微甲08-13班 統一教學期中考解答

1. (12%) (a) Use logarithmic differentiation to find the derivative of the function $y = x^{\frac{1}{x}}$, $x > 0$.

   (b) Find the tangent line of the function $y = \tan^{-1}(e^x)$ at $x = 0$.

   Sol:

   (a)

$$
\begin{aligned}
\because \quad & \ln y = \frac{1}{x}\ln x \\
\Longrightarrow \quad & \frac{1}{y}\cdot\frac{dy}{dx} = \frac{\frac{1}{x}x - \ln x}{x^2} \\
\Longrightarrow \quad & \frac{dy}{dx} = x^{\frac{1}{x}}\cdot\frac{1-\ln x}{x^2}.
\end{aligned}
$$

   (b) When $x = 0$, $y(0) = \tan^{-1} 1 = \frac{\pi}{4}$,

$$
\begin{aligned}
\Longrightarrow \quad & \frac{dy}{dx} = \frac{e^x}{1+e^{2x}} \\
\Longrightarrow \quad & \frac{dy}{dx}\Big|_{x=0} = \frac{1}{2},
\end{aligned}
$$

   the tangent line:

$$
y - \frac{\pi}{4} = \frac{x}{2}.
$$

2. (10%) Find the highest and the lowest points of the curve given by $x^2 + xy + 2y^2 = 28$.

   Sol:

   First note that the curve is an elliptic (for example, by examining the discriminant $= 1-4\cdot1\cdot2 < 0$), so it does have an unique highest and an unique lowest points.

   i) Implicit differentiation with respect to $x \Rightarrow 2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$.

   ii) At the highest and the lowest points, $\frac{dy}{dx} = 0$, so $2x + y = 0$.

   iii) By ii) and the equation of the elliptic, we get $x = \pm 2$, and then $y = \mp 4$.

   So the highest point is $(-2, 4)$, and the lowest point is $(2, -4)$.

3. (14%) Evaluate the limits.

(a) $\lim_{x\to\infty}(1+\frac{3}{x}+\frac{5}{x^2})^x$, (b) $\lim_{x\to 0}\frac{\int_{\cos x}^{1}\frac{2}{t}dt-x^2}{x^4}$.

Sol:

(a) Taking logarithm.

$$\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = x\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right) = \log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)\Big/\left(\frac{1}{x}\right)$$

To find the limit above. Consider the following limit

$$\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)'\Big/\left(\frac{1}{x}\right)' = \left(-\frac{3}{x^2}-\frac{15}{x^3}\right)\Big/\left(\left(1+\frac{3}{x}+\frac{5}{x^2}\right)\left(-\frac{1}{x^2}\right)\right) \to 3$$

as $x\to\infty$. Hence by l'Hospital Rule, we conclude that

$$\lim_{x\to\infty}\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = 3$$

This is equivalent to

$$\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = e^3$$

(b) Since this limit is of the form 0/0. So we consider the following limit

$$\left(\int_{\cos x}^{1}\frac{2}{t}dt-x^2\right)'\Big/(x^4)' = \frac{2\tan x-2x}{4x^3}$$

The last equality holds by the Fundamental Theorem of Calculus. Again this limit is of the form 0/0, we consider the limit

$$\frac{(2\tan x-2x)'}{(4x^3)'} = \frac{2\sec^2 x-2}{12x^2}$$

This is still a indeterminated form. Hence we differentiate again. We arrived that

$$\frac{4\sec^2 x\tan x}{24x}$$

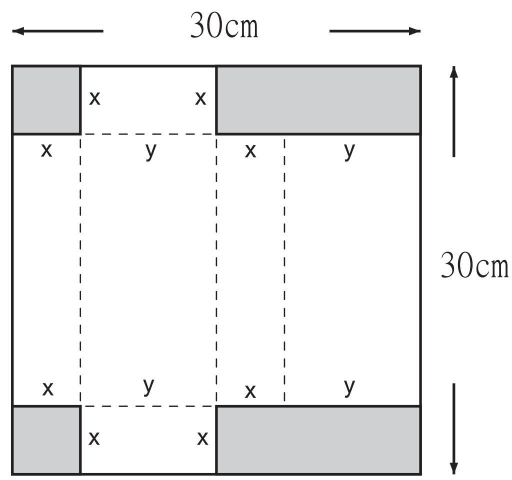
We meet 0/0, so differentiate it again

$$\frac{4\sec^4 x+8\sec^2 x\tan^2 x}{24} \to \frac{1}{6} \quad \text{as} \quad x\to 0$$

Therefore, by l'Hospital Rule, we conclude that

$$\lim_{x\to 0}\left(\left(\int_{\cos x}^{1}\frac{2}{t}dt-x^2\right)/(x^4)\right) = \frac{1}{6}$$

4. (10%) See the figure below. A box with cover is to be constructed from a square piece of cardboard, 30cm wide, by cutting out a square or a rectangle (shaded region) from each of the four corners and bending up the remaining cardboard (unshaded region) along the dotted lines. What is the largest volume that such a box can have? Justify that the volume you obtain actually is the maximum volume.



Sol:

$2y + 2x = 30 \Rightarrow y = 15 - x$

Volume= $V(x) = x(15 - x)(30 - 2x) = 2x^3 - 60x^2 + 450x$ on $[0, 15]$

$V'(x) = 6x^2 - 120x + 450 = 6(x - 15)(x - 5)$

發生最大值的位置可能點有端點和 critical point $0, 5, 15$

$V(0) = 0 = V(15), V(5) = 5 \cdot 10 \cdot 20 = 1000.$

Then the largest volume is 1000.

5. (20%) Study the function $y = f(x) = 20x^3 - 3x^5$ on $\mathbb{R}$, answer the following questions, and sketch the graph of this function. Write down all necessary calculation.

(a) Is $f(x)$ an odd function? ____________________.

Is $f(x)$ an even function? ____________________.

(b) x-intercept(s) is(are) ____________________, y-intercept is ____________________.

(c) $f'(x) =$ ____________________

f is increasing on interval(s) ____________________.

f is decreasing on interval(s) ____________________.

the coordinate(s) of local maximum point(s) is(are) $(x, y) =$ ____________________.

the coordinate(s) of local minimum point(s) is(are) $(x, y) =$ ____________________.

(d) $f''(x) =$ ____________________.

f is concave upward on interval(s) ____________________.

f is concave downward on interval(s) ____________________.

the coordinate(s) of inflection point(s) is(are) $(x, y) =$ ____________________.

(e) The equation(s) of asymptote(s) of $y = f(x)$ is(are) ____________________.

(Answer **none** if there is no asymptote.)

(f) Sketch the graph of $y = f(x)$.

Sol:

(a) $f(-x) = 20(-x)^3 - (-x)^5 = -(20x^3 - 3x^5) = -f(x)$,

hence $f(x)$ is odd not even.

(b) $0 = 20x^3 - 3x^5 \Rightarrow x$-intercepts are $0, \sqrt{\dfrac{20}{3}}, -\sqrt{\dfrac{20}{3}}$.

$f(0) = 0 \Rightarrow y$-intercept is 0.

(c) Since $f'(x) = 60x^2 - 15x^4 = 15x^2(2 - x)(2 + x)$, hence we have

$f'(x) > 0$ on $(-2, 0)$, $(0, 2) \Rightarrow f(x)$ is increasing on $(-2, 0)$, $(0, 2)$.

$f'(x) < 0$ on $(-\infty, -2)$, $(2, \infty) \Rightarrow f(x)$ is decreasing on $(-\infty, -2)$, $(2, \infty)$.

And it is easy to get $f(x)$ has local maximum at $(x, y) = (2, 64)$, and local minimum at $(x, y) = (-2, -64)$.

(d) $f'(x) = 120x - 60x^2 = 60x(\sqrt{2} - x)(\sqrt{2} + x)$,

hence $f''(x) > 0$ on $(-\infty, -\sqrt{2}))$, $(0, \sqrt{2}) \Rightarrow f(x)$ is concave upward on $(-\infty, -\sqrt{2}))$, $(0, \sqrt{2})$.

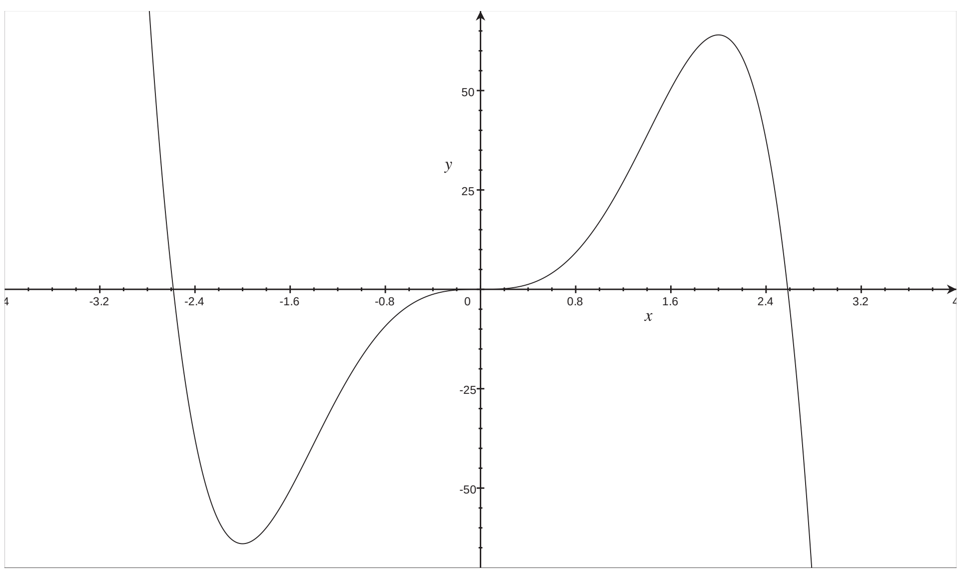
$f''(x) < 0$ on $(0, \infty)$, $(-\sqrt{2}, 0) \Rightarrow f(x)$ is concave downward on $(0, \infty)$, $(-\sqrt{2}, 0)$.

hence the inflection points are $(0, 0)$, $(\sqrt{2}, 28\sqrt{2})$, $(-\sqrt{2}, -28\sqrt{2})$.

Claim: $f(x)$ has no asymptote.

Proof: $\lim_{x\to\infty} f(x) - (ax + b) = \infty$, and $\lim_{x\to-\infty} f(x) - (ax + b) = -\infty$, for all $a,\ b \in \mathbb{R}$.

(e) As follows.



6. (10%) A particle moves along the curve with equation $x^3 + y^4 + xy = 3$. Let $(x(t), y(t))$ be the position coordinate, measured in meters, of the particle at time t, measured in seconds, and $S(t) = \sqrt{x(t)^2 + y(t)^2}$ be the distance between the particle and the origin at time t. Suppose that at $t = 1$ the particle is located at $(x, y) = (1, 1)$ with $\left.\frac{dS}{dt}\right|_{t=1} = \sqrt{2}$ m/sec.

(a) Find $x'(1)$ and $y'(1)$.

(b) Find the speed, $\sqrt{x'(t)^2 + y'(t)^2}$, of the particle at $t = 1$.

Sol:

(a)

$$s^2 = x^2 + y^2$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$\text{input } t = 1, x(1) = 1, y(1) = 1 \Rightarrow s = \sqrt{2}, \frac{ds}{dt} = \sqrt{2}$$
$$2\sqrt{2} \cdot \sqrt{2} = 2\frac{dx}{dt} + 2\frac{dy}{dt} \Rightarrow 2 = \frac{dx}{dt} + \frac{dy}{dt} \tag{1}$$

$$x^3 + y^4 + xy = 3$$
$$3x^2\dot{x} + 4y^3\dot{y} + \dot{x}y + x\dot{y} = 0$$
$$\text{input} t = 1 \Rightarrow 3\dot{x} + 4\dot{y} + \dot{x} + \dot{y} = 0 \Rightarrow 4\dot{x} + 5\dot{y} = 0 \tag{2}$$

solve (1) and (2), then we obtain $\dot{y} = -8, \dot{x} = 10$.

(b)

$$\sqrt{8^2 + 10^2} = 2\sqrt{41}$$

7. (10%) Consider the limit $\lim_{n\to\infty} n^2 \sum_{k=1}^{n} \frac{k}{n^4+k^4}$.

(a) Explain carefully why the limit exists. Express the limit as a definite integral.

(b) Evaluate this definite integral.

Sol:

(a)

$$\lim_{n\to\infty} n^2 \sum_{k=1}^{n} \frac{k}{n^4+k^4} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k \cdot n^{-1}}{1+(k \cdot n^{-1})^4}$$

This is exact the Riemann Sum of the function $f(x) = \frac{x}{1+x^4}$ on $[0, 1]$. Note that $f(x)$ is continuous on $[0, 1]$. Hence integrable. We conclude the limit exists and is equal to the integral of $f(x)$ on $[0, 1]$, i.e.,

$$\lim_{n\to\infty} n^2 \sum_{k=1}^{n} \frac{k}{n^4+k^4} = \int_0^1 \frac{x}{1+x^4} dx$$

(b) Evaluate the integral. Let $u = x^2$. $du = 2xdx$ and when x varies from 0 to 1, u varies from 0 to 1. Hence

$$\int_0^1 f(x)dx = \int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \int_0^1 \frac{1}{1+u^2} du = \frac{1}{2} \tan^{-1} u \Big|_0^1 = \frac{\pi}{8}$$

8. (14%) Evaluate the following integrals.

(a) $\displaystyle\int \frac{e^{\sin x}}{\sec x}dx$, (b) $\displaystyle\int_1^{64} \frac{4\sqrt{x}+7\sqrt[3]{x}}{\sqrt[6]{x}}dx$.

Sol:

(a) Let $u = \sin x$, then $\displaystyle\int \frac{e^{\sin x}}{\sec x}dx = \int e^u du = e^u + C = e^{\sin x} + C$

(b) $\displaystyle\int_1^{64} \frac{4x^{\frac{1}{2}}+7x^{\frac{1}{3}}}{x^{\frac{1}{6}}}dx = \int_1^{64} 4x^{\frac{1}{3}} + 7x^{\frac{1}{6}} = 3x^{\frac{4}{3}} + 6x^{\frac{7}{6}}|_1^{64} = (768+768)-(3+6) = 1527$