一、求極限:

(a) (10%) $\lim_{x \to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}$

sol.

Using L'Hopital's Rule.
$\lim_{x \to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3} = \lim_{x \to 0} \frac{xe^x-e^x+1}{3x^2} = \lim_{x \to 0} \frac{xe^x}{6x} = \lim_{x \to 0} \frac{e^x}{6} = \frac{1}{6}.$

(b) (10%) $\lim_{x \to 0} (\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}}$

sol.
Consider $\lim_{x \to 0} (\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}} = \lim_{x \to 0} e^{\ln (\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}}}$.
Using L'Hopital's Rule to compute $\lim_{x \to 0} \frac{\ln (\frac{1+\tan x}{1+\sin x})}{x^3}$. $\lim_{x \to 0} \frac{\ln (\frac{1+\tan x}{1+\sin x})}{x^3}$

$$= \lim_{x \to 0} \frac{(\frac{1+\sin x}{1+\tan x}) \times \frac{\sec^2 x(1+\sin x) - \cos x(1+\tan x)}{(1+\sin x)^2}}{3x^2}$$

$$= \lim_{x \to 0} \frac{\sec^2 x(1+\sin x) - \cos x(1+\tan x)}{3x^2(1+\sin x)(1+\tan x)}$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \tan x(1+\sin x) + \sin x(1+\tan x)}{6x(1+\sin x)(1+\tan x) + 3x^2 \cos x(1+\tan x) + 3x^2 \sec^2 x(1+\sin x)}$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \tan x \cos x + (4\sec^2 x \tan^2 x + 2\sec^4 x)(1+\sin x)}{6(1+\sin x)(1+\tan x) + \cdots}$$

$$+ \frac{\sec^2 x \sin x + \cos x(1+\tan x)}{6(1+\sin x)(1+\tan x) + \cdots}$$

$$= \frac{1}{2}.$$

Therefore, $\lim_{x \to 0} (\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}} = \lim_{x \to 0} e^{\ln (\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}}} = e^{\frac{1}{2}}$.

二、求導函數:

(a) (10%) $y = 10^{\sin^{-1} x}$

sol.

$y = 10^{sin^{-1}(x)}$

$ln(y) = sin^{-1}(x)ln(10)$

$\frac{y'}{y} = ln(10)\frac{1}{\sqrt{1-x^2}}$

$y' = ln(10)\frac{1}{\sqrt{1-x^2}}10^{sin^{-1}(x)}$ 對數微分法

或

$y = 10^{sin^{-1}(x)}$

$y = exp(sin^{-1}(x)ln(10))$

$y' = exp(sin^{-1}(x)ln(10)) * (ln(10)\frac{1}{\sqrt{1-x^2}})$ 鏈鎖率

(b) (10%) $y = (\ln x)^{\frac{x}{2}}$, $x > 1$。

sol.

$y = (ln(x))^{\frac{x}{2}}$

$ln(y) = \frac{x}{2}(ln(ln(x)))$

$\frac{y'}{y} = \frac{1}{2}(ln(ln(x))) + \frac{x}{2}(\frac{\frac{1}{x}}{ln(x)})$萊布尼茲法則

$y' = (ln(x))^{\frac{x}{2}}(\frac{1}{2}(ln(ln(x))) + \frac{1}{2}(\frac{1}{ln(x)}))$對數微分法

或

$y = (ln(x))^{\frac{x}{2}}$

$y = exp(\frac{x}{2}(ln(ln(x))))$

$y' = exp(\frac{x}{2}(ln(ln(x)))) * (\frac{1}{2}(ln(ln(x))) + \frac{x}{2}(\frac{\frac{1}{x}}{ln(x)}))$

$y' = (ln(x))^{\frac{x}{2}}(\frac{1}{2}(ln(ln(x))) + \frac{1}{2}(\frac{1}{ln(x)}))$

三、(10%)

有兩曲線 $y=x^2+1$ 及 $y=-\dfrac{4}{9}x^3+2x$。求出與這兩曲線同時相切的所有直線。

sol.

曲線一: $y=x^2+1$ 上一點 (t,t^2+1) 之切線為

$$y-(t^2+1)=2t(x-t) \qquad \text{即} \qquad y=2tx+(1-t^2)$$

曲線二: $y=-\frac{4}{9}x^3+2x$ 上一點 $(s,-\frac{4}{9}s^3+2s)$ 之切線為

$$y-(-\tfrac{4}{9}s^3+2s)=(-\tfrac{4}{3}s^2+2)(x-s) \qquad \text{即} \qquad y=(-\tfrac{4}{3}s^2+2)x+\tfrac{8}{9}s^3$$

兩切線為同一直線, 故

$$2t = 2-\tfrac{4}{3}s^2 \qquad \text{即} \qquad 1-t = \tfrac{2}{3}s^2 \qquad (1)$$

$$1-t^2 = \tfrac{8}{9}s^3 \qquad (2)$$

若$s=0$, 則 $t=1$

若$s\neq 0$, 兩式相除得 $\quad 1+t = \frac{4}{3}s \qquad (3)$

(1) (3) 二式相加得 $\quad s^2+2s-3=0 \qquad$ 即 $\quad s=1 \ , t=\frac{1}{3}$

或 $\quad s=-3 \ , t=-5$

故共同切線有三條, 分別為

$$y=\tfrac{2}{3}(x-\tfrac{1}{3})+\tfrac{10}{9} \qquad (\text{ i.e } \ y=\tfrac{2}{3}x+\tfrac{8}{9}\ )$$

$$y=-10(x+5)+26 \qquad (\text{ i.e } \ y=-10x-24\ )$$

$$y=2x$$

四、(10%)

若 x, y 滿足 $x^2 - y^3 - xy = 1$, 試求出在點 $(2,1)$ 的切線方程式 及二階導數 $\left.\dfrac{d^2y}{dx^2}\right|_{(2,1)}$。

sol.

$$x^2 - y^3 - xy = 1$$

$$2x - 3y^2y\prime - y - xy\prime = 0$$

$$y\prime = \frac{2x - y}{x + 3y^2}|_{(x=2,y=1)} = \frac{3}{5}$$

Hence the tangent line is $y - 1 = \frac{3}{5}(x - 2)$

$$y\prime(x + 3y^2) = 2x - y$$

$$y\prime\prime(x + 3y^2) + y\prime(1 + 6yy\prime) = 2 - y\prime$$

$$y\prime\prime = \frac{2 - 2y\prime - 6y(y\prime)^2}{x + 3y^2}|_{x=2,y=1,y\prime=\frac{3}{5}} = -\frac{34}{125}$$

五、(10%)

有一三次多項式 $f(x)$, 其首項係數爲 1, $y = f(x)$ 圖形通過 $(0, 4)$, 它在 $1 < x < 3$ 是遞減的, 而在 $x > 3$ 及 $x < 1$ 是遞增的, 試求 $f(x)$ 及其反曲點。

sol.

Let $f(x) = x^3 + ax^2 + bx + c$. $f(0) = 4 \Rightarrow c = 4$

Since $f(x)$ is decreasing on $(1, 3)$ and increasing on $(3, \infty)$ and $(-\infty, 1)$, $f'(1) = 0$ and $f'(3) = 0$.

Hence $a = -6, b = 9$.

Thus $f(x) = x^3 - 6x^2 + 9x + 4$ $f''(x) = 6x - 12 \Rightarrow f''(2) = 0$.

Hence inflection point may be at $x = 2$.

Since $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$, inflection point is at $x = 2$.

Hence the inflection point is $(2, 6)$.

六、(10%)

(如圖) 你在教室中倚牆而立, 並注視前方之黑板。該黑板寬 9 公尺, 且距此牆 3 公尺。令 x 爲你距教室前方之距離,α 爲你對黑板之視角。求 x, 使視角 α 爲最大。

sol.

$$\begin{aligned}
\alpha &= \cot^{-1}\frac{x}{12} - \cot^{-1}\frac{x}{3} \\
\frac{d\alpha}{dx} &= \frac{-1}{1+\left(\frac{x}{12}\right)^2}\cdot\frac{1}{12} - \frac{(-1)}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3} \\
&= -\frac{12}{144+x^2} + \frac{3}{9+x^2} \\
&= \frac{-108-12x^2+432+3x^2}{(144+x^2)(9+x^2)} \\
&= \frac{324-9x^2}{(144+x^2)(9+x^2)} \\
&= -9\frac{(x-6)(x+6)}{(144+x^2)(9+x^2)}
\end{aligned}$$

Solve $\frac{d\alpha}{dx} = 0$ for critical points.

$$\begin{aligned}
&\frac{d\alpha}{dx} = 0 \\
&\Rightarrow -9\frac{(x-6)(x+6)}{(144+x^2)(9+x^2)} \\
&\Rightarrow x = \pm 6
\end{aligned}$$

$\frac{d\alpha}{dx}$ positive on $(-6, 6)$, negative on $(6, \infty)$, therefore the angle α have maximum value at $x = 6$.

七、(20%)

令 $y = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$。

(a) 求此函數圖形的所有漸近線。

sol.

Let $f(x) = x^{1/3}(x-3)^{2/3}$,

then $f'(x) = \frac{1}{3}x^{-2/3}(x-3)^{2/3} + \frac{2}{3}x^{1/3}(x-3)^{-1/3} = \frac{x-1}{x^{2/3}(x-3)^{1/3}}$.

Since $lim_{x\to\infty}f'(x) = lim_{x\to\infty}\frac{x-1}{x^{2/3}(x-3)^{1/3}} = lim_{x\to\infty}\frac{1-1/x}{(1-3/x)^{1/3}} = 1$,

and $lim_{x\to\infty}f(x)-x = lim_{x\to\infty}x^{1/3}(x-3)^{2/3}-x = lim_{x\to\infty}\frac{x(x-3)^2-x^3}{x^{2/3}(x-3)^{4/3}+x^{1/3}(x-3)^{2/3}x+x^2} =$
$lim_{x\to\infty}\frac{-6x^2+9x}{x^{2/3}(x-3)^{4/3}+x^{1/3}(x-3)^{2/3}x+x^2} = lim_{x\to\infty}\frac{-6+9/x}{(1-3/x)^{4/3}+(1-3/x)^{2/3}+1} = -2$,

we obtain a asymptote y = x - 2 at $x \to \infty$.

(Similarly, we can also find a asymptote y = x - 2 at $x \to -\infty$)

(b) 求其圖形遞增及遞減的範圍。

sol.

According to (a) , we have f'(x) = $\frac{x-1}{x^{2/3}(x-3)^{1/3}}$.

We may say that f is increasing on$(-\infty, 1)\cup(3,\infty)$, and decreasing on (1,3).

(c) 求其圖形凹向上 (concave up) 及凹向下 (concave down) 的範圍。

sol.

According to (a) , we have $\ln f'(x) = ln(x-1) - \frac{2}{3}\ln x - \frac{1}{3}\ln(x-3)$

$\Rightarrow \frac{f''(x)}{f'(x)} = \frac{1}{x-1} - \frac{2}{3x} - \frac{1}{3(x-3)}$

$\Rightarrow f''(x) = \frac{-2}{x^{5/3}(x-3)^{4/3}}$.

Then we obtain that f is concave up on $(-\infty, 0)$ and concave down on $(0,3)$

,and $(3,\infty)$

(d) 求此函數的所有局部極值及反曲點。

sol.

According to (b) , $f' = 0 \Rightarrow x = 1$, and f' doestn't exist at $x = 3$.

It is easy to observe that $f(1) = 2^{2/3}$ is a local maximum , and $f(3) = 0$ is a local minimum .

According to (c) , we may find a inflection point $(0,0)$.

(e) 作此函數之簡圖。

sol.

(Skip)