1. (10%) 令曲線由 $\mathbf{r}(t) = (t + \sin t \cos t)\mathbf{i} + (\sin^2 t)\mathbf{j} + (2\cos t)\mathbf{k}$, $t \in \mathbb{R}$, 所定義。

   (a) 求曲線上的曲率 κ 。

   (b) 求 κ 發生最大值之點, 以及在這些點的單位切向量 (unit tangent) $\mathbf{T}$ 及單位主法向量 (principal unit normal) $\mathbf{N}$。

Sol.

$$r'(t) = (1 + \cos 2t)i + \sin 2tj + (-2\sin t)k \Rightarrow T = e_1 = \frac{r'(t)}{|r'(t)|} = \frac{1}{2}r'(t).$$

Since

$$r"(t) = (-2\sin 2t)i + (2\cos 2t)j + (-2\cos t)k,$$

$$n = r"(t) - < r"(t), e_1 > e_1 = (-2\sin 2t)i + (2\cos 2t)j + (-2\cos t)k$$

and

$$|n| = 2\sqrt{\cos^2 t + 1},$$

it follows that

$$N = \frac{1}{\sqrt{\cos^2 t + 1}}\left((-\sin 2t)i + (\cos 2t)j + (-\cos t)k\right).$$

We have

$$\kappa = \frac{< e_1', N >}{|r'(t)|} = \frac{1}{2}\sqrt{\cos^2 t + 1}.$$

Therefore,when $t = m\pi, m \in Z, \kappa$ has mamimum. Moreover,$T = (1, 0, 0), N = \frac{1}{\sqrt{2}}(0, 1, -1)$(n is even) ,$N = \frac{1}{\sqrt{2}}(0, 1, 1)$(n is odd).

2. (18%) 判斷以下級數是絕對收斂 (absolute convergence)、條件收斂 (conditional convergence) 或發散 (divergence)?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}-\sqrt{n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$

Sol.

(a)(1) $\lim\limits_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} = 2$. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergence, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ is divergence.

(2) Since $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ is decreasing and $\lim\limits_{x\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ is convergence.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ is conditional convergence.

(b) Since $\frac{1}{\sqrt{n+1}-\sqrt{n}}$ is increasing, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$ is divergence.

(c) $\lim\limits_{n\to\infty} \frac{\frac{1}{\sqrt{n^3}}}{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}} = 2$. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ is convergence, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$ is convergence.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$ is absolutely convergence.

3. (10%)

(a) 將 $f(x) = \tan^{-1} x$ 表成在 $x = 0$ 的冪級數 (包括一般項及收斂半徑)。

(b) 估計積分 $\int_0^{\frac{1}{4}} \frac{\tan^{-1} x}{\sqrt{x}} dx$ 之値, 使其誤差 $< \frac{1}{10^4}$(須明確解釋你的估計值之誤差確實滿足題目中的要求)。

Sol.

Sol: (a)

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \sum_{n=0}^{\infty}(-t^2)^n\, dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

Let $a_n = \frac{(-1)^n}{2n+1}$, by Ratio Test, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1 \;\Rightarrow\; R = 1$

(b)

$$\frac{\tan^{-1} x}{\sqrt{x}} = x^{1/2} - \frac{1}{3}x^{5/2} + \frac{1}{5}x^{9/2} - \frac{1}{7}x^{13/2} + \;\cdots$$

$$\int_0^{\frac{1}{4}} \frac{\tan^{-1} x}{\sqrt{x}}\, dx = [\; \frac{2}{3}x^{3/2} - \frac{2}{21}x^{7/2} + \frac{2}{55}x^{11/2} - \frac{2}{105}x^{15/2} + \;\cdots\;]_0^{1/4}$$

$$= \tfrac{2}{3}(\tfrac{1}{2})^3 - \tfrac{2}{21}(\tfrac{1}{2})^7 + \tfrac{2}{55}(\tfrac{1}{2})^{11} - \tfrac{2}{105}(\tfrac{1}{2})^{15} + \;\cdots$$

$$(\; \equiv u_1 - u_2 + u_3 - u_4 + \;\cdots)$$

$\because\; \tfrac{2}{55}(\tfrac{1}{2})^{11} < 10^{-4}$ and $u_n \geq 0$, $u_n \searrow 0 \;\; \forall n$

By The Alternating Series Estimation Theorem,

$$\int_0^{\frac{1}{4}} \frac{\tan^{-1} x}{\sqrt{x}}\, dx \approx \frac{2}{3}(\frac{1}{2})^3 - \frac{2}{21}(\frac{1}{2})^7 = \frac{37}{448} \quad , \quad |\text{Error}| \;\leq 10^{-4}$$

4. (10%) 考慮冪級數 $\sum_{n=2}^{\infty} \frac{1}{n \ln n} x^n$ 。

(a) 求收斂半徑 r 。

(b) 討論在 $x = r$ 及 $x = -r$ 的收斂性。

Sol.

(a)

由比率審斂法，

$$\lim_{n \to \infty} \left| \frac{\frac{1}{(n+1)\ln(n+1)} x^{n+1}}{\frac{1}{(n)\ln(n)} x^n} \right| = |x| < 1$$

收斂半徑 r =1 .

(b)

r=1,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$$

收斂性與

$$\int_2^{\infty} \frac{1}{x(\ln x)} dx = \ln(\ln(n))|_2^{\infty} \to \infty$$

一致，由積分審斂法，級數發散 .

r=-1,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)} (-1)^n$$

為一交錯級數，

且 $\frac{1}{n(\ln n)}, n = 2, 3, 4 \ldots$ 正向遞減收斂至零，

由交錯級數審斂法, 級數收斂, 且為條件收斂. ✠

5. (10%) 令 $z=f(x,y)$ 爲一連續可微函數, 利用代換 $x=r\cos\theta,\quad y=r\sin\theta$ 可得一個恆等關係

$$(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2=A(r,\theta)(\frac{\partial z}{\partial r})^2+B(r,\theta)(\frac{\partial z}{\partial \theta})^2 \quad 。$$

試求 $A(r,\theta)$ 及 $B(r,\theta)$ 。

Sol.

$x=r\cos\theta, y=r\sin\theta$

$z=f(x,y)$

$$\begin{cases} \frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta \\ \frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta) \end{cases}$$

左右取平方

$$\begin{cases} (\frac{\partial z}{\partial r})^2=(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta)^2\ldots\ldots[1] \\ (\frac{\partial z}{\partial \theta})^2=(\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta))^2\ldots\ldots[2] \end{cases}$$

$\Rightarrow r^2[1]+[2]$ 左右相加

$\Rightarrow r^2(\frac{\partial z}{\partial r})^2+(\frac{\partial z}{\partial \theta})=r^2((\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2)$

$\Rightarrow (\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2=(\frac{\partial z}{\partial r})^2+(\frac{1}{r^2})(\frac{\partial z}{\partial \theta})$

$\Rightarrow A(r,\theta)=1, B(r,\theta)=\frac{1}{r^2}$ ✠

6. (10%) 令 $z = ye^x + \cos\dfrac{y}{x}$, 求該曲面上之點 $(x, y, z) = (1, \frac{\pi}{2}, \frac{e\pi}{2})$ 的切平面及法線方程式。

7. (10%) 設函數 $z = f(x, y)$ 在 $(x, y, z) = (\pi, \pi, \pi)$ 的附近滿足方程式

$$\sin(x+y) + \sin(y+z) + \sin(z+x) = 0 \text{ 。}$$

求函數 $z = f(x, y)$ 在 (π, π) 點, 沿著哪個方向的方向導數 (directional derivative) 最大? 最大值爲何?

8. (10%) 求函數 $z = x^3 - 2x + xy^2 + y + 1$ 的所有臨界點 (critical point), 並判斷它是極大、極小或是鞍點 (saddle point)。

Sol.

$$\begin{cases} \frac{\partial z}{\partial x} = 3x^2 - 2 + y^2 = 0 \\ \frac{\partial z}{\partial y} = 2xy + 1 = 0 \end{cases} \tag{1}$$

$\Rightarrow y = -\frac{1}{2x} \Rightarrow 3x^2 + \frac{1}{4x^2} = 2 \Rightarrow 12x^4 - 8x^2 + 1 = 0 \Rightarrow (6x^2 - 1)(2x^2 - 1) = 0$

$\Rightarrow x = \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{2}} \Rightarrow (x, y) = \pm(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{2}), \pm(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ are critical points

$z_{xx} = 6x, z_{yy} = 2x, z_{xy} = 2y \Rightarrow D = z_{xx}z_{yy} - z_{xy}^2 = 12x^2 - 4y^2$

At $\pm(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{2}) \Rightarrow D = 12 \cdot \frac{1}{6} - 4 \cdot \frac{6}{4} = 2 - 6 = -4 < 0$ are saddle points

At $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \Rightarrow D = 12 \cdot \frac{1}{2} - 4 \cdot \frac{2}{4} = 6 - 2 = 4 > 0, z_{xx} = \frac{6}{\sqrt{2}} > 0$ is local minimum point

At $-(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \Rightarrow D = 12 \cdot \frac{1}{2} - 4 \cdot \frac{2}{4} = 6 - 2 = 4 > 0, z_{xx} = -\frac{6}{\sqrt{2}} < 0$ is local maximum point

9. (12%) 平面 $x + y + \sqrt{2}z = 0$ 與橢球面 (ellipsoid) $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{2} = 1$ 相交成一橢圓。

(a) 求橢圓上之點與原點的最長距離及最短距離。

(b) 求橢圓面積。

Sol.

$$\begin{aligned} \text{Let} \quad & f(x,y,z) = x^2 + y^2 + z^2 \\ & g(x,y,z) = x + y + \sqrt{2}z \\ & h(x,y,z) = \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{2} - 1 \end{aligned}$$

Then solve the equations

$$\begin{cases} \triangledown f = \lambda \triangledown g + \mu \triangledown h & -(1) \\ g = 0 & -(2) \\ h = 0 & -(3) \end{cases}$$

¿From (1), we have

$$\begin{cases} 2x = \lambda + \mu\frac{x}{2} & -(1.1) \\ 2y = \lambda + \mu\frac{y}{2} & -(1.2) \\ 2z = \lambda + \mu z & -(1.3) \end{cases} \Rightarrow \begin{cases} (4-\mu)x = \lambda \\ (4-\mu)y = \lambda \\ (2-\mu)z = \lambda \end{cases}$$

If $\mu = 4, \lambda = 0$, then $z = 0 \Rightarrow x + y = 0\,(\because g = 0)$

substitute y, z with $-x, 0$ in (3), we have $(x,y,z) = \pm(\sqrt{2}, -\sqrt{2}, 0)$

distance $= \sqrt{4} = 2$

If $\mu \neq 4$, then $x = y = \frac{\lambda}{4-\mu} \Rightarrow z = -\sqrt{2}x\,(\because g = 0)$

substitute y, z with $x, -\sqrt{2}x$ in (3), we have $(x,y,z) = \pm(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{4}{3}})$

distance $= \sqrt{\dfrac{8}{3}}$

So the longest distance $= 2$, and the shortest distance $= \sqrt{\dfrac{8}{3}}$ $\qquad -(4)$

(2) Since the center of the ellipse is at $O(0,0,0)$,

the semimajor axis $= 2$, and the semiminor axis $= \sqrt{\dfrac{8}{3}}$

$$\text{Area} = 2\sqrt{\frac{8}{3}}\pi$$