94學年度下學期微積分甲一組期中考

(20%) 1. Let

$$\mathbf{r}(u) = \frac{1-u^2}{1+u^2}\,\mathbf{i} + \frac{2u}{1+u^2}\,\mathbf{j}.$$

(i) Find the arc length $s(t)$ of the curve $\mathbf{r}(u)$ from $u=0$ to $u=t$, $t>0$.

(ii) Use the result in (i) to reparametrize the curve with respect to arc length measured from the point $(1,0)$ in the direction of increasing u.

(iii) Show that the set $\{\mathbf{r}(u) | 0 \le u < \infty\}$ is the semicircle $\{(r,\theta) | r = 1,\ 0 \le \theta < \pi\}$.

<u>Solution:</u> (i)

$$\vec{\mathbf{r}}(u) = \left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right) = \left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}, \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) = (\cos\theta, \sin\theta).$$

$$u = \tan\frac{\theta}{2},\ du = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta,\ \frac{d\theta}{du} = \frac{2}{1+u^2}$$

$$\mathbf{r}'(u) = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{du} = (-\sin\theta, \cos\theta)\frac{2}{1+u^2}$$

$$|\mathbf{r}'(u)| = \frac{2}{1+u^2}$$

$$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = 2\int_0^t \frac{1}{1+u^2}\,du = 2\tan^{-1} t.$$

(ii)

$$s = 2\tan^{-1} u$$

$$u = \tan\frac{s}{2}$$

$$\vec{\mathbf{r}}(u) = \left(\frac{1-\tan^2\frac{s}{2}}{1+\tan^2\frac{s}{2}}, \frac{2\tan\frac{s}{2}}{1+\tan^2\frac{s}{2}}\right) = (\cos s, \sin s).$$

(iii)

$$u = \tan\frac{s}{2}$$

$$0 \le u < \infty \ \rightarrow\ 0 \le s < \pi$$

$$\vec{\mathbf{r}}(u) = (\cos s, \sin s).$$

(20%) 2. Let

$$f(x,y) = \begin{cases} \dfrac{xy^2}{2x^2+3y^4}, & \text{if } (x,y) \ne (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(i) Is f continuous at $(0,0)$? Give your reason.

(ii) Let $\vec{u} =< \cos\theta, \sin\theta >$ be a unit vector. Find the directional derivative $D_{\vec{u}} f(0,0)$ if exists, or show that it does not exist.

(iii) Is f differentiable at $(0,0)$? Give your reason.

Solution: (ii)

$$D_{\vec{u}}f(0,0) = \begin{cases} \dfrac{\sin^2\theta}{2\cos\theta}, & 當\ \cos\theta \neq 0; \\ 0, & 當\ \cos\theta = 0. \end{cases}$$

(20%) 3. Suppose that $w = f(u,v)$ is a differentiable function with continuous second partial derivatives. Let $u = 2xy$ and $v = x^2 - y^2$.

(i) Express $w_x(=\frac{\partial w}{\partial x})$, w_y, w_{xx}, and w_{yy} in terms of x, y, w_u, w_v, w_{uu}, w_{vv} and w_{uv}.

(ii) Find the relationship between $w_{xx} + w_{yy}$ and $w_{uu} + w_{vv}$.

Solution:
$w_x = 2yw_u + 2xw_v$
$w_y = 2xw_u - 2yw_v$
$w_{xx} = 4y^2w_{uu} + 4x^2w_{vv} + 8xyw_{uv} + 2w_v$
$w_{yy} = 4x^2w_{uu} + 4y^2w_{vv} - 8xyw_{uv} - 2w_v$
$w_{xx} + w_{yy} = 4(x^2+y^2)(w_{uu}+w_{vv}) = 4\sqrt{u^2+v^2}(w_{uu}+w_{vv})$

(20%) 4. Let

$$z = f(x,y) = x^2y + xy^2 - xy - x^2 - y^2.$$

(i) Find all the critical points of f, and determine whether they are local maximum points, local minimum points, or saddle points?

(ii) Find the absolute maximum and absolute minimum of f on the region $R = \{(x,y)|x^2 + xy + 2y^2 \leq 8\}$.

Solution: (i)

$$\begin{aligned}
\nabla f(x,y) &= \ < 2xy + y^2 - y - 2x, x^2 + 2xy - x - 2y > \\
&= \ < (y-1)(2x+y), (x-1)(x+2y) > \\
\nabla f(x,y) &= 0 \Rightarrow (x,y) = (1,1),\ (-2,1),\ (1,-2), (0,0). \\
D(x,y) &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \begin{aligned} f_{xx} &= 2y - 2 \\ f_{yy} &= 2x - 2 \\ f_{xy} &= f_{yx} = 2x + 2y - 1 \end{aligned}
\end{aligned}$$

The Second Derivatives Test:

$$\begin{aligned}
&D(0,0) = 4 - 1 > 0,\ f_{xx}(0,0) < 0 \text{ (local max)} \\
&D(1,-2) = D(-2,1) = 0 - 9 < 0 \text{ (saddle)} \\
&D(1,1) = 0 - 9 < 0 \text{ (saddle)}.
\end{aligned}$$

Answer. critical points: $(0,0)$, $(1,1)$, $(-2,1)$, $(1,-2)$.
local maximum point: $(0,0)$.
local minimum points: none.
saddle points: $(1,1)$, $(-2,1)$, $(1,-2)$.

(ii) The absolute extrema are located at the critical points in the interior of the region R or located on the boundary of R.
Let $g(x,y) := x^2 + xy + 2y^2$, clearly

$$g(0,0) < 8,\ g(1,1) < 8,\ g(1,-2) = 7 < 8,\ g(-2,1) = 4 < 8.$$

Use the method of Lagrange multipliers to find extreme value of f on $g(x,y) = 8$.

$$\begin{aligned}\nabla g(x,y) &= \ < 2x + y, x + 4y > \\ \nabla f = \lambda \nabla g \ \Rightarrow & \left\{ \begin{array}{l} (y-1)(2x+y) = \lambda(2x+y) \\ (x-1)(x+2y) = \lambda(x+4y) \end{array} \right.\end{aligned}$$

(1) If $y = -2x$, then $x^2 - 2x^2 + 8x^2 = 8 \Rightarrow (x,y) = (\pm\sqrt{\frac{8}{7}}, \mp\sqrt{\frac{32}{7}})$
(2) If $y \neq -2x$, then $\lambda = y - 1$. Thus,

$$\begin{aligned}& x^2 - x - 2y + 2xy = xy + 4y^2 - x - 4y \\ & \Rightarrow \left\{ \begin{array}{l} x^2 + xy + 2y - 4y^2 = 0 \\ x^2 + xy + 2y^2 = 8 \end{array} \right. \\ & \Rightarrow 6y^2 - 2y = 8 \Rightarrow (3y-4)(y+1) = 0 \Rightarrow (x,y) = (\ ,\frac{4}{3}), (3,-1), (-2,-1).\end{aligned}$$

$$\begin{aligned}& f(0,0) = 0 \\ & f(\partial R) = x^2y + xy^2 - 8 + y^2 \\ & f(\pm\sqrt{\frac{8}{7}}, \mp\sqrt{\frac{32}{7}}) = \pm\frac{16}{7}\sqrt{\frac{8}{7}} - \frac{24}{7} \\ & f(3,-1) = f(-2,-1) = -13 \\ & f(\ ,\frac{4}{3}) = (8 - 2y^2)y - 8 + y^2 = -\frac{8}{27}\end{aligned}$$

Answer. absolute maximum$= 0$, at the point $(0,0)$.
absolute minimum$= -13$, at the points $(3,-1)$, $(2,-1)$.

(20%) 5. (i) Evaluate the integral $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.

(ii) Find the volume of the solid that is above the cone $z = \sqrt{x^2 + y^2}$ and inside $x^2 + y^2 + 2z^2 = 4$.

(iii) Find the surface area of the solid given in (ii).

<u>Solution:</u> (i)

$$\begin{aligned}\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy &= \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\, dy\, dx = \int_0^3 ye^{x^2}\Big|_{y=0}^{y=\frac{x}{3}}\, dx \\ &= \int_0^3 \frac{x}{3}e^{x^2}\, dx = \frac{1}{6}e^{x^2}\Big|_0^3 = \frac{1}{6}(e^9 - 1)\end{aligned}$$

(ii) $x^2+y^2+2(x^2+y^2)=4 \Rightarrow x^2+y^2=\frac{4}{3}$, $r=\frac{2}{\sqrt{3}}$.

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi}\int_0^{\frac{2}{\sqrt{3}}}\left[\sqrt{\frac{4-r^2}{2}}-r\right] r\,dr\,d\theta \\ &= 2\pi\cdot\left(-\frac{2}{3}\cdot\left(\frac{4-r^2}{2}\right)^{\frac{3}{2}}-\frac{r^3}{3}\right)\Big|_0^{\frac{2}{\sqrt{3}}} = \left(\frac{8\sqrt{2}}{3}-\frac{16\sqrt{3}}{9}\right)\pi\end{aligned}$$

(iii) For $z=\sqrt{x^2+y^2}$ part:
$z_x=\frac{x}{\sqrt{x^2+y^2}}$, $z_y=\frac{y}{\sqrt{x^2+y^2}}$

$$\text{Area} = \int_0^{2\pi}\int_0^{\frac{2}{\sqrt{3}}}\sqrt{1+\frac{r^2}{r^2}}r\,dr\,d\theta = 2\pi\cdot\frac{\sqrt{2}}{2}r^2\Big|_0^{\frac{2}{\sqrt{3}}} = \frac{4}{3}\sqrt{2}\pi$$

For $z=\sqrt{\frac{4-x^2-y^2}{2}}$ part:
$z_x=\frac{-x}{\sqrt{2(4-x^2-y^2)}}$, $z_y=\frac{-y}{\sqrt{2(4-x^2-y^2)}}$

$$\begin{aligned}\text{Area} &= \int_0^{2\pi}\int_0^{\frac{2}{\sqrt{3}}}\sqrt{1+\frac{r^2}{8-2r^2}}r\,dr\,d\theta = 2\pi\int_0^{\frac{2}{\sqrt{3}}}\sqrt{\frac{8-r^2}{8-2r^2}}r\,dr \\ &= 2\pi\int_0^{\sin^{-1}\frac{1}{\sqrt{3}}}2\sqrt{2}\sin t\sqrt{2-\sin^2 t}\,dt \quad \begin{pmatrix} r=2\sin t \\ dr=2\cos t\,dt\end{pmatrix} \\ &= 2\pi\int_{\sqrt{\frac{2}{3}}}^{1}2\sqrt{2}\sqrt{1+u^2}\,du \quad \begin{pmatrix} u=\cos t \\ du=-\sin t\,dt\end{pmatrix} \\ &= 4\sqrt{2}\pi\int_{\tan^{-1}\sqrt{\frac{2}{3}}}^{\frac{\pi}{4}}\sec^3 v\,dv \quad \begin{pmatrix} u=\tan v \\ du=\sec^2 v\,dv\end{pmatrix} \\ &= 2\sqrt{2}\pi(\sec v\tan v+\ln|\sec v+\tan v|)\Big|_{\tan^{-1}\sqrt{\frac{2}{3}}}^{\frac{\pi}{4}} \\ &= 4\pi-\frac{4}{3}\sqrt{5}\pi+2\sqrt{2}\pi\ln\frac{\sqrt{3}+\sqrt{6}}{\sqrt{2}+\sqrt{5}}\end{aligned}$$