3}}dx \quad (\text{let } t = x^{2/3}) \\
&= \int_0^1 2\pi(1-t)^{\frac{3}{2}} \cdot \frac{3}{2}dt \quad (dt = \frac{2}{3}x^{-1/3}dx) \\
&= 3\pi\int_0^1 (1-t)^{\frac{3}{2}}dt \\
&= 3\pi\,[-\frac{2}{5}(1-t)^{\frac{5}{2}}]\Big|_0^1 \\
&= \frac{6\pi}{5}
\end{aligned}$$

(c) Suppose that the coordinate of the centroid is (m_x, m_y), then

$$m_x = \frac{\int_{x=0}^{x=1} x\,ds}{3/2} = \frac{2}{3}\int_0^1 x^{2/3}dx = \frac{2}{3}\cdot\frac{3}{5} = \frac{2}{5}$$

By Pappus's Theorem, the area of the surface generated by revolving the curve is

$$S = 2\pi r L = 2\pi[\frac{2}{5} - (-1)]\frac{3}{2} = \frac{21\pi}{5}$$

This value is the same as

$$\int_0^1 2\pi[(1 - y^{2/3})^{3/2} + 1]y^{-1/3}dy \quad (\int 2\pi(x+1)ds)$$

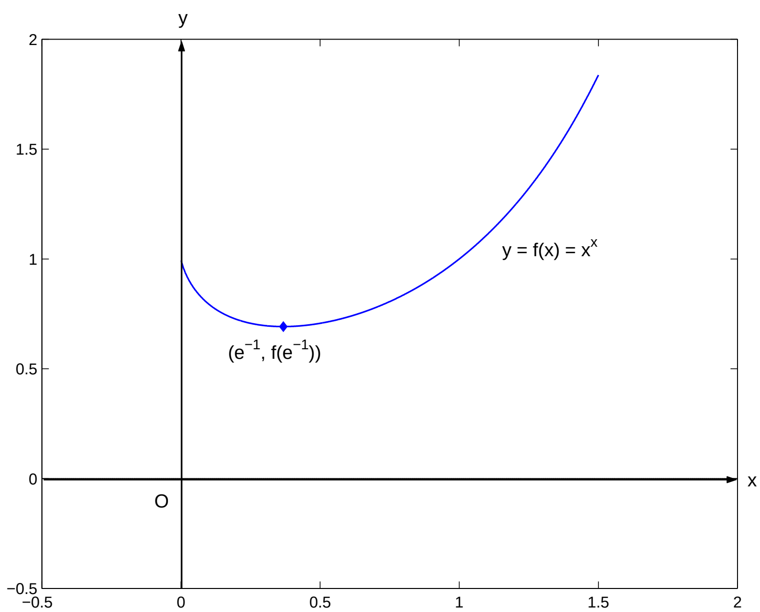
2. Consider $f(x) = x^x$, $x > 0$, evaluate $f'(x)$, $f''(x)$, and $\lim_{x\to 0^+} f(x)$. Give a full discussion of the monotonicity, concavity, local extrema of $f(x)$, and sketch the graph.

Sol : $f(x) = x^x = e^{x\ln x}$.

$$\begin{aligned} f'(x) &= e^{x\ln x}\cdot\frac{d}{dx}(x\ln x) = x^x(\ln x+1) \\ f''(x) &= (\ln x+1)\cdot\frac{d}{dx}x^x + x^x\cdot\frac{d}{dx}(\ln x+1) = x^x(\ln x+1)^2 + x^{x-1} \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} e^{x\ln x} = \exp(\lim_{x\to 0^+} x\ln x) = \exp(\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}) \\ &= \exp(\lim_{x\to 0^+}\frac{1/x}{-x^{-2}}) \quad \text{(by L'Hôpital's rule)} \\ &= \exp(\lim_{x\to 0^+}(-x)) = 1 \end{aligned}$$

Since $x > 0$, $x^x > 0$, we know that $f''(x) > 0$ for all $x > 0$, *i.e.*, f is always concave up.
$f'(x) > (<) 0 \Leftrightarrow (\ln x + 1) > (<) 0 \Leftrightarrow \ln x > (<) -1 \Leftrightarrow x > (<) e^{-1}$.
Hence f is increasing if $x > e^{-1}$, and is decreasing if $0 < x < e^{-1}$.
$f'(x) = 0 \Leftrightarrow x = e^{-1}$. Since $f''(x) > 0$ for all $x > 0$, we know that f attains minimum at $x = e^{-1}$.
Note that since the limit of $f(x)/x$ as x tends to infinity does not exists, there is not aysmptote.

The graph of $f(x) = x^x$:



3. Find the limits

$$\text{(a)} \quad \lim_{x\to 1^+}\left(\frac{1}{x-1} - \frac{1}{\ln x}\right), \qquad \text{(b)} \quad \lim_{x\to 0^+}\frac{\exp(-1/x^2)}{x}.$$

Sol : (a)

$$\begin{aligned} \lim_{x\to 1^+}\left(\frac{1}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to 1^+}\frac{\ln x-(x-1)}{(x-1)\ln x} \\ &= \lim_{x\to 1^+}\frac{1/x-1}{\ln x+(x-1)/x} \quad \text{(by L'Hôpital's rule)} \\ &= \lim_{x\to 1^+}\frac{1-x}{x\ln x+x-1} \end{aligned}$$

$$= \lim_{x\to 1^+} \frac{-1}{\ln x + x \cdot 1/x + 1} \quad \text{(by L'Hôpital's rule)}$$
$$= \frac{-1}{2}$$

(b) Let $y = 1/x$, then as x tends to 0^+, y tends to infinity. Then

$$\lim_{x\to 0^+} \frac{\exp(-1/x^2)}{x} = \lim_{y\to\infty} \frac{\exp(-y^2)}{1/y}$$
$$= \lim_{y\to\infty} \frac{y}{\exp(y^2)}$$
$$= \lim_{y\to\infty} \frac{1}{2y\exp(y^2)} \quad \text{(by L'Hôpital's rule)}$$
$$= 0$$

4. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. How far from the front of the room should you sit so that your viewing angle of the blackboard is maximal (see the included figure below)?

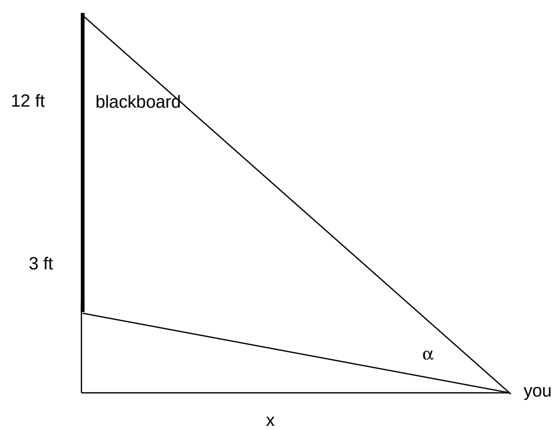


Figure 1: Figure for Problem #4

Sol : As the figure shows, let x be the distance between I and the wall. Then

$$\alpha = f(x) = \tan^{-1}\frac{15}{x} - \tan^{-1}\frac{3}{x}$$

The problem is equivalent to maximize α.

$$f'(x) = \frac{1}{1+(15/x)^2}\frac{d}{dx}(\frac{15}{x}) - \frac{1}{1+(3/x)^2}\frac{d}{dx}(\frac{3}{x}) = \frac{-12x^2+540}{(x^2+9)(x^2+225)}$$

$f'(x) = 0 \Leftrightarrow -12x^2 + 540 = 0 \Leftrightarrow x^2 = 45 \Leftrightarrow x = \pm 3\sqrt{5}$. Here we choose $x = 3\sqrt{5}$.
For $x < 3\sqrt{5}$, $f'(x) > 0$, and for $x > 3\sqrt{5}$, $f'(x) < 0$.
So f attains its maximum at $x = 3\sqrt{5}$, *i.e.*, the viewing angle is maximal when I sit $3\sqrt{5}$-ft from the front of the room.

5. In an oilrefinery a storage tank contains 2000 gallons of gasoline that initially has 100lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2lb of additive per gallon is pumped into the tank at the rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. Find the amount of additive in the tank 20 min after the process starts.

Sol : Suppose at time t (min) there are $x(t)$ lb additive in the tank. Then $x(0) = 100$. At time t, the rate of additive pumped into the tank is $40 \cdot 2 = 80$ lb/min, and since there are $2000 + (40-45)t = 2000 - 5t$ gallons of gasoline in the tank, and the rate of additive pumped out of the tank is $45 \cdot \frac{x(t)}{2000-5t}$. Hence we have the differential equation:

$$\frac{dx}{dt} = 80 - 45 \cdot \frac{x}{2000-5t} = 80 - \frac{9x}{400-t} \quad \text{with initial condition } x(0) = 100$$

Then

$$\frac{dx}{dt} + \frac{9}{400-t}x = 80 \Rightarrow \frac{d}{dt}\frac{x}{(400-t)^9} = \frac{80}{(400-t)^9} \Rightarrow$$

$$\frac{x}{(400-t)^9} = \int \frac{80}{(400-t)^9}dt = 10(400-t)^{-8} + C \quad \text{, or } x = 10(400-t) + C(400-t)^9$$

$x(0) = 100$ implies $C = \frac{100-4000}{400^9} = -\frac{3900}{400^9}$. Hence

$$\begin{aligned} x(20) &= -\frac{3900}{400^9} \cdot (400-20)^9 + 10 \cdot (400-20) \\ &= -3900 \cdot (\frac{380}{400})^9 + 10 \cdot 380 \\ &= -3900 \cdot 0.95^9 + 3800 \qquad (*) \\ &\approx 1342 \text{ (lb)} \end{aligned}$$

Rmk: The teacher said that if you have got the correct formula, then you will obtain the points if you have (*).

6. Evaluate the following integrals:

$$\text{(a)} \int x^{3/2} \ln x \, dx, \qquad \text{(b)} \int_0^3 \frac{x^2}{\sqrt{9+x^2}}\, dx,$$

$$\text{(c)} \int_1^\infty \left[\frac{x}{2(x^2+1)} - \frac{1}{2x}\right] dx, \qquad \text{(d)} \int \frac{x+1}{(x-1)^2(x^2+2)}\, dx.$$

Sol : (a) Set $u = \ln x$, $dv = x^{3/2}dx$, then $v = \frac{2}{5}x^{5/2}$, hence

$$\begin{aligned} \int x^{3/2} \ln x dx &= \ln x \cdot \frac{2}{5}x^{5/2} - \int \frac{2}{5}x^{5/2}\left(\frac{d}{dx}\ln x\right)dx \\ &= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{5/2} \cdot \frac{1}{x}dx \\ &= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}dx \\ &= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\cdot\frac{2}{5}x^{5/2} + C \\ &= \frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2} + C \end{aligned}$$

(b) Set $x = 3\sinh y$, then $dx = 3\cosh y dy$, and $x = 0 \to y = 0$, $x = 3 \to y = \ln(1+\sqrt{2})$, hence

$$\int_0^3 \frac{x^2}{\sqrt{9+x^2}}dx = \int_0^{\ln(1+\sqrt{2})} \frac{9\sinh^2 y}{\sqrt{9(1+\sinh^2 y)}} \cdot 3\cosh y dy$$

$$
\begin{aligned}
&= 9\int_0^{\ln(1+\sqrt{2})} \frac{\sinh^2 y}{3\cosh} \cdot 3\cosh y dy \\
&= 9\int_0^{\ln(1+\sqrt{2})} \sinh^2 y dy \\
&= 9\int_0^{\ln(1+\sqrt{2})} \frac{\cosh 2y - 1}{2} dy \\
&= \frac{9}{2}\left[\frac{1}{2}\sinh 2y\Big|_0^{\ln(1+\sqrt{2})} - y\Big|_0^{\ln(1+\sqrt{2})}\right] \\
&= \frac{9}{2}\left[\frac{e^{2\ln(1+\sqrt{2})} - e^{-2\ln(1+\sqrt{2})}}{4} - \ln(1+\sqrt{2})\right] \\
&= \frac{9}{2}\left[\frac{(1+\sqrt{2})^2 - (1+\sqrt{2})^{-2}}{4} - \ln(1+\sqrt{2})\right] \\
&= \frac{9}{2}\left[\sqrt{2} - \ln(1+\sqrt{2})\right]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\int_1^{\infty}\left[\frac{x}{2(x^2+1)} - \frac{1}{2x}\right]dx &= \lim_{t\to\infty}\int_1^t \left[\frac{x}{2(x^2+1)} - \frac{1}{2x}\right]dx \\
&= \lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{4}\Big|_1^t - \frac{\ln x}{2}\Big|_1^t\right] \\
&= \lim_{t\to\infty}\frac{1}{4}\left[\ln(t^2+1) - \ln 2 - \ln t^2\right] \\
&= \frac{1}{4}\lim_{t\to\infty}\ln\frac{t^2+1}{2t^2} \\
&= \frac{1}{4}\ln\left(\lim_{t\to\infty}\frac{t^2+1}{2t^2}\right) \\
&= \frac{1}{4}\ln\frac{1}{2} = -\frac{\ln 2}{4}
\end{aligned}
$$

(d) Assume that

$$
\frac{x+1}{(x-1)^2(x^2+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+2}
$$

Then $[ax+(b-a)](x^2+2)+(x-1)^2(cx+d) = x+1$

Then $\begin{cases} a+c=0 \\ -a+b-2c+d=0 \\ 2a+c-2d=1 \\ 2(b-a)+d=1 \end{cases} \Rightarrow \begin{cases} a=-\frac{1}{9} \\ b=\frac{2}{3} \\ c=\frac{1}{9} \\ d=-\frac{5}{9} \end{cases}$ Hence

$$
\begin{aligned}
&\int \frac{x+1}{(x-1)^2(x^2+2)}dx = \int\left[-\frac{1}{9(x-1)} + \frac{2}{3(x-1)^2} + \frac{x-5}{9(x^2+2)}\right]dx \\
&= -\frac{1}{9}\ln(x-1) - \frac{2}{3}\cdot\frac{1}{x-1} + \frac{1}{18}\int\frac{2x dx}{x^2+2} - \frac{5}{9}\int\frac{dx}{x^2+2} \\
&= -\frac{1}{9}\ln(x-1) - \frac{2}{3}\cdot\frac{1}{x-1} + \frac{1}{18}\ln(x^2+2) - \frac{5}{9\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C \\
&= -\frac{1}{9}\ln(x-1) - \frac{2}{3}\cdot\frac{1}{x-1} + \frac{1}{18}\ln(x^2+2) - \frac{5\sqrt{2}}{18}\tan^{-1}\frac{x}{\sqrt{2}} + C
\end{aligned}
$$

7. (a) Find the Taylor formula for the logarithmic function $\ln(1-t)$ about $t=0$,

(b) Let $F(x)=\int_0^x [\ln(1-t)/t]\,dt$, find a polynomial that will approximate $F(x)$ throughout the interval $[0,1/5]$ with an error of magnitude less than 10^{-3}.

Sol : (a) Let $f(t)=\ln(1-t)$, then $f^{(k)}(t)=-(k-1)!(1-t)^{-k}$, then by Taylor formula, there exists a ξ between 0 and t such that

$$f(t)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}t^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}t^{n+1}=\sum_{k=1}^{n}\frac{-t^k}{k}-\frac{t^{n+1}}{(n+1)(1-\xi)^{-(n+1)}}$$

(b) From (a), for $t\in(0,\ 0.2)$, there exists an $\xi\in(0,\ t)$ such that

$$\frac{\ln(1-t)}{t}=\sum_{k=1}^{n}\frac{-t^{k-1}}{k}-\frac{t^n}{(n+1)(1-\xi)^{-(n+1)}}$$

To approximate $F(x)$ in $(0,\ 0.2)$ with error less than 10^{-3}, choose n large enough so that the integral of remainder term is less than 10^{-3}. Since

$$\left|-\frac{t^n}{(n+1)(1-\xi)^{(n+1)}}\right|\le\frac{5^{n+1}t^n}{(n+1)4^{n+1}}$$

$$\left|\int_0^x\left[\frac{\ln(1-t)}{t}-\left(\sum_{k=1}^{n}\frac{-t^{k-1}}{k}\right)\right]dt\right|\le\int_0^{0.2}\left|\frac{\ln(1-t)}{t}-\left(\sum_{k=1}^{n}\frac{-t^{k-1}}{k}\right)\right|dt$$

$$\le\int_0^{0.2}\frac{t^n}{(n+1)(1-\xi)^{n+1}}dt\le\int_0^{0.2}\frac{5^{n+1}t^n}{(n+1)4^{n+1}}dt\le\frac{5^{n+1}0.2^{n+1}}{4^{n+1}(n+1)^2}\le\frac{1}{4^n}$$

Choose $n\ge\dfrac{3}{\log_{10}4}\approx 4.9829$, or $n=5$

We have

$$F(x)=\int_0^x\frac{\ln(1+t)}{t}dt\approx\int_0^x\sum_{k=1}^{5}\frac{-t^{k-1}}{k}dt=\sum_{k=1}^{5}\frac{-x^k}{k^2}=-x-\frac{x^2}{4}-\frac{x^3}{9}-\frac{x^4}{16}-\frac{x^5}{25}$$

with error less than $\dfrac{1}{4^6 6^2}\approx 6.782\times10^{-6}$.

Rmk : From the analysis, when $n\ge 3$, the error is less than 1.563×10^{-4}. So for all $n\ge3$ will get the credit.