(95上微甲一組 部分期末考參考答案)

1. (15%)

   (a) Find a nonzero number $a \neq 0$ that satisfies $\lim_{x\to\infty}\left(\frac{3x+a}{3x-a}\right)^{2x+1} = 4.$

   (b) Determine whether the improper integral $I_0 = \int_0^1 \frac{\cos t}{t^{4/3}}\,dt$ is convergent or divergent.

   (c) Evaluate the limit $L_0 = \lim_{x\to 0^+} x^{1/6}\int_{\sqrt{x}}^1 \frac{\cos t}{t^{4/3}}\,dt.$

Ans. (a) $a =$ ______________________________ ,

(b) I_0 is ____________(convergent/divergent), (c) $L_0 =$ ____________ .

Solution:

(a) Method 1:

$\lim_{x\to\infty}(\frac{3x+a}{3x-a})^{2x+1}$

$=\lim_{x\to\infty}(1+\frac{2a}{3x-a})^{2x+1} = \lim_{x\to\infty}(1+\frac{2a}{3x-a})^{\frac{3x-a}{2a}\cdot\frac{2a(2x+1)}{(3x-a)}} = [\lim_{x\to\infty}(1+\frac{2a}{3x-a})^{\frac{3x-a}{2a}}]^{\lim_{x\to\infty}\frac{2a(2x+1)}{3x-a}}$

$=e^{\frac{4}{3}a} = 4.$

$\Rightarrow \frac{4}{3}a = 2\ln 2 \Rightarrow a = \frac{3}{2}\ln 2$

Method 2:

$\lim_{x\to\infty}[(1+\frac{2a}{3x-a})^{3x-a}]^{\frac{2}{3}}\cdot(1+\frac{2a}{3x-a})^{1+\frac{2}{3}a}$

$= [\lim_{x\to\infty}(1+\frac{2a}{3x-a})^{3x-a}]^{\frac{2}{3}}\cdot\lim_{x\to\infty}(1+\frac{2a}{3x-a})^{1+\frac{2}{3}a}$

$= (e^{2a})^{\frac{2}{3}}\cdot 1 = e^{\frac{4}{3}a} = 4.$

$\Rightarrow \frac{4}{3}a = 2\ln 2 \Rightarrow a = \frac{3}{2}\ln 2.$

Method 3:

$\lim_{x\to\infty}[(1+\frac{2a}{3x-a})^{3x-a}]^{\frac{2}{3}}\cdot(1+\frac{2a}{3x-a})^{1+\frac{2}{3}a}e^{\lim_{x\to\infty}(2x+1)\ln(\frac{3x+a}{3x-a})} = 4$

By L' Hôpital Rule $\lim_{x\to\infty} 3a\frac{4x^2+4x+1}{9x^2-a^2} = \frac{4a}{3} = \ln 4$

$\Rightarrow a = \frac{3}{2}\ln 2.$

(b) $\lim_{t\to 0^+}\frac{\cos t}{t^{4/3}} = \infty.$

Method 1:

Apply limit comparison test

$\lim_{x\to 0^+}\frac{\frac{\cos x}{x^{4/3}}}{\frac{1}{x^{4/3}}} = \lim_{x\to 0^+}\cos x = 1$

Since $\int_0^1 \frac{dx}{x^p}$ div. $\Leftrightarrow p \geq 1.$ so $\int_0^1 \frac{\cos t}{t^{4/3}}\,dt$ div.

Method 2:
Apply comparison test $\frac{\cos x}{x^{4/3}} \le \frac{\cos 1}{x^{4/3}}$, $x \in (0,1]$
Since $\int_0^1 \frac{dx}{x^p}$ div. $\Leftrightarrow p \ge 1$. so $\int_0^1 \frac{\cos t}{t^{4/3}}\, dt$ div.

(c) $\lim\limits_{x\to 0^+} \frac{\int_{\sqrt{x}}^1 \frac{\cos t}{t^{4/3}} dt}{x^{-\frac{1}{6}}}$ (由(b) 知爲$\frac{\infty}{\infty}$ type)
$= \lim\limits_{x\to 0^+} \frac{-\frac{\cos\sqrt{x}}{(\sqrt{x})^{4/3}}\cdot\frac{1}{2}x^{-1/2}}{-\frac{1}{6}x^{-7/6}}$ (L'Hôpital's rule)
$= \lim\limits_{x\to 0^+} 3\cdot\cos\sqrt{x}$
$= 3.$

2. (24%) Evaluate the following three integrals.

$$I_1 = \int_e^{e^3} (\ln x)^2\, dx, \qquad I_2 = \int_0^{1/4} \sqrt{\frac{x}{1-x}}\, dx,$$
$$I_3 = \int \frac{\sin x(1-\cos x)}{(\sin^2 x + 2\cos^2 x)(1+\cos x)^2}\, dx\ .$$

Ans. $I_1 =$ ________________________________ ,

$I_2 =$ ________________________________ ,

$I_3 =$ ________________________________ .

Solution:

(1) (Let $u = (\ln x)^2, dv = dx, \Rightarrow du = 2\ln x\cdot\frac{1}{x}dx, v = x.$)
$I_1 = \int_e^{e^3}(\ln x)^2 dx = x(\ln x)^2|_e^{e^3} - \int_e^{e^3} 2\ln x dx = 9e^3 - e - 2\int_e^{e^3}\ln x dx$
(Let $u = \ln x, dv = dx, \Rightarrow du = \frac{1}{x}, v = x.$)
$=9e^3 - e - 2(x\ln x|_e^{e^3} - \int_e^{e^3} 1dx)$ $=9e^3 - e - 2(3e^3 - e - e^3 + e) = 5e^3 - 3e.$

(2) $I_2 = \int_0^{\frac{1}{4}}\sqrt{\frac{x}{1-x}}$ (Let $x = \sin^2\theta, \sqrt{x} = \sin\theta, \sqrt{1-x} = \cos\theta$)
$=\int_0^{\frac{\pi}{6}}\frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta d\theta$ $=2\int_0^{\frac{\pi}{6}}\sin^2\theta d\theta$ $=2\int_0^{\frac{\pi}{6}}\frac{1-\cos 2\theta}{2}d\theta$
$=[\theta - \frac{\sin\sin 2\theta}{2}]|_0^{\frac{\pi}{6}} = \frac{\pi}{6} - \frac{\sqrt{3}}{4}.$

(3) $I_3 = \int \frac{\sin x(1-\cos x)}{(\sin^2 x + 2\cos^2 x)(1+\cos x)^2}dx$
$= -\int \frac{1-\cos x}{(1+\cos^2 x)(1+\cos x)^2}\, d\cos x$ (Let $u = \cos x$)
$= -\int \frac{\frac{1}{2}}{1+u} + \frac{1}{(1+u)^2} + \frac{-\frac{1}{2}(u+1)}{1+u^2}du$
$= -\frac{1}{2}\ln|1+u| - \frac{1}{1+u} + \frac{1}{2}\int\frac{u}{1+u^2}du + \frac{1}{2}\int\frac{1}{1+u^2}du$
$= -\frac{1}{2}\ln|1+u| - \frac{1}{1+u} + \frac{1}{4}\ln|1+u^2| + \frac{1}{2}\tan^{-1}u + C$
$= -\frac{1}{2}\ln|1+\cos x| + \frac{1}{1+\cos x} + \frac{1}{2}\ln|1+\cos^2 x| + \frac{1}{2}\tan^{-1}(\cos x) + C$

3. (10%) Find the area of the region in polar coordinates that is inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 3$.

Ans. The area = ____ .

Solution:

The curves intersect when $r = 2 + 2\cos\theta = 3$

$\Leftrightarrow 2\cos\theta = 1 \Leftrightarrow \cos\theta = \frac{1}{2} \Leftrightarrow \theta = \pm\frac{\pi}{3}$.

Let $f(\theta) = 3$, $g(\theta) = 2 + 2\cos\theta$

Then $A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [g^2(\theta) - f^2(\theta)]d\theta$

$= \int_0^{\frac{\pi}{3}} [g^2(\theta) - f^2(\theta)]d\theta$

$= \int_0^{\frac{\pi}{3}} [4 + 8\cos\theta + 4\cos^2\theta - 9]d\theta$

$= \int_0^{\frac{\pi}{3}} [8\cos\theta + 4\cos^3\theta - 5]d\theta$

$= \int_0^{\frac{\pi}{3}} [8\cos\theta + 4(\frac{1+\cos 2\theta}{2}) - 5]d\theta$

$= \int_0^{\frac{\pi}{3}} (8\cos\theta + 2\cos 2\theta - 3)d\theta$

$= [8\sin\theta + \sin 2\theta - 3\theta]_0^{\frac{\pi}{3}}$

$= \frac{9\sqrt{3}}{2} - \pi$.

4. (10%) Find the arc length parameter $s = s(\theta)$ along the curve (in polar coordinates) $r = 3\cos^2\left(\frac{\theta}{2}\right)$, $-\pi/2 \le \theta \le \pi/2$, from the point where $\theta = 0$. Then solve for θ as a function of $s : \theta = \theta(s)$.

Ans. $s(\theta) =$ ____ ,

$\theta(s) =$ ____ .

Solution:

$r(u) = 3\cos^2\frac{u}{2}$, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$s(\theta) = \int_0^\theta \sqrt{r^2 + (\frac{dr}{du})^2}du$

$= \int_0^\theta \sqrt{(3\cos^2\frac{u}{2})^2 + (-3\cos\frac{u}{2}\sin\frac{u}{2})^2}du$

$= \int_0^\theta \sqrt{9\cos^4\frac{u}{2} + 9\cos^2\frac{u}{2}\sin^2\frac{u}{2}}du$

$= \int_0^\theta \sqrt{9\cos^2\frac{u}{2}(\cos^2\frac{u}{2} + \sin^2\frac{u}{2})}du$

$= \int_0^\theta \sqrt{9\cos^2\frac{u}{2}}du$

$= \int_0^\theta 3\cos\frac{u}{2}du$

$= 6\sin\frac{\theta}{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta(s) = 2\sin^{-1}(\frac{s}{6})$.

5. (15%) Find the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$, and the curvature κ of the curve $\mathbf{r}(t) = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + (2e^t)\mathbf{k}$, $t \in \mathbb{R}$.

Ans. $\mathbf{T} =$ ______________________________,

$\mathbf{N} =$ ______________________________,

$\kappa =$ ______________________________.

Solution:

$V = \frac{dr}{dt} = e^t(\sin 2t + 2\cos 2t)i + e^t(\cos 2t - 2\sin 2t)j + 2e^t k,$

$|V| = \sqrt{e^{2t}(\sin 2t + 2\cos 2t)^2 + e^{2t}(\cos 2t - 2\sin 2t)^2 + 4e^{2t}} = 3e^t$

$T = \frac{V}{|V|} = \frac{1}{3}(\sin 2t + 2\cos 2t)i + \frac{1}{3}(\cos 2t - 2\sin 2t)j + \frac{2}{3}k,$

$\frac{dT}{dt} = \frac{2}{3}(\cos 2t - 2\sin 2t)i + \frac{2}{3}(-\sin 2t - 2\cos 2t)j,$

$|\frac{dT}{dt}| = \sqrt{\frac{4}{9}(\cos 2t - 2\sin 2t)^2 + \frac{4}{9}(-\sin 2t - 2\cos 2t)^2} = \frac{2}{3}\sqrt{5},$

$N = \frac{\frac{dT}{dt}}{|\frac{dT}{dt}|} = \frac{1}{\sqrt{5}}(\cos 2t - 2\sin 2t)i + \frac{1}{\sqrt{5}}(-\sin 2t - 2\cos 2t)j,$

$\kappa = \frac{1}{|v|}|\frac{dT}{dt}| = \frac{1}{3e^t} \cdot \frac{2}{3}\sqrt{5} = \frac{2\sqrt{5}}{9e^t}.$

Hence $T = \frac{1}{3}(\sin 2t + 2\cos 2t)i + \frac{1}{3}(\cos 2t - 2\sin 2t)j + \frac{2}{3}k,$

$N = \frac{1}{\sqrt{5}}(\cos 2t - 2\sin 2t)i + \frac{1}{\sqrt{5}}(-\sin 2t - 2\cos 2t)j,$

$\kappa = \frac{2\sqrt{5}}{9e^t}.$

6. (16%) Let $y = f(x) = \dfrac{1}{2}(x-1)^2$ and $\mathbf{r}(x) = (x, f(x)), 1 \le x \le 2$, be the curve representing the graph of the function $y = f(x)$ for $x \in [1, 2]$.
(a) Find the arc length of this curve.
(b) Find the area of the surface generated by revolving this curve about the y-axis.

Ans. (a) The arc length = ______________________________,

(b) The area of the surface of revolution = ______________________________.

Solution:

(a) Method 1:

Arc length$=\int_1^2 \sqrt{1+(x-1)^2}dx$

$=\int_0^1 \sqrt{1+u^2}du$ (Let $u = x-1$).

$=\int \sec\theta \sec^2\theta d\theta = \int \sec^3\theta d\theta$. (Let $u = tan\theta$, $du = \sec^2\theta d\theta$).

$=\sec\theta\tan\theta - \int(\sec^2\theta - 1)\sec\theta d\theta$

$=\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}ln|\sec\theta + \tan\theta|$

$=\frac{1}{2}\sqrt{u^2+1}\cdot u + \frac{1}{2}\ln|\sqrt{u^2+1}+u|_0^1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}+1).$

Method 2:

Arc length$= \int_1^2 \sqrt{1+f'(x)^2}dx = \int_1^2 \sqrt{1+(x-1)^2}dx = \int_0^1 \sqrt{1+t^2}dt$

$= \frac{1}{2}\{t\sqrt{1+t^2} + \ln|t+\sqrt{t^2+1}|\}_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2}).$

(b) Surface area$=\int 2\pi s ds = \int_1^2 2\pi x\sqrt{1+f'(x)^2}dx = \int_1^2 2\pi x\sqrt{1+(x-1)^2}dx$

$\overset{x-1=t}{=} \int_0^1 2\pi(t+1)\sqrt{1+t^2}dt = 2\pi\int_0^1 t\sqrt{1+t^2}dt + 2\pi\int_0^1 \sqrt{1+t^2}dt$

$= 2\pi\cdot\frac{1}{3}(1+t^2)^{\frac{3}{2}}|_0^1 + 2\pi\cdot\frac{1}{2}\{\sqrt{2}+\ln(1+\sqrt{2})\}$

$= \frac{2}{3}\pi\{2^{\frac{3}{2}}-1\} + \pi\{\sqrt{2}+\ln(1+\sqrt{2})\}$

7. (10%) Find the area of the **infinite** region bounded by the positive x- and y-axis and the graph of $f(x) = \dfrac{1}{\sqrt{x}(2+x)}, 0 \le x < \infty$.

Ans. The area = ____________________________________ .

Solution:

Area$=\int_0^\infty \frac{dx}{\sqrt{x}(2+x)} = \lim_{M\to\infty}\int_0^M \frac{dx}{\sqrt{x}(2+x)} \overset{x=t^2,dx=2tdt}{=} \int_0^\infty \frac{2tdt}{t(2+t^2)} = 2\int_0^\infty \frac{dt}{t^2+(\sqrt{2})^2}$

$= 2\{\frac{1}{\sqrt{2}}\tan^{-1}(\frac{t}{\sqrt{2}})\}_0^\infty = \sqrt{2}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}$