94學年上學期微甲一組期中考

(20%) 1. Evaluate the limits.

(a) $\lim\limits_{x\to\infty} x(\sqrt{x^2-x}-x)\sin\dfrac{1}{x}$.

Solution:

$$\begin{aligned}\lim_{x\to\infty} x\left(\sqrt{x^2-x}-x\right)\sin\frac{1}{x} &= \lim_{x\to\infty}\left(\sqrt{x^2-x}-x\right)\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}+x}\cdot\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} \\ &= -\frac{1}{2}\cdot 1 = -\frac{1}{2}\end{aligned}$$

(b) $\lim\limits_{a\to 0^+}\dfrac{1}{\sqrt{a}}\left(\lim\limits_{x\to a}\dfrac{1}{x-a}\displaystyle\int_{\sin a}^{\sin x}\sqrt[4]{t^2+a^2}\ dt\right)$.

Solution:

$$\begin{aligned}&\lim_{a\to 0^+}\frac{1}{\sqrt{a}}\left(\lim_{x\to a}\frac{1}{x-a}\int_{\sin a}^{\sin x}\sqrt[4]{t^2+a^2}\ dt\right) \\ =\ &\lim_{a\to 0^+}\frac{1}{\sqrt{a}}\left(\frac{d}{dx}\int_{.}^{\sin x}\sqrt[4]{t^2+a^2}\ dt\bigg|_{x=a}\right) \\ =\ &\lim_{a\to 0^+}\frac{1}{\sqrt{a}}\cdot\sqrt[4]{\sin^2 a+a^2}\cdot\cos a \\ =\ &\lim_{a\to 0^+}\sqrt[4]{\left(\frac{\sin a}{a}\right)^2+1}\cdot\lim_{a\to 0^+}\cos a \\ =\ &\sqrt[4]{2}\cdot 1=\sqrt[4]{2}\end{aligned}$$

(c) $\lim\limits_{n\to\infty}\dfrac{\pi}{n}\left[\sin\dfrac{\pi}{n}\cos\left(\cos\dfrac{\pi}{n}\right)+\sin\dfrac{2\pi}{n}\cos\left(\cos\dfrac{2\pi}{n}\right)+\cdots+\sin\pi\cos\left(\cos\pi\right)\right]$

Solution:

$$\begin{aligned}&\lim_{n\to\infty}\frac{\pi}{n}\left[\sin\frac{\pi}{n}\cos\left(\cos\frac{\pi}{n}\right)+\sin\frac{2\pi}{n}\cos\left(\cos\frac{2\pi}{n}\right)+\cdots+\sin\pi\cos\left(\cos\pi\right)\right] \\ =\ &\int_0^\pi \sin x\cos\left(\cos x\right)\ dx \\ =\ &-\sin\left(\cos x\right)\big|_0^\pi = 2\sin 1\end{aligned}$$

(20%) 2. (a) Find $f(9)$ if

i. $\displaystyle\int_0^{f(x)} t^2\ dt = x\cos\pi x.$

ii. $\displaystyle\int_0^{x^2} f(t)\ dt = x\sin\pi x.$

Solution:

i. $x\cos\pi x = \dfrac{1}{3}t^3\Big|_0^{f(x)} = \dfrac{1}{3}f(x)^3,$

$9\cdot\cos 9\pi = \dfrac{1}{3}f(9)^3,$

$f(9)^3 = -27,$

$f(9) = -3.$

ii. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} f(t)\ dt = \dfrac{d}{dx} x\sin\pi x,$
$2x\cdot f(x^2) = \sin\pi x + \pi x\cos\pi x,$
$x = 3 \Rightarrow 2\cdot 3\cdot f(9) = \sin 3\pi + 3\pi\cdot\cos 3\pi = -3\pi \Rightarrow f(9) = -\dfrac{\pi}{2}.$
$\left(x = -3 \Rightarrow 2\cdot(-3)\cdot f(9) = \sin(-3\pi) - 3\pi\cos(-3\pi) = 3\pi \Rightarrow f(9) = -\dfrac{\pi}{2}\right)$

(b) Evaluate $\displaystyle\int_{\pi^2/9}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t\sin\sqrt{t}}}\ dt.$

Solution:

Let $u = \sin\sqrt{t}$, then $du = \cos\sqrt{t}\cdot\dfrac{1}{2\sqrt{t}}\ dt = \dfrac{\cos\sqrt{t}\ dt}{2\sqrt{t}}.$

$t = \dfrac{\pi^2}{4} \Rightarrow u = \sin\dfrac{\pi}{2} = 1,$
$t = \dfrac{\pi^2}{9} \Rightarrow u = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}.$

Hence $\displaystyle\int_{\pi^2/9}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t\sin\sqrt{t}}}dt = \int_{\frac{\sqrt{3}}{2}}^{1} \frac{1}{\sqrt{u}}\cdot 2du = 2\int_{\frac{\sqrt{3}}{2}}^{1} u^{-\frac{1}{2}}du = 4\sqrt{u}\Big|_{\frac{\sqrt{3}}{2}}^{1} = 4\left(1 - \frac{\sqrt[4]{3}}{\sqrt{2}}\right).$

(10%) 3. Define $f(x) = \begin{cases} x^3, & x > 0, \\ 0, & x \le 0. \end{cases}$

(a) Show that f is differentiable at $x = 0$.

(b) For $a < 0 < b$ the mean value theorem states that there exists $\theta \in (0,1)$ such that

$$f(b) - f(a) = f'(a + \theta\cdot(b-a))\cdot(b-a).$$

Find an expression of θ in terms of a and b.

Solution:

(a) $\displaystyle\lim_{\Delta x\to 0^+}\frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0^+}(\Delta x)^3 = 0,$
$\displaystyle\lim_{\Delta x\to 0^-}\frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{0-0}{\Delta x} = 0$

f 在 $x = 0$ 點的右導數等於左導數,故 f 在 $x = 0$ 點可微分.

(b) 因為 $a < 0$, $b > 0$, 所以 $f(a) = 0$, $f(b) = b^3$,平均値定理變成

$$b^3 = f'(a + \theta\cdot(b-a))\cdot(b-a) \tag{1}$$

今因

$$f'(x) = \begin{cases} 3x^2 & 當\ x > 0 \\ 0 & 當\ x \le 0 \end{cases}$$

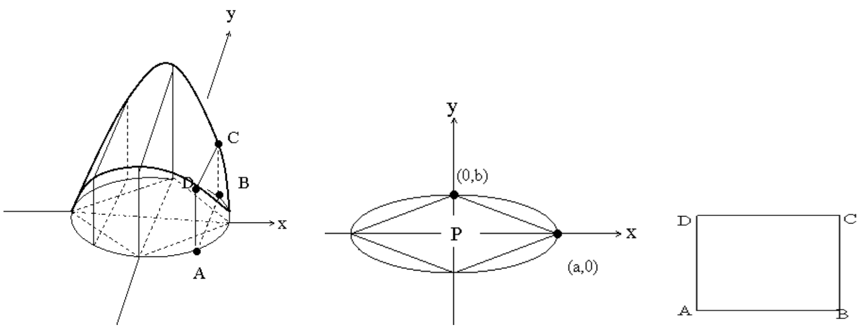
故若 $a + \theta\cdot(b-a) \le 0$, 則 $f'(a + \theta\cdot(b-a)) = 0$,這跟 (1) 式抵觸.因此 $a + \theta\cdot(b-a) > 0$,由 (1) 式得到

$$b^3 = 3(a + \theta\cdot(b-a))^2\cdot(b-a)$$

解得

$$\theta = \frac{-a}{b-a} + \sqrt{\frac{b^3}{3(b-a)^3}} \text{ 或 } \frac{1}{b-a}\left(\sqrt{\frac{b^3}{3(b-a)}} - a\right).$$

(10%) 4. For each $a > 0$, $b > 0$ consider a solid whose base is an elliptic region given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ and whose cross-sections perpendicular to the x-axis are squares. Let P be the inscribed polygon as shown in figures below.



Assume that the inscribed polygon has fixed perimeter 4. Show that the maximum of the volume of the solid occurs when $a = \sqrt{\frac{1}{3}}$ and $b = \sqrt{\frac{2}{3}}$. Also find the maximum volume.

Solution:

$$A(x) = \left(2\sqrt{\frac{a^2b^2 - b^2x^2}{a^2}}\right)^2 = 4b^2\left(1 - \frac{x^2}{a^2}\right).$$

$$V = \int_{-a}^{a} 4b^2\left(1 - \frac{x^2}{a^2}\right)\,dx = 8b^2\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{16}{3}ab^2 \text{ 或 } \frac{16}{3}\left(a - a^3\right)$$

$$a^2 + b^2 = 1 \;\Rightarrow\; V = \frac{16}{3}a(1-a^2) = \frac{16}{3}(a - a^3) \;\text{ (Note } 0 < a < 1)$$

$$\Rightarrow \frac{dV}{da} = \frac{16}{3}(1 - 3a^2) = 0 \text{ when } a = \sqrt{\frac{1}{3}} \;\left(\text{thus } b = \sqrt{\frac{2}{3}}\right)$$

The maximum of V is $\frac{16}{3}\sqrt{\frac{1}{3}} \cdot \frac{2}{3} = \frac{32}{9\sqrt{3}}$ 或 $\frac{32}{27}\sqrt{3}$

(10%) 5. Using a linear approximation to estimate the value $\cot 46^\circ$. Is your approximation accurate to within 0.1?

Solution:

$$\begin{aligned} L(46^\circ) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(46^\circ - \frac{\pi}{4}\right) \\ &= 1 - 2\left(\frac{46\pi}{180} - \frac{\pi}{4}\right) \\ &= 0.966. \end{aligned}$$

$$\text{By } f''(x) > 0,\; x \in \left[\frac{\pi}{4}, 46^\circ\right],$$

$$L(46^\circ) < f(46^\circ),$$

$$\text{By } f'(x) < 0,\ x \in \left[\frac{\pi}{4}, 46^\circ\right],$$

$$f(46^\circ) - 0.1 < f\left(\frac{\pi}{4}\right) - 0.1 = 0.9 < 0.966 = L(46^\circ).$$

(10%) 6. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius $R > r$ centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

Solution:

cone under water

$$V = \frac{\pi}{3}\left(\frac{r}{h}\right)^2 x^3,$$

relation

$$\pi R^2 y = \text{water} + \frac{\pi}{3}\left(\frac{r}{h}\right)^2 x^3,$$

chain rule

$$\frac{dy}{dt} = \left(\frac{r}{R}\right)^2 \frac{dx}{dt},$$

relation

$$\frac{dy}{dt} + 1 = \frac{dx}{dt}.$$

Thus answer is

$$\frac{dy}{dt} = \frac{r^2}{R^2 - r^2}.$$

(20%) 7. 畫圖題分 A、B 兩部份。

A 為填充題, 最後一小題為畫圖。

B 為空白頁, 供同學記錄其計算過程之用。請於計算所得各項答案處以底線標記並註明其所對應之填充格之號碼。沒有計算記錄或未以底線標記並加註填充題格號者, 該小題不予計分。

A. Study the function $y = f(x) = x^{2/3}(x-2)^2$ and answer the following questions.

i. The domain of $y = f(x)$ is $a)$____________________

ii. $f'(x) = b)$____________________ ,

$y = f(x)$ has critical points at $x = c)$____________________

iii. $f''(x) = d)$____________________

iv. $y = f(x)$ is increasing on intervals $e)$____________________

$y = f(x)$ is decreasing on intervals $f)$____________________

v. $y = f(x)$ is concave upward on intervals $g)$____________________

$y = f(x)$ is concave downward on intervals $h)$____________________

vi. $y = f(x)$ has local maximum at $(x, y) = i)$____________________

$y = f(x)$ has local minimum at $(x, y) = j)$____________________

$y = f(x)$ has inflection points at $x = k)$____________________

vii. Does $y = f(x)$ have any asymptote? $l)$____________________

viii. Sketch the graph of $y = f(x)$ $m)$:

<u>Solution:</u>

$a)\mathbb{R}$

$b)\frac{4}{3}x^{-\frac{1}{3}}(x-2)(2x-1)$

$c)0, \frac{1}{2}, 2$

$d)\frac{8}{9}x^{-\frac{4}{3}}\left(5x^2 - 5x - 1\right)$

$e)(0, \frac{1}{2}),\ (2, \infty)$

$f)(-\infty, 0),\ (\frac{1}{2}, 2)$

$g)(-\infty, \frac{5-3\sqrt{5}}{10}),\ (\frac{5+3\sqrt{5}}{10}, \infty)$

$h)(\frac{5-3\sqrt{5}}{10}, 0),\ (0, \frac{5+3\sqrt{5}}{10})$

$i)(\frac{1}{2}, \frac{9}{4}\frac{1}{\sqrt[3]{4}})$

$j)(0,0),\ (2,0)$

$k)\frac{5 \pm 3\sqrt{5}}{10}$

$l)$ No

$m)$