95學年度下學期微積分甲統一教學二組期末考參考答案

1.(10%) 求積分 $\iint\limits_R \frac{dA}{(1+x^2+y^2)^2}$, 其中 R 是雙紐線 $r^2 = \cos 2\theta$ 右半邊封閉曲線所圍成的區域。

<u>參考解答:</u>

$R: -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, 0 \le r \le \sqrt{\cos 2\theta}$

$$
\begin{aligned}
\text{原式} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} \frac{r\,dr d\theta}{(1+r^2)^2} = \frac{-1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\frac{1}{1+r^2}\,\Big|_0^{\sqrt{\cos 2\theta}})\,d\theta \\
&= \frac{-1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\frac{1}{1+\cos 2\theta} - 1)\,d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - \frac{1}{2}\sec^2\theta)\,d\theta \\
&= \frac{1}{2}(\theta - \frac{1}{2}\tan\theta\,\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}) = \frac{\pi}{4} - \frac{1}{2}
\end{aligned}
$$

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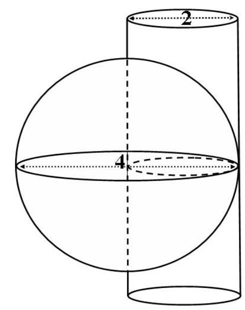
2.(10%) 求積分 $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{1+x^4}\,dxdy$。

<u>參考解答:</u>

$R: 0 \le x \le 2, 0 \le y \le x^3$

原式$=\int_0^2 \int_0^{x^3} \sqrt{1+x^4}\,dydx = \int_0^2 x^3\sqrt{1+x^4}\,dx = \frac{1}{6}(x^4+1)^{\frac{3}{2}} \mid_0^2 = \frac{1}{6}(17^{\frac{3}{2}}-1)$

3.(10%) 如圖。一個半徑為 2 之球體與直徑為 2 之直圓柱體相切。求兩立體相交部份的體積。



<u>參考解答:</u>

取方程式為 $x^2+y^2+z^2=4$ 及 $(x-1)^2+y^2=1$

Let $x = r\cos\theta$, $y = r\sin\theta$

原方程式可改寫為 $0 \le z \le \sqrt{4-r^2}$ 及 $r = 2\cos\theta$

$$
\begin{aligned}
V &= 4\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \sqrt{4-r^2}\cdot r\,drd\theta = -2\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \sqrt{4-r^2}\,d(4-r^2)d\theta \\
&= -2\int_0^{\frac{\pi}{2}} \frac{2}{3}(4-r^2)^{\frac{3}{2}} \mid_0^{2\cos\theta} d\theta = -\frac{4}{3}\int_0^{\frac{\pi}{2}} [4(1-\cos^2\theta)]^{\frac{3}{2}} - 4^{\frac{3}{2}}\,d\theta \\
&= -\frac{4}{3}\int_0^{\frac{\pi}{2}} 8(\sin^3\theta - 1)\,d\theta = -\frac{32}{3}\int_0^{\frac{\pi}{2}} \sin^3\theta\,d\theta + \frac{32}{3}\int_0^{\frac{\pi}{2}} d\theta \\
&= -\frac{32}{3}\int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\,d\cos\theta + \frac{32}{3}\int_0^{\frac{\pi}{2}} d\theta \\
&= -\frac{32}{3}\cdot(\cos\theta - \frac{\cos^3\theta}{3}) \mid_0^{\frac{\pi}{2}} + \frac{32}{3}\cdot\theta \mid_0^{\frac{\pi}{2}} \\
&= -\frac{32}{3}(\frac{2}{3}-\frac{\pi}{2}) = \frac{48\pi-64}{9}
\end{aligned}
$$

4.(10%) 求積分 $\int_0^{\frac{2}{3}}\int_y^{1-\frac{y}{2}}(2x+y)e^{y-x}\,dxdy$。

<u>參考解答:</u>

令 $\begin{cases} u = x-y \\ v = 2x+y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}(u+v) \\ y = \frac{1}{3}(-2u+v) \end{cases}$, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$

所以邊界變爲 $u=0,\ v=2,\ v=2u.$

$$\begin{aligned}
\text{原式} &= \int_0^2\int_0^{\frac{v}{2}}\frac{1}{3}v\cdot e^{-u}\,dudv = \frac{1}{3}\int_0^2 v\cdot(-e^{-u})\,\Big|_0^{\frac{v}{2}}\ dv \\
&= -\frac{1}{3}\int_0^2 v\cdot(e^{-\frac{v}{2}}-1)\,dv = \frac{2}{3}\int_0^2 v\,de^{-\frac{v}{2}} + \frac{1}{3}\int_0^2 v\,dv \\
&= \frac{1}{3}(2ve^{-\frac{v}{2}}+4e^{-\frac{v}{2}}+\frac{v^2}{2})\,\Big|_0^2 = \frac{1}{3}(4e^{-1}+4e^{-1}+2-4) \\
&= \frac{8}{3}e^{-1}-\frac{2}{3}.
\end{aligned}$$

5.(10%) 考慮 $x^2+y^2+z^2=1$ 與 $y=1-x^2$ 在第一卦限部分的交線, C 爲其上從 $(\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2})$ 到 $(1,0,0)$ 的曲線段。令 $\mathbf{F}=-3z\,\mathbf{i}+\frac{3}{2}x\,\mathbf{j}$, 求 $\int_C \mathbf{F}\cdot d\mathbf{r}$。

<u>參考解答:</u>

$$
\begin{aligned}
\mathbf{r}(x) &= \langle x, 1-x^2, \sqrt{x^2-x^4}\,\rangle \\
\int_C \mathbf{F}\,d\mathbf{r} &= \int_{\frac{\sqrt{2}}{2}}^{1} -3\sqrt{x^2-x^4}\,dx + \frac{3}{2}x\cdot(-2x)\,dx \\
&= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{3}{2}\sqrt{1-x^2}\,d(1-x^2) - \int_{\frac{\sqrt{2}}{2}}^{1} 3x^2\,dx \\
&= (1-x^2)^{\frac{3}{2}} - x^3 \,\Big|_{\frac{\sqrt{2}}{2}}^{1} \\
&= -1
\end{aligned}
$$

6.(10%) 令 $\mathbf{F} = (\sqrt{x^2+y^2} - \frac{x}{1+y^2})\,\mathbf{i} + (e^x + \tan^{-1} y)\,\mathbf{j}$, C 爲心臟線 $r = 1+\cos\theta$. 求 $\oint_C \mathbf{F}\cdot\mathbf{n}ds$。

<u>參考解答:</u>

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \iint_{r\le 1+\cos\theta} \Big(\frac{x}{\sqrt{x^2+y^2}} - \frac{1}{1+y^2} + \frac{1}{1+y^2}\Big)\,dA \\
&= \iint_{r\le 1+\cos\theta} \frac{x}{\sqrt{x^2+y^2}}\,dA \\
&= 2\int_0^{\pi}\int_0^{1+\cos\theta} r\cos\theta\,dr d\theta \\
&= \int_0^{\pi} (1+\cos\theta)^2\cos\theta\,d\theta \\
&= \int_0^{\pi} 2\cos^2\theta\,d\theta \\
&= \pi.
\end{aligned}$$

7.(10%) (a) 令向量場 $\mathbf{F} = e^{yz}\,\mathbf{i} + (xze^{yz} + z\cos y)\,\mathbf{j} + (xye^{yz} + \sin y)\,\mathbf{k}$。求此向量場的位勢函數(potential function)。

(b) 令 C 爲從 $(1,0,1)$ 經由 $(1,\frac{\pi}{2},1)$, 再到 $(1,\frac{\pi}{2},2)$ 的折線段。$\mathbf{F}$ 將一物體沿著 C 移動, 求所作的功。

<u>參考解答:</u>

(a)

$$
\begin{aligned}
&f_x = e^{yz} \Rightarrow f(x,y,z) = xe^{yz} + g(y,z)\\
&f_y = xze^{yz} + \frac{\partial g}{\partial y}(y,z) = xze^{yz} + z\cos y \Rightarrow g(y,z) = z\sin y + h(z)\\
&f_z = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y \Rightarrow h(z) = C\\
&\therefore f(x,y,z) = xe^{yz} + z\sin y + C
\end{aligned}
$$

(b) 所作的功$=f(1,\frac{\pi}{2},2) - f(1,0,1) = e^{\pi} + 1.$

8.(10%) 求輪胎面 (torus) $\mathbf{r}(u,v) = (R + r\cos u)\cos v\,\mathbf{i} + (R + r\cos u)\,\sin v\,\mathbf{j} + r\sin u\,\mathbf{k}$, $0 \le u, v \le 2\pi$ 的表面積, 其中 $r < R$ 爲常數。

<u>參考解答:</u>

$$
\begin{aligned}
&\mathbf{r}_u = -r\sin u\cos v\,\mathbf{i} - r\sin u\sin v\,\mathbf{j} + r\cos u\,\mathbf{k}\\
&\mathbf{r}_v = -(R + r\cos u)\sin v\,\mathbf{i} + (R + r\cos u)\cos v\,\mathbf{j} + 0\mathbf{k}\\
&\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = -(R + r\cos u)(r\cos v\cos u)\,\mathbf{i} - (R + r\cos u)(r\sin v\cos u)\,\mathbf{j}\\
&\qquad\qquad + (-r\sin u)(R + r\cos u)\,\mathbf{k}\\
&|\mathbf{r}_u \times \mathbf{r}_v| = r(R + r\cos u)\\
&\Rightarrow A = \int_0^{2\pi}\int_0^{2\pi} (rR + r^2\cos u)\,dudv = 4\pi^2 rR.
\end{aligned}
$$

9.(10%) 令 S 爲平面 $z = x + 2$ 在圓柱 $x^2 + y^2 = 1$ 之內的部份, $\mathbf{n}$ 爲向上的法向量, $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + (x + 2y)\,\mathbf{k}$。求 $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, d\sigma$。

<u>參考解答:</u>

$\text{curl}\mathbf{F} = (2 - x)\,\mathbf{i} - \mathbf{j} + (z - 1)\,\mathbf{k}, \mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$.

$\text{curl}\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{2}}(-2 + x + z - 1) = \frac{1}{\sqrt{2}}(-2 + x + x + 2 - 1) = \frac{1}{\sqrt{2}}(2x - 1)$

$$\begin{aligned}\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, d\sigma &= \frac{1}{\sqrt{2}} \iint_S (2x - 1)\, d\sigma \\ &\quad (\text{Let } x = r\cos\theta, y = r\sin\theta, z = 2 + r\cos\theta) \\ &= \frac{1}{\sqrt{2}} \int_0^{2\pi} \int_0^1 (2r\cos\theta - 1) \cdot \sqrt{1 + r^2}\, r\, dr d\theta = -\pi\end{aligned}$$

另解: $C : \langle \cos\theta, \sin\theta, 2 + \cos\theta \rangle, \theta \in [0, 2\pi]$

由 Stokes 定理

$$\begin{aligned}\text{原式} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} [-\sin\theta\sin\theta + \cos\theta(2 + cos\theta)\cos\theta + (\cos\theta + 2\sin\theta + 2\sin^2\theta\cos\theta)]\, d\theta \\ &= \int_0^{2\pi} [-3\sin^2\theta + 2\cos^2\theta]\, d\theta \\ &= \int_0^{2\pi} -3(\frac{1 - \cos 2\theta}{2}) + 2(\frac{1 + \cos 2\theta}{2})\, d\theta \\ &= -\frac{1}{2} \cdot 2\pi = -\pi\end{aligned}$$

10.(10%) 令 D 爲立方體 $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le 1$。$S$ 爲 D 的六個面, $\mathbf{F} = (-x^2 - 4xy)\,\mathbf{i} - 6yz\,\mathbf{j} + 12z\,\mathbf{k}$。試求 a, b 之值, 使得通量 (flux) $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ 爲最大, 並求其值。

參考解答:

由 Divergence 定理

$$\begin{aligned}
\text{Flux} &= \iiint_D \text{div}\mathbf{F}\,dV \\
&= \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12)\,dzdydx \\
&= \int_0^a \int_0^b (-2x - 4y + 9)\,dydx = \int_0^a [(-2x+9)\cdot b - 2b^2]\,dx \\
&= -a^2b - 2ab^2 + 9ab := \texttt{f(a,b)}
\end{aligned}$$

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0 \Rightarrow \begin{cases} -2a - 2b + 9 = 0 \\ -a - 4b + 9 = 0 \end{cases} \Rightarrow a = 3, b = \tfrac{3}{2}, f(3, \tfrac{3}{2}) = \tfrac{27}{2}.$$