94學年上學期微甲二組期末考

(12%) 1. Evaluate the integral $\displaystyle\int \frac{2x^2+5x+3}{(x^2+2x+2)(x-1)}\,dx$.

Ans: $\tan^{-1}(x+1)+2\ln|x-1|+C$.

(10%) 2. Evaluate the integral $\displaystyle\int e^{\sin^{-1}x}\,dx$.

Solution:

$$\begin{aligned}
\int e^{\sin^{-1}x}\,dx &\overset{y=\sin^{-1}x}{=} \int e^y\,d\sin y = e^y\sin y - \int \sin y\cdot e^y\,dy \\
&= e^y\sin y + \int e^y\,d\cos y = e^y\sin y + e^y\cos y - \underbrace{\int \cos y\cdot e^y\,dy}_{\int e^y\,d\sin y}
\end{aligned}$$

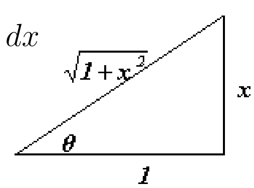
So

$$\int e^y\,d\sin y = \frac{e^y}{2}(\sin y+\cos y)+C = \frac{e^{\sin^{-1}x}}{2}\left(x+\sqrt{1-x^2}\right)+C.$$

(12%) 3. Evaluate the integral $\displaystyle\int_1^2 \frac{dx}{x^2\sqrt{1+x^2}}$.

Solution:

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}\ ,\ \tan\theta = x \Rightarrow \sec^2\theta\,d\theta = dx$$



$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\cos\theta\cdot\sec^2\theta\,d\theta}{\tan^2\theta} \\
&= \int \frac{\frac{1}{\cos^2\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\cos\theta\,d\theta \\
&= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\
&= \int \csc^2\theta\cos\theta\,d\theta \\
&\quad \text{Let } u=\cos\theta \Rightarrow du = -\sin\theta\,d\theta \\
&\quad dv = \csc^2\theta\,d\theta \Rightarrow v = -\cot\theta \\
&= -\cos\theta\cdot\cot\theta - \int(-\cot\theta)(-\sin\theta)\,d\theta \\
&= -\cos\theta\cot\theta - \sin\theta + C \\
&= -\frac{\sqrt{1+x^2}}{x}
\end{aligned}$$

(12%) 4. Find the length of the curve $y = \int_1^{\frac{1}{x^2}} \sqrt{t^3+1}\,dt$, $\frac{1}{2} \le x \le 1$.

Solution:

$y' = -2x^{-3}\sqrt{1+x^{-6}}$, $1 + y'^2 = 4x^{-12} + 4x^{-6} + 1$

$$s = \int_{\frac{1}{2}}^{1} \sqrt{1+y'^2}\,dx = \int_{\frac{1}{2}}^{1} (2x^{-6}+1)\,dx = \left(x - \frac{2}{5}x^{-5}\right)\Big|_{\frac{1}{2}}^{1} = \frac{129}{10}.$$

(12%) 5. For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

Solution:

$$L = \lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = \lim_{x\to 0}\frac{\sin 2x + ax^3 + bx}{x^3} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2\cos 2x + 3ax^2 + b}{3x^2}.$$

As $x \to 0$, $3x^2 \to 0$, and $(2\cos 2x + 3ax^2 + b) \to b + 2$, so the last limit exists only if $b + 2 = 0$, that is $b = -2$. Thus,

$$\lim_{x\to 0}\frac{2\cos 2x + 3ax^2 - 2}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-4\sin 2x + 6ax}{6x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-8\cos 2x + 6a}{6} = \frac{6a-8}{6},$$

which is equal to 0 if and only if $a = \frac{4}{3}$. Hence, $L = 0$ if and only if $b = -2$ and $a = \frac{4}{3}$.

(12%) 6. Let $f(x) = \sin^2 x$

(a) Find the Maclaurin series for $f(x)$.

Solution:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2x) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$\begin{aligned}\sin^2 x &= \frac{1}{2}\left[1 - \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}\right] \\ &= -\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} 2^{2n-1} x^{2n}}{(2n)!}\end{aligned}$$

(b) Find $f^{(94)}(0)$.

(10%) 7. Discuss the convergence of the the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^{n-1}}$.

Solution:

ratio test

$$a_n = \frac{n!}{n^{n-1}}$$

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(n+1)^n}}{\frac{n!}{n^{n-1}}}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{n^{n-1}}{(n+1)^n} \\
&= \lim_{n\to\infty} (n+1) \frac{n^{n-1}}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n-1} \\
&= e^{-1}
\end{aligned}$$

(10%) 8. Discuss the convergence of the the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$.
(absolutely convergent, conditionally convergent, or divergent)

(10%) 9. Discuss the convergence of the the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$.
(absolutely convergent, conditionally convergent, or divergent)