06 Spring Final Exam

1. (20%)Evaluate the following two integrals.

   (a) $I_1 = \int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy.$

   (b) $I_2 = \iint_\Gamma y\,dS$, where the surface Γ is the part of the cylinder $x^2+z^2=2z$ with $y \geq 0$ and is inside the cone $z=\sqrt{x^2+y^2}$.

   Solution:

   (a) The region of integration is the solid hemisphere $x^2+y^2+z^2 \leq 4$, $x \geq 0$.

$$\begin{aligned}
&\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\pi}\int_{0}^{2} (\rho\sin\phi\sin\theta)^2(\sqrt{\rho^2})\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta \int_0^\pi \sin^3\phi\,d\phi \int_0^2 \rho^5\,d\rho \\
&= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^\pi \left[\frac{1}{6}\rho^6\right]_0^2 \\
&= (\frac{\pi}{2})(\frac{2}{3}+\frac{2}{3})(\frac{32}{3}) = \frac{64}{9}\pi
\end{aligned}$$

   (b) The surface Γ is parametrized by $r(\theta, y) = (\cos\theta, y, 1+\sin\theta)$, $D = \{(\theta, y) | 0 \leq \theta \leq \pi, 0 \leq y \leq \sqrt{(1+\sin\theta)^2 - \cos^2\theta}\}$. Thus $r_\theta \times r_y = -(\cos\theta, 0, \sin\theta)$ and

$$\begin{aligned}
&\iint_\Gamma y dS \\
=\ &\iint_D y|r_\theta \times r_y| dA \\
=\ &\int_0^\pi \int_0^{\sqrt{(1+\sin\theta)^2-\cos^2\theta}} y dy d\theta \\
=\ &\int_0^\pi \left((1+\sin\theta)^2 - \cos^2\theta\right) d\theta \\
=\ &2 + \frac{\pi}{2}
\end{aligned}$$

2. (20%)Consider the vector field $\mathbf{H}(x,y,z) = \langle y^2 z, 2xyz - z^2\sin y, 2z\cos y + xy^2\rangle$.

   (a) Evaluate $curl\,\mathbf{H}$.

   (b) Find all possible potential functions of the field $\mathbf{H}$ if it is conservative. Otherwise, explain why it is not conservative.

(c) Evaluate the line integral $\int_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r}$ along the curve $\mathcal{C}$: $\mathbf{r}(t) = \langle \sin\frac{\pi t}{2}, t^2 - t, \frac{\pi t}{2} \rangle$, $0 \le t \le 1$.

Solution:

(a)

$$
\begin{aligned}
&\mathit{curl}\,\mathbf{H} = [\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}]\,\mathbf{i} + [\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}]\,\mathbf{j} + [\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}]\,\mathbf{k} \\
&\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(2z\cos y + xy^2) = -2z\sin y + 2xy \\
&\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(2xyz - z^2\sin y) = 2xy - 2z\sin y \\
&\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(y^2 z) = y^2, \quad \frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(2z\cos y + xy^2) = y^2 \\
&\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2xyz - z^2\sin y) = 2yz, \quad \frac{\partial P}{\partial y}(y^2 z) = 2yz \\
&\therefore \mathit{curl}\,\mathbf{H} = 0
\end{aligned}
$$

(b) **H 為一個守恆場, 故存在一個位勢函數 (純量場)** φ **使得** $\mathbf{H} = \nabla\varphi$, **亦即**
$\frac{\partial \varphi}{\partial x} = y^2 z$, $\frac{\partial \varphi}{\partial y} = 2xyz - z^2\sin y$, $\frac{\partial \varphi}{\partial z} = 2z\cos y + xy^2$
於是 $\varphi = xy^2 z + f(y,z)$, 代入第二式得到

$$2xyz + \frac{\partial f}{\partial y} = 2xyz - z^2\sin y$$

所以 $\frac{\partial f}{\partial y} = -z^2\sin y$, 解得 $f(y,z) = z^2\cos y + g(z)$. 因此

$$\varphi = xy^2 z + z^2\cos y + g(z)$$

代入第三式得到

$$xy^2 + 2z\cos y + g'(z) = 2z\cos y + xy^2$$

於是 $g'(z) = 0$, 從而 $g(z) = C$. 因此, 位勢函數為

$$\varphi = xy^2 z + z^2\cos y + C$$

(c)

$$\int_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathcal{C}} \nabla\varphi \cdot d\mathbf{r} = \varphi(\mathbf{r}(1)) - \varphi(\mathbf{r}(0)) = \varphi(1, 0, \frac{\pi}{2}) - \varphi(0,0,0) = \frac{\pi^2}{4}$$

3. (20%)Consider the vector field $\mathbf{W}(x,y,z) = \langle x^3 + 3y + \tan z, y^3, x^2 + y^2 + 3z^2 \rangle$.

(a) Find $\mathit{div}\,\mathbf{W}$.

(b) Find the flux of the vector field $\mathbf{W}$ across S, which is the part of the surface $1 - z = x^2 + y^2$ with $0 \le z \le 1$ and is oriented upward.

Solution:

By Divergnece Theorem

$$\iint_S \mathbf{W}\cdot d\mathbf{S} + \iint_{S_1} \mathbf{W}\cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{W} dV.$$

Where S_1 is $\{(r\cos\theta, r\sin\theta, 0)|0\le r\le 1, 0\le\theta\le 2\pi\}$.

$$\begin{aligned}
& \iint_{S_1} \mathbf{W}\cdot d\mathbf{S} \\
= & \iint_{S_1} \mathbf{W}\cdot \mathbf{n}\, dS \\
= & \iint_{S_1} (x^3+3y, y^3, x^2+y^2)\cdot(0,0,-1)\, d\mathbf{S} \\
= & -\int_0^{2\pi}\int_0^1 r^3 dr d\theta \\
= & -\frac{\pi}{2}
\end{aligned}$$

On the other hand, $E = \{(x,y,z)|(x,y)\in R_{x,y}, 0\le z\le 1-(x^2+y^2)\}$ and $R_{x,y} = \{(r\cos\theta, r\sin\theta)|0\le r\le 1, 0\le\theta\le 2\pi\}$. Hence

$$\begin{aligned}
& \iiint_E \text{div}\, \mathbf{W}\, dV \\
= & \iint_{R_{x,y}}\int_0^{1-(x^2+y^2)} (3x^2+3y^2+6z)\, dz\, dx\, dy \\
= & \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 3r^3+6rz\, dz\, dr\, d\theta \\
= & \int_0^{2\pi}\int_0^1 3r^3(1-r^2)+3r(1-r^2)\, dr\, d\theta \\
= & 6\pi\int_0^1 3r-3r^3\, dr \\
= & \frac{3\pi}{2}
\end{aligned}$$

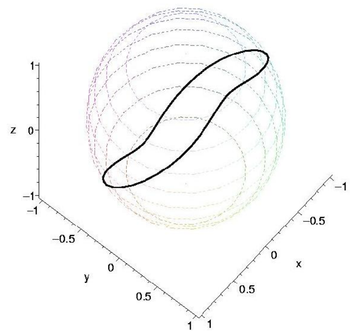
So the answer is 2π.

4. (15%) Let $\mathbf{F}(x,y,z) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z^2\rangle$ be a vector field on $\mathcal{D}_1 = \{(x,y,z)\,|\,x^2+y^2\ne 0\}$. Let $r = \sqrt{x^2+y^2+z^2}$ and $\mathbf{G}(x,y,z) = \langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle$ be a vector field on $\mathcal{D}_2 = \{(x,y,z)\,|\,x^2+y^2+z^2\ne 0\}$. Let C be the simple closed curve $\mathbf{r}(t) = \langle\cos(\sin t)\cos t, \cos(\sin t)\sin t, \sin(\sin t)\rangle, 0\le t\le 2\pi$. See the figure below.

   (a) Compute *curl* $\mathbf{F}$ on $\mathcal{D}_1$ and *curl* $\mathbf{G}$ on $\mathcal{D}_2$.

   (b) Evaluate the line integrals $\int_C \mathbf{F}\cdot d\mathbf{r}$ and $\int_C \mathbf{G}\cdot d\mathbf{r}$.

(c) Is $\mathbf{F}$ conservative on $\mathcal{D}_1$? Is $\mathbf{G}$ conservative on $\mathcal{D}_2$? State your reasons.



Solution:

(a) $curl\ \mathbf{F} = curl\ \mathbf{G} = 0$.

(b) Since there is a surface $S_1 \subset \mathcal{D}_1$, such that $\partial S_1 = C \cup (-C_1)$. Where C_1 is $\{(a\cos t, a\sin t, 0) | a < 1, 0 \le t \le 2\pi\}$. By Stokes' Theorem, we have

$$\int_C \mathbf{F}\cdot dr + \int_{-C_1} \mathbf{F}\cdot dr = \iint_{S_1} curl\ \mathbf{F}\cdot d\mathbf{S} = 0.$$

Hence

$$\begin{aligned} & \int_C \mathbf{F}\cdot dr \\ = & \int_{C_1} \mathbf{F}\cdot dr \\ = & \int_0^{2\pi} (-\frac{a\sin t}{a^2}, \frac{a\cos t}{a^2}, 0)\cdot(-a\sin t, a\cos t, 0)dt \\ = & \ 2\pi \end{aligned}$$

On the other hand, there exists a surface $S_2 \subset \mathcal{D}_2$, such that $\partial S_2 = C$. Hence by Stokes' Theorem, we obtain

$$\int_C \mathbf{G}\cdot dr = \iint_{S_1} curl\ \mathbf{G}\cdot d\mathbf{S} = 0.$$

(c) $\mathbf{F}$ is not conservative since $\int_{C_1} \mathbf{F} \cdot dr \neq 0$. Since any closed curve γ in $\mathcal{D}_2$ bounds a surface $\Omega \subset \mathcal{D}_2$. Therefore by Stokes' Theorem, we obtain

$$\int_\gamma \mathbf{G} \cdot dr = \iint_\Omega curl\ \mathbf{G} \cdot d\mathbf{S} = 0.$$

So $\mathbf{G}$ is conservative.

5. (15%) Let E_0 be the bounded region in $\mathbb{R}^3$ that is bounded by the surface $\sqrt{x}+\sqrt{y}+\sqrt{z} = 1$ and the coordinate planes.

(a) Find the volume of E_0.

(b) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by $L(u, v, w) = (x, y, z)$, where

$$\begin{cases} x(u,v,w) &= a_1 u + a_2 v + a_3 w, \\ y(u,v,w) &= b_1 u + b_2 v + b_3 w, \\ z(u,v,w) &= c_1 u + c_2 v + c_3 w, \end{cases}$$

and $a_i, b_i, c_i \in \mathbb{R}, i = 1, 2, 3$. Evaluate the Jacobian of this transformation. Is it a constant?

(c) Define a transformation $T(u, v, w) = (-w, \frac{u}{2} + v + w, -u + 2v + w)$. Let E_1 be the image of E_0 under the transformation T, E_2 be the image of E_1 under T, E_3 be the image of E_2 under T, and so on. That is, E_n is the image of E_{n-1} under $T, n \geq 1$. Find the volume of E_n.

<u>Solution:</u>

(a) Let $x = u^2$, $y = v^2$, $z = w^2$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 8uvw$$

$$\begin{aligned} V &= \iiint_{E_0} dx, dy, dz \\ &= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dw \, dv \, du \\ &= \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2 \, dv \, du \\ &= \int_0^1 \int_0^{1-u} 4u(1-u)^2 v - 8u(1-u)^2 v^2 + 4uv^3 \, dv \, du \\ &= \int_0^1 \frac{1}{3} u(1-u)^4 \, du = \frac{1}{90} \end{aligned}$$

(b)

$$\det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(c) Let $T(u, v, w) = (x, y, z)$,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 0 & -1 \\ \frac{1}{2} & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -1 \cdot (1+1) = -2$$

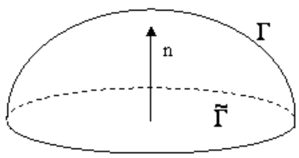
$$\Rightarrow \iiint_{E_1} dV = \iiint_{E_0} 2\,dV = 2 \cdot \frac{1}{90}$$

$$\Rightarrow \iiint_{E_2} dV = \iiint_{E_1} 2\,dV = 2^2 \cdot \frac{1}{90}$$

$$\Rightarrow \text{volume}(E_n) = \frac{2^n}{90}$$

6. (10%)Evaluate $\iint_\Gamma curl\,\mathbf{U}\cdot\, d\mathbf{S} = \iint_\Gamma curl\,\mathbf{U}\cdot\mathbf{n}\,dS$, where $\mathbf{U}(x,y,z) = \langle ye^z, x+y^2e^z, ze^{xy}\rangle$ and Γ is the part of the surface $z = 1-x^2-2y^2$ with $z \geq 0$ and upward orientation.

Solution



[解一]

$$curl\,\mathbf{U} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ ye^z & x+y^2e^z & ze^{xy} \end{vmatrix} = \left\langle xze^{xy} - y^2e^z, ye^z - zye^{xy}, 1-e^z \right\rangle$$

$$\iint_\Gamma curl\,\mathbf{U}\cdot d\mathbf{S} = \iint_{\tilde{\Gamma}} curl\,\mathbf{U}\cdot\mathbf{n}\,dS = \iint_{\tilde{\Gamma}} 1-e^z \overset{z=0}{=} \iint_{\tilde{\Gamma}} 1-1 = 0,$$

where $\mathbf{n} =< 0,0,1 >$ and $1-e^z = 0$ for $z=0$ $X-Y$ plane.

[解二]

$$\iint_\Gamma curl\,\mathbf{U}\cdot d\mathbf{S} = \int_{\partial\Gamma} \mathbf{U}\cdot d\mathbf{r} = \int_0^{2\pi} \frac{-1}{\sqrt{2}}\sin^2\theta + \frac{1}{\sqrt{2}}\cos^2\theta + \frac{1}{2\sqrt{2}}\cos\theta\sin^2\theta\,d\theta = 0$$

$$\partial\Gamma : x^2+2y^2 = 1,\ z = 0 (x = \cos\theta,\ y = \frac{1}{\sqrt{2}}\sin\theta)$$