972 微甲08-13班期中考解答

1. (12%) Determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty}(-1)^n\left(e^{\frac{1}{n}}-1\right)$.

   (b) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$.

   Sol:

   (a) $\sum_{n=1}^{\infty}(-1)^n(e^{\frac{1}{n}}-1)$

   $\because\ e^{\frac{1}{n}}-1\to\ 0$ as $n\to\infty$

   and $e^{\frac{1}{x}}-1$ is decreasing in x,

   (since $f(x)=e^{\frac{1}{x}}-1,\ f'(x)=-\frac{1}{x^2}e^{\frac{1}{x}}<0$ for $x\geq 1$)

   By alternating series test, the series converges.

   (b) $\sum_{n=2}^{\infty}\frac{1}{n(lnn)^3}$

   Let $g(x)=\frac{1}{x(lnx)^3}$. Then $g(x)\geq 0$, is decreasing to 0, and continuous for $x\geq 2$

   (since the function $x(lnx)^3$ is increasing.)

   $g(n)=\frac{1}{n(lnn)^3}$ and the improper integral

   $\int_2^{\infty}\frac{dx}{x(lnx)^3}=-\frac{1}{2}(lnx)^{-2}\Big|_2^{\infty}=\frac{1}{2(ln2)^2}$ converges.

   By integral test, the series converges.

2. (12%)

   (a) Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

   (b) Evaluate $\lim_{n\to\infty}\frac{1-\cos\left(\frac{1}{n}\right)}{1-\cos\left(\frac{1}{n+1}\right)}$.

   (c) Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)x^n$.

   Sol:

   (a) By L'Hopital's rule,

   $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}.$$

(b) By (a),

$$\begin{aligned}
\lim_{n\to\infty}\frac{1-\cos(\frac{1}{n})}{1-\cos(\frac{1}{n+1})} &= \lim_{n\to\infty}\frac{1-\cos(\frac{1}{n})}{(\frac{1}{n})^2}\cdot\frac{(\frac{1}{n+1})^2}{1-\cos(\frac{1}{n+1})}\\
&= \lim_{n\to\infty}\frac{1-\cos(\frac{1}{n})}{(\frac{1}{n})^2}\cdot\lim_{n\to\infty}\frac{(\frac{1}{n+1})^2}{1-\cos(\frac{1}{n+1})}\\
&= \frac{(\frac{1}{2})}{(\frac{1}{2})}\\
&= 1.
\end{aligned}$$

(c) By (b), let $a_n = (1-\cos(\frac{1}{n}))x^n$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1-\cos(\frac{1}{n})}{1-\cos(\frac{1}{n+1})}x\right| = |x|$$

Hence, the radius $R = 1$.

For $x=-1$, since $1-\cos(\frac{1}{n})$ decreases to 0, so according to alternating test, $\sum_{n=1}^{\infty}(-1)^n(1-\cos(\frac{1}{n}))$ converges.

For $x = 1$, since $\lim_{n\to\infty}\frac{1-\cos(\frac{1}{n})}{(\frac{1}{n})^2} = \frac{1}{2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so by comparison test, $\sum_{n=1}^{\infty}(1-\cos(\frac{1}{n}))$ converges.

Therefore, the interval of convergence is $[-1, 1]$.

3. (10%) Let $f(x) = \sin(x^3)$. Find $f^{(15)}(0)$.

Sol:

$$\begin{aligned}
\sin(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots\\
\Rightarrow \sin(x^3) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} - \cdots\\
\Rightarrow f^{15}(0) &= \frac{15!}{5!}.
\end{aligned}$$

4. (12%) Find the curvature κ, the unit tangent vector $\mathbf{T}$, and the unit normal vector $\mathbf{N}$ of the curve $\mathbf{r}(t) = \langle\cos t, \sin t, \ln(\cos t)\rangle$ at $\mathbf{r}(0) = (1, 0, 0)$.

Sol:

$r'(t) = (-\sin t, \cos t, \frac{-\sin t}{\cos t}) \Longrightarrow |r'(t)| = \sqrt{1+\tan^2 t} = \frac{1}{\cos t}$ (because $\cos t > 0$)

$\Longrightarrow T(t) = (-\sin t\cos t, \cos^2 t, -\sin t)$, so $T(0) = (0, 1, 0)$

$T'(t) = (-\cos^2 t + \sin^2 t, -2\sin t\cos t, -\cos t)$

$\Longrightarrow k(0) = |dT/ds|_{t=0} = |T'(0)|/|r'(0)| = \sqrt{2};\ N(0) = \dfrac{T'(0)}{|T'(0)|} = \dfrac{1}{\sqrt{2}}(-1, 0, -1)$

5. (10%) Where does the tangent plane of the surface $z = e^{x-y}$ at $(1, 1, 1)$ intersect the z axis?

Sol:

$$\frac{\partial z}{\partial x} = e^{x-y},\quad \frac{\partial z}{\partial y} = -e^{x-y} \tag{1}$$

So, the tangent plane is

$$z - 1 = \frac{\partial z}{\partial x}(1,1)(x-1) + \frac{\partial z}{\partial y}(1,1)(y-1). \tag{2}$$

$$z - axis \Rightarrow x = y = 0 \text{ input in equation (2), then we obtain } z = 1 \tag{3}$$

Therefore, the intersect point is $(0, 0, 1)$.

6. (12%) You are wandering around in a strange desert where the temperature at the point (x, y) is given by the function $T(x, y) = e^{y-x^2}$.

   (a) You have stopped at the point $(2, 1)$. Suddenly you are feeling chilled and want to warm up – in what direction should you go to warm up as rapidly as possible?

   (b) Figure out the coordinates of all the points where there is no increase or decrease in temperature in the $\langle 1, 1\rangle$ direction.

Sol:

(a) $f(2, 1)$ is the direction we need (you show that you known this property)
$f(2, 1) = e^{-3}(-4, 1)$ (this answer can be replace by $t(-4, 1)$, for any $t > 0$)

(b) $f(x, y) \cdot (1, 1) = 0 \Rightarrow x = \dfrac{1}{2},\ y \in \mathbb{R}$

7. (10%) Let $z = f(x, y)$ such that all the second partial derivatives of f are continuous. Let $x = r\cos\theta$ and $y = r\sin\theta$.

   (a) Evaluate $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial \theta}{\partial x}$, and $\dfrac{\partial \theta}{\partial y}$. Express the results in (functions of) r and θ.

   (b) Express f_x in terms of (functions of) r, θ, f_r, f_θ.

(c) Express f_{xx} in terms of (functions of) r, θ, f_r, f_θ, f_{rr}, $f_{r\theta}$, $f_{\theta\theta}$.

Sol:

(a) Since $r = \sqrt{x^2+y^2}$

Hence $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{r\cos\theta}{r} = \cos\theta.$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{r\sin\theta}{r} = \sin\theta.$

Since $\theta = \tan^{-1}\dfrac{y}{x}$

Hence $\dfrac{\partial \theta}{\partial x} = \dfrac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2} = \dfrac{-y}{x^2+y^2} = \dfrac{-\sin\theta}{r}.$

Similarly $\dfrac{\partial \theta}{\partial y} = \dfrac{\frac{1}{x}}{1+(\frac{y}{x})^2} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}.$

(b) Follows chain rule $\Rightarrow f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial f}{\partial \theta}\dfrac{\partial \theta}{\partial x} = f_r\cos\theta + f_\theta\dfrac{-\sin\theta}{r}$

(c)

$$
\begin{aligned}
f_{xx} &= \frac{\partial(f_r\cos\theta - f_\theta\frac{\sin\theta}{r})}{\partial x} \\
&= \cos\theta\frac{\partial}{\partial x}f_r + f_r\frac{\partial}{\partial x}\cos\theta - \frac{\sin\theta}{r}\frac{\partial}{\partial x}f_\theta - f_\theta\frac{\partial}{\partial x}\frac{\sin\theta}{r} \\
&= \cos\theta\left(f_{rr}\frac{\partial r}{\partial x} + f_{r\theta}\frac{\partial \theta}{\partial x}\right) - f_r\sin\theta\frac{\partial \theta}{\partial x} \\
&\quad -\frac{\sin\theta}{r}\left(f_{\theta r}\frac{\partial r}{\partial x} + f_{\theta\theta}\frac{\partial \theta}{\partial x}\right) \\
&\quad -f_\theta\left(\frac{\partial}{\partial r}\frac{\sin\theta}{r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\sin\theta}{r}\frac{\partial \theta}{\partial x}\right) \\
&= f_{rr}\cos^2\theta + f_{r\theta}\cos\theta\left(-\frac{\sin\theta}{r}\right) - f_r\sin\theta\left(-\frac{\sin\theta}{r}\right) \\
&\quad -f_{\theta r}\left(\frac{\sin\theta}{r}\right)\cos\theta + f_{\theta\theta}\left(\frac{\sin\theta}{r}\right)^2 \\
&\quad -f_\theta\left(\frac{-\sin\theta}{r^2}\cos\theta + \frac{\cos\theta}{r}\left(-\frac{\sin\theta}{r}\right)\right) \\
&= f_{rr}\cos^2\theta - f_{r\theta}\frac{\sin 2\theta}{r} + f_r\frac{\sin^2\theta}{r} + f_{\theta\theta}\left(\frac{\sin\theta}{r}\right)^2 + f_\theta\frac{\sin 2\theta}{r^2}
\end{aligned}
$$

8. (12%) Find and classify the critical points of $f(x,y) = xye^{-x^2-y^2}$.

Sol:

$f_x(x,y)=0, f_y(x,y)=0 \Rightarrow (x,y)=(0,0),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}),(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ (5分)

$f_{xx}(x,y)=(-6xy+4x^3y)e^{-x^2-y^2}$,

$f_{yy}(x,y)=(-6xy+4xy^3)e^{-x^2-y^2}$,

$f_{xy}=(1-2x^2)(1-2y^2)e^{-x^2-y^2}$

$D(x,y)=f_{xx}f_{yy}-(f_{xy})^2$

$\Rightarrow D(0,0)<0, (0,0)$ is a saddle point

$\Rightarrow D(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})>0,\ f_{xx}(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})<0, (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ attains maximum

$\Rightarrow D(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})>0,\ f_{xx}(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})<0, (\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$ attains maximum

$\Rightarrow D(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})>0,\ f_{xx}(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})>0, (\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$ attains minimum

$\Rightarrow D(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})>0,\ f_{xx}(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})>0, (\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$ attains minimum

9. (10%) Use the method of Lagrange multipliers to find the extreme values of z on the curve of intersection of $x^2+z^2=1$ and $y^2+z^2+z=1$.

Sol:

we want to find the extreme value of function: $f(x,y,z)=z$, Constrain functions: $g(x)=x^2+z^2-1$ and $h(x)=y^2+z^2+z-1$

by the Lagrange multiplier, we have $\nabla f=\lambda\nabla g+\mu\nabla h$ That is

$$\begin{cases} -2\lambda x=0 \\ -2\mu y=0 \\ \lambda(2z)+\mu(2z+1)=1 \end{cases}$$

Then we discuss for conditions based on λ and μ

(1) $\lambda=0,\mu=0$ Contradiction to the equation $\lambda(2z)+\mu(2z+1)=1$

(2) $\lambda=0,\mu\neq 0$ we can get $y=0\to z^2+z-1=0\to z=\frac{-1\pm\sqrt{5}}{2}$. By examine $x^2+z^2=1$, we find that $z=\frac{-1+\sqrt{5}}{2}$

(3) $\lambda\neq 0,\mu=0$ we can get $x=0\to z=\pm 1$. By examine $y^2+z^2+z=1$, we find that

$z = -1$

(4) $\lambda \neq 0, \mu \neq 0$ we can get $x = 0$ and $y = 0$ which are contradict to the constrain $x^2 + z^2 = 1$ and $y^2 + z^2 + z = 1$

Extreme values:

Maximum: $z = \dfrac{-1 + \sqrt{5}}{2}$

Minimum: $z = -1$