微積分甲八十九學年度上學期期末考

1. 求下列之極限值 (10分)

(a) $\lim_{x\to\pi} \frac{e^{\sin x}-1}{x-\pi}$

(b) $\lim_{x\to 0^+} \frac{\int_0^x \sin(t^2)\,dt}{x\,\sin(x^2)}$

解答: (a) $\lim_{x\to\pi} \frac{e^{\sin x}-1}{x-\pi} = \lim_{x\to\pi} \frac{e^{\sin x}\cos x}{1}$ (3分) $= -1$ (5分)

(b) $\lim_{x\to 0^+} \frac{\int_0^x \sin(t^2)\,dt}{x\,\sin(x^2)} = \lim_{x\to 0^+} \frac{\sin x^2}{\sin x^2 + 2x^2\cos x^2}$ (1分)

$= \lim_{x\to 0^+} \frac{1}{1+2\frac{x^2}{\sin x^2}\cos x^2}$ (3分) $= \frac{1}{3}$ (5分)

2. 設 $f(x) = x^{(1/x^2)}$, $x > 0$, 試求 (10分)

(a) $f(x)$ 的最大値

(b) $\lim_{x\to\infty} f(x)$

解答: (a) 取 $\ln f(x) = \dfrac{\ln x}{x^3}$ 或寫成 $f(x) = e^{\frac{\ln x}{x^3}}$,
然後算出 $f'(x)$ 或 $\dfrac{d}{dx}\left(\dfrac{\ln x}{x^3}\right) = \dfrac{1-3x}{x^{\mathbf{4}}}$ 給2分.
正確求出臨界點 $x = e^{1/3}$ 再給3分, 說明 $x = e^{1/3}$ 確爲極大値 (而不是臨界點或極小値), 再給1分. 有兩法:
(1) 說明: 對 $x < e^{1/3}$, $f'(x) > 0$ 而對 $x > e^{1/3}$, $f'(x) < 0$.
(2) 說明在 $x = e^{1/3}$點時, $f''(x) < 0$.
嚴格說來 (2) 法只指出 $x = e^{1/3}$爲局部極大, 須再說明. (譬如說 $x = e^{1/3}$ 爲唯一之臨界點.) 但因只有1分,(2) 亦採計. 共6分, 小錯不扣.

(b) 用 L'Hopital's Rule 說明 $\lim_{x\to\infty} \dfrac{\ln x}{x^3} = \lim_{x\to\infty} \dfrac{1/x}{3x^2} = 0$ 給全分4分. 以下的解均給2分.
(1) 用 $\lim_{x\to\infty} x^{1/x} = 1$ 而未說明此等式爲何成立, 如:
(i)$1 \le x^{1/x^3} < x^{1/x} \Rightarrow \lim_{x\to\infty} x^{1/x^3} = \lim_{x\to\infty} x^{1/x} = 1$ (夾擠)
(ii) $\lim_{x\to\infty} x^{1/x^3} = \lim_{x\to\infty} (x^{1/x})^{1/x^2} = 1^0$
(iii) $\lim_{x\to\infty} x^{1/x^2} = \{\lim_{x\to\infty} (x^3)^{1/x^2}\}^{1/3} = 1^{1/3}$
(2) 利用 $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$ 或寫成 $\ln x =$ xmall order of x 而未說明此等式爲何成立, 如:
$\lim_{x\to\infty} \dfrac{\ln x}{x^3} = \lim_{x\to\infty} \dfrac{\ln x}{x} \cdot \lim_{x\to\infty} \dfrac{1}{x^2} = 0$
但只說: x 增加得比 $\ln x$ 快, 不給分.
(3) 設 $x^{1/x^3} = 1 + \alpha$, $\alpha > 0$. 則 $x = (1+\alpha)^{x^3} = 1 + x^3\alpha + \cdots$ (二項式定理). 所以 $x > x^3\alpha$ 或 $\dfrac{1}{x^2} > \alpha > 0$. 故 $\alpha \to 0$. 此法之缺點在於 x 非正整數時, $(1+\alpha)^{x^3}$ 之二項式展開, 只對 $|\alpha| < 1$成立.

3. 解微分方程:$\begin{cases} \dfrac{dy}{dx} = -\dfrac{\sqrt{1-x^2}}{x}, \quad 0 < x \leq 1 \\ y(1) = 0 \end{cases}$ (10分)

解答: $y = -\int \frac{1-x^2}{x}\,dx$ 令 $t = \sqrt{1-x^2}$

$$
\begin{aligned}
&= \int \frac{t}{\sqrt{1-t^2}} \cdot \frac{t}{\sqrt{1-t^2}} dt \\
&= \int \frac{t^2}{1-t^2} dt \\
&= \int -1 + \frac{1}{1-t^2} dt \\
&= -t + \frac{1}{2} \int \frac{1}{1-t} + \frac{1}{1+t} dt \\
&= -t - \frac{1}{2} \ln|1-t| + \frac{1}{2} \ln|1+t| + C \\
&= -\sqrt{1-x^2} + \frac{1}{2} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} + C
\end{aligned}
$$

$y(1) = 0 \Rightarrow 0 = \frac{1}{2} \ln \left(\frac{1+0}{1-0} \right) + C$

$\Rightarrow y = -\sqrt{1-x^2} + \frac{1}{2} \ln \left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \right)$

4. 有一桶子, 一開始裝有100公升的清水. 桶子上方有一龍頭, 以每分鐘1公升的速度將濃度爲 每公升10公克的食鹽水注入桶中. 有一攪拌器將桶中的溶液隨時攪拌均勻. 桶底有一出口, 以每分鐘3公升的速度 流出溶液. (10分)

(a) 以 $y(t)$ 表時間爲 t 時桶中的食鹽含量, 導出函數 $y(t)$ 所需滿足的微分方程式.

(b) 問從全是清水到桶內食鹽含量最多, 耗時多少?

解答: (a) $y(t)$ 爲 t 時刻的食鹽量

t 時刻的水量$=100+t-3t=100-2t$

$$y(t+\triangle t)=y(t)+10\triangle t-\frac{y(t)}{100-2t}\times 3\triangle t$$

$$\frac{y(t+\triangle t)-y(t)}{\triangle t}=\frac{-3y(t)}{100-2t}+10$$

$y(t)$ 滿足 $y'(t)=10-\dfrac{3y}{100-2t}$ 和 $y(0)=0$

(b) $\begin{cases} y'(t)+\dfrac{3y}{100-2t}=10 \\ y(0)=0 \end{cases}$

let $p(t)=\dfrac{3}{100-2t}\Rightarrow \displaystyle\int p(t)dt=\int\frac{3}{100-2t}dt=-\frac{3}{2}\ln(100-2t)$

let $q(t)=-\dfrac{3}{2}\ln(100-2t)$

$e^{q(t)}y(t)'+e^{q(t)}\cdot q'(t)y(t)=e^{q(t)}\cdot 10=10\cdot(100-2t)^{-3/2}$

$\Rightarrow (e^{q(t)}y)'=10\cdot(100-2t)^{-3/2}$

$\Rightarrow e^{q(t)}y=e^{q(0)}y(0)+10\cdot\displaystyle\int(100-2t)^{-3/2}\,dt$

$\Rightarrow y(t)=10e^{-q(t)}(100-2t)^{-1/2}+C\cdot e^{-q(t)}=10\cdot(100-2t)+C(100-2t)^{3/2}$

$y(0)=0\Rightarrow 0=1000+1000C\Rightarrow C=-1$

$y(t)=10\cdot(100-2t)-(100-2t)^{3/2}$

$y'(t)=0\Rightarrow 10=3\cdot 10-3(100-2t)^{1/2}$

$\Rightarrow t=\dfrac{250}{9}$

5. 求下列積分:

(a) $\displaystyle\int x^2 \sin(3x)\, dx$ (7分)

(b) $\displaystyle\int \frac{2}{x^3\sqrt{x^2-1}}\, dx, \quad x > 1$ (8分)

解答: (a) $u = x^2, \quad dv = \sin(3x)dx \Rightarrow du = 2xdx, \quad v = \dfrac{-1}{3}\cos(3x)$

$$\int x^2 \sin(3x)\, dx = \frac{-1}{3}x^2\cos(3x) + \frac{2}{3}\int x\cos(3x)dx$$

$$= \frac{-1}{3}x^2\cos(3x) + \frac{2}{3}\left[\frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin(3x)\, dx\right]$$

$$= \frac{-1}{3}x^2 + \cos(3x) + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos(3x) + C$$

(b) let $x = \sec\theta, \quad 0 < \theta < \dfrac{\pi}{2}$

$$\Rightarrow x^2 = \sec^2\theta - 1 = \tan^2\theta, \quad dx = \sec\theta\tan\theta d\theta$$

$$\int \frac{2}{x^3\sqrt{x^2-1}}dx = \int \frac{2}{\sec^3\theta\tan\theta} \times \sec\theta\tan\theta d\theta = \int \frac{2}{\sec^2\theta}d\theta = \int 2\cos^2\theta d\theta$$

$$= \int (1+\cos 2\theta)d\theta = \theta + \frac{\sin 2\theta}{2} + C$$

$$= \sec^{-1}x + \frac{\sin(2(\sec^{-1}x))}{2} + C$$

6. 求下列積分

(a) $\int_1^\infty \frac{\ln y}{y^3} dy$ (7分)

(b) $\int_0^{\pi/2} \frac{1}{2+\cos x} dx$ (8分)

解答: (a) $u = \ln y, \quad dv = y^{-3} dy$

$du = \frac{1}{y} dy, \quad v = \frac{-1}{2} y^{-2}$

(a) $\int_1^\infty \frac{\ln y}{y^3} dy = \lim_{s\to\infty} \int_1^s \frac{\ln y}{y^3} dy$

$= \lim_{s\to\infty} \left(1 - \frac{\ln y}{2y^2}\right)\Big|_1^s + \frac{1}{2}\int_1^s \frac{1}{y^3} dy$ (2分)

$= \lim_{s\to\infty} \left(1 - \frac{\ln y}{2y^2}\right)\Big|_1^s - \frac{1}{4y^2}\Big|_1^s$ (4分)

$= \lim_{s\to\infty} \left(\frac{-\ln s}{2s^2} - \frac{1}{4s^2} + \frac{1}{4}\right)$(6分)

$= \frac{1}{4}$ (7分)

(b) $z = \tan\frac{x}{2}$, $dx = \frac{2}{1+z^2} dz$, $\cos x = \frac{1-z^2}{1+z^2}$

$\int_0^{\pi/2} \frac{1}{2+\cos x} dx$

$= \int_0^1 \frac{\frac{2}{1+z^2} dz}{2 + \frac{1-z^2}{1+z^2}}$ (2分)

$= \int_0^1 \frac{2}{z^2+3} dz$ (4分)

$= 2\left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{z}{\sqrt{3}}\right)\Big|_0^1$ (六分)

$= \frac{\sqrt{3}\pi}{9}$ (8分)

7. 請就 $a=1, a=2$ 分別討論級數 $\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\cdots n}{(a+1)(a+2)(a+3)\cdots(a+n)}$ 的收斂性. (10分)

解答: $a=1 \Rightarrow \frac{1\cdot 2\cdot 3\cdots n}{(1+1)(1+2)(1+3)\cdots(1+n)} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent

Since $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$ is a decreasing positive sequence.

We can use integral test

$\int_2^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} (\ln b) - \ln 2 = \infty$

$a=2 \Rightarrow \frac{1\cdot 2\cdot 3\cdots n}{(3)(4)(5)\cdots(2+n)} = \frac{2}{(n+2)(n+1)}$

$\frac{2}{(n+2)(n+1)} = \frac{2}{n^2+3n+2} < \frac{2}{n^2}$

$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}$ is convergent

since $\left\{\frac{2}{(n+2)(n+1)}\right\}_{n=1}^{\infty}$ is a positive sequence

and $\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+1)} < \sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent by integral test.

8, 規定 $a \geq 0$. 問 a 在什麼範圍時, 級數 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^a}$ 收斂. (10分)

解答: 用 integral test 列出 $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^a}$ (給兩分)

$a \neq 1$ 算出 $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^a} dx = \frac{1}{1-a} \left.\frac{1}{(\ln x)^{a-1}}\right|_2^{\infty}$

$\displaystyle= \lim_{b\to\infty} \left[\frac{1}{1-a} \frac{1}{(\ln b)^{a-1}} - \frac{1}{1-a} \frac{1}{(\ln 2)^{a-1}}\right]$ 累積給5分

$\left.\begin{array}{l} a > 1 \Rightarrow \text{級數收斂} \\ a < 1 \Rightarrow \text{級數發散} \end{array}\right\}$ 累積給九分

$a = 1 \Rightarrow \displaystyle\int_{\ln 3}^{\infty} \frac{1}{y} dy = \lim_{y\to\infty} \ln y - \ln(\ln 3) = \infty \Rightarrow$ 級數發散 (累積得十分)

9. (a) 求 $\sqrt{2}\cos x$ 對 $x = \dfrac{\pi}{4}$ 展開的泰勒級數. (5分)

(b) 證明: 當 x 爲任一實數時, 前述泰勒級數收斂到 $\sqrt{2}\cos x$. (5分)

解答: (a) $\sqrt{2}\cos x = \sqrt{2}\cos(x-\pi/4+\pi/4) = \sqrt{2}[\cos(x-\pi/4)\cos(\pi/4) - \sin(x-\pi/4)\sin(\pi/4)]$

$= 2\left[\cos(x-\pi/4) - \sin(x-\pi/4)\right]$

$\displaystyle= 2\left[1 - \frac{(x-\pi/4)^2}{2!} + \frac{(x-\pi/4)^4}{4!} - \cdots\right] - 2\left[\frac{(x-\pi/4)}{1!} - \frac{(x-\pi/4)^3}{3!} + \cdots\right]$

$\displaystyle= \sum_{k=0}^{\infty} 2(-1)^k \left[\frac{(x-\pi/4)^{2k}}{(2k)!} - \frac{(x-\pi/4)^{2k+1}}{(2k+1)!}\right]$

(b) 由 $\displaystyle\lim_{k\to 0} \frac{a^k}{k!} = 0$

$\displaystyle\lim_{k\to 0} R_{2k} = \lim_{k\to 0} (-1)^k \frac{(x-\pi/4)^{2k}}{(2k)!} = 0$

$\displaystyle\lim_{k\to 0} R_{2k+1} = \lim_{k\to 0} (-1)^{k+1} \frac{(x-\pi/4)^{2k+1}}{(2k+1)!} = 0$

10. 求 $f(x) = e^{-x^2}$ 的 Maclaurin 級數, 及利用此級數求 $\displaystyle\int_0^1 e^{-x^2} dx$ 之近似值. (必須確時確定其誤差小於 10^{-3}) (10分)

解答: $\displaystyle f(x) = \sum_{k=0}^{\infty} \frac{(-x^2)}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$

$\displaystyle\int_0^1 e^{-x^2} dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} dx = \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k x^{2k}}{k!} dx = \left.\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}\right|_0^1$

for $0 < x < 1$

$\displaystyle R_k(x) = \frac{x^{2k+1}}{(2k+1)k1} < \frac{1}{(2k+1)k!} < 10^{-3}$

ie $10^3 < (2k+1)k!$ if $k \geq 5$

$\displaystyle\Rightarrow \int_0^1 e^{-x^2} dx \approx \left.\sum_{k=0}^{4} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}\right|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = \frac{5651}{7560}$