94學年上學期微甲1組期末考

1. (20%)Evaluate the following integrals.

$$(a)\int_0^4 \frac{dx}{\sqrt{x(4-x)}}\,, \quad (b)\int_2^\infty \frac{x^2-2x-4}{(x^2-x)(x^2+4)}\,dx\,, \quad (c)\int_5^\infty \left(\ln\frac{x+5}{x} - \frac{5}{x+5}\right)dx\,.$$

Solution:

(b)

$$\begin{aligned} & \int_2^\infty \frac{x^2-2x-4}{(x^2-x)(x^2+4)}\,dx \\ = & \int_2^\infty \left(\frac{1}{x} - \frac{1}{x-1} + \frac{2}{x^2+4}\right)dx \\ = & \lim_{c\to\infty}\left(\ln x - \ln(x-1) + \ \texttt{arctan}\ \frac{x}{2}\right)|_2^c \\ = & \frac{\pi}{4} - \ln 2 \end{aligned}$$

(c)

$$\begin{aligned} & \int_5^\infty \left(\ln\frac{x+5}{x} - \frac{5}{x+5}\right)dx \\ = & \lim_{b\to\infty}\int_5^b \left(\ln(x+5) - \ln x - \frac{5}{x+5}\right)dx \\ = & \lim_{b\to\infty}\left((x+5)\ln(x+5) - x\ln x - 5\ln(x+5)\right)|_5^b \\ = & \lim_{b\to\infty}\left(x\ln(x+5) - x\ln x\right)|_5^b \\ = & \lim_{b\to\infty}\frac{\ln\frac{b+5}{b}}{\frac{1}{b}} - 5\ln 2 \\ = & 5 - 5\ln 2 \end{aligned}$$

2. (20%)Let $y = f(x)$ be an increasing function that passes through the origin. Suppose that the arc length of the curve from $(0,0)$ to $(x, f(x))$ is given by $s(x) = \int_0^x \sqrt{1+e^t}\,dt$.

   (i) Identify the function $y = f(x)$.

   (ii) Denote $s(2) = A_1 + A_2\sqrt{2} + A_3\sqrt{1+e^2} + B\ln(1+\sqrt{2}) + C\ln(1+\sqrt{1+e^2})$, $A_1, A_2, A_3, B, C \in \mathbb{Q}$. Determine A_1, A_2, A_3, B, and C.

   (iii) The arc of the curve from $(0,0)$ to $(2, f(2))$ is rotated about the x-axis. Let $u(2)$ be the area of the resulting surface. Denote $u(2) = D - 4\pi s(2)$. Determine D.

Solution:

(i) $f'(x) = e^{\frac{x}{2}}$
$f(x) = 2e^{\frac{x}{2}} - 2$

(ii)

$$
\begin{aligned}
s(2) &\overset{\sqrt{1+e^t}=y}{=} \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{2y^2}{y^2-1}\,dy \\
&= (2y + \ln(y-1) - \ln(y+1))|_{\sqrt{2}}^{\sqrt{1+e^2}} \\
&= 2(\sqrt{1+e^2} - \sqrt{2}) - 2\ln(\sqrt{1+e^2}+1) + 2 + 2\ln(\sqrt{2}+1)
\end{aligned}
$$

$A_1 = 2,\ A_2 = -2,\ A_3 = 2,\ B = 2,\ C = -2.$

(iii)

$$
\text{area} = 4\pi \int_0^2 e^{\frac{t}{2}}\sqrt{1+e^t}\,dt - 4\pi s(2)
$$

$$
\begin{aligned}
4\pi \int_0^2 e^{\frac{t}{2}}\sqrt{1+e^t}\,dt &\overset{e^{\frac{t}{2}}=\tan\theta}{=} 8\pi \int_{\frac{\pi}{4}}^{\tan^{-1} e} \sec^3\theta\,d\theta \\
&= 4\pi(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)|_{\frac{\pi}{4}}^{\tan^{-1} e} \\
&= 4\pi\{e\sqrt{e^2+1} + \ln(\sqrt{e^2+1}+e) - \sqrt{2} - \ln(\sqrt{2}+1)\}
\end{aligned}
$$

3. (15%)Determine whether each series is convergent or divergent. State your reason. If the series converges, find its sum.

$$
(a)\sum_{n=3}^{\infty} \frac{1}{n\sqrt[3]{\ln n}}, \qquad (b)\sum_{n=1}^{\infty} \frac{(-\ln 3)^n}{n!}, \qquad (c)\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{\sqrt{n}}}}.
$$

<u>Solution:</u>

(a) Let $f(x) = \dfrac{1}{x\sqrt[3]{\ln x}}$, then $f(x)$ is continuous, positive, and decreasing on $[3,\infty)$. So the Integral Test applies

$$
\begin{aligned}
\int_3^{\infty} f(x)\,dx &= \lim_{t\to\infty}\int_3^t \frac{1}{x\sqrt[3]{\ln x}}\,dx = \lim_{t\to\infty}\int_{\ln 3}^{\ln t} u^{-\frac{1}{3}}\,du \\
&= \lim_{t\to\infty} \frac{3}{2}u^{\frac{2}{3}}\Big|_{\ln 3}^{\ln t} = \lim_{t\to\infty}\left(\frac{3}{2}(\ln t)^{\frac{2}{3}} - \frac{3}{2}(\ln 3)^{\frac{2}{3}}\right) = \infty.
\end{aligned}
$$

So the series diverges.

(b) $e^x - 1 = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$ converges for every $x \in \mathbb{R}$.
The sum is $e^{-\ln 3} - 1 = e^{\ln\frac{1}{3}-1} = \dfrac{1}{3} - 1 = -\dfrac{2}{3}$

(c) Use the Limit Comparison Test with $a_n = \dfrac{1}{n^{1+\frac{1}{\sqrt{n}}}}$ and $b_n = \frac{1}{n}$,

$$
\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{n^{1+\frac{1}{\sqrt{n}}}} = \lim_{n\to\infty}\frac{1}{n^{1/\sqrt{n}}} = 1,
$$

since $\lim\limits_{x\to\infty} x^{\frac{1}{\sqrt{x}}} = 1$ by L'Hospital's Rule.
The harmonic series $\sum \frac{1}{n}$ diverges, so $\sum a_n$ diverges.

4. (10%)Given a power series $\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^{3n}}{\sqrt{2n+1}}$, find

(i) the radius of convergence;

(ii) the interval of convergence;

(iii) the value(s) of x for which the series converges absolutely;

(iv) the value(s) of x for which the series converges conditionally.

Solution:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{3n+3}}{\sqrt{2n+3}} \cdot \frac{\sqrt{2n+1}}{(x-1)^{3n}}\right| = |x-1|^3 \sqrt{\frac{2n+1}{2n+3}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow |x-1|^3 < 1 \Rightarrow |x-1| < 1 \Rightarrow 0 < x < 2$$

At $x = 0$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$ diverges by comparison to p-series $\sum \frac{1}{n^{\frac{1}{2}}}$.

At $x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$ converges by Alternating Series Test.

(i) $R = 1$

(ii) $(0, 2]$

(iii) $0 < x < 2$

(iv) $x = 2$

5. (15%)Consider the indefinite integral $\int \frac{e^x - 1}{x}\, dx$.

(i) Represent this indefinite integral as a power series centered at 0. Also find the values of x for which this representation holds.

(ii) To evaluate $\int_{-\frac{1}{2}}^{0} \frac{e^x - 1}{x}\, dx$, let $\alpha_3 \in \mathbb{R}$ be the approximation obtained by using the first **three** (nonzero) terms of the power series found in (i). Thus $\int_{-\frac{1}{2}}^{0} \frac{e^x - 1}{x}\, dx$ $= \alpha_3 +$ error. Estimate the error and express the answer in *reduced fraction.* **NO** need to compute α_3.

(iii) Deduce the identity $\int \frac{dx}{\ln x} = C + \ln|\ln x| + \ln x + \frac{(\ln x)^2}{2 \cdot 2!} + \frac{(\ln x)^3}{3 \cdot 3!} + \cdots$, $x > 0$. (Hint. You may start from the result obtained in (i))

Solution:

$\frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \Rightarrow \int \frac{e^x - 1}{x}\, dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$ with radius of convergence equal to infinity. Let $\ln x = u$, then

$$\begin{aligned}\int \frac{1}{\ln x}\,dx &= \int \frac{e^u}{u}\,du = C + \ln|u| + \sum_{n=1}^{\infty}\frac{u^n}{n\cdot n!}\\ &= C + \ln|\ln x| + \ln x + \frac{(\ln x)^2}{2\cdot 2!} + \frac{(\ln x)^3}{3\cdot 3!} + \cdots\end{aligned}$$

6. (20%)Define

$$f(x) = \begin{cases} (1+x)^{\frac{1}{x}} & \text{if } x > -1\,, x \neq 0\,,\\ e & \text{if } x = 0\,.\end{cases}$$

(i) Show that $f(x)$ is continuous at 0.

(ii) Find $f'(x)$ for $x > -1\,, x \neq 0$.

(iii) Evaluate $f'(0)$.

(iv) Let $\sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series of $f(x)$. Determine c_0, c_1 and c_2.

Solution:

(i) $\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} \overset{L'H}{=} e^{\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}} = e^1 = e = f(0)$

(ii) Let $y = (1+x)^{\frac{1}{x}}$ then $\ln y = \frac{1}{x}\ln(1+x)$, $y'\cdot\frac{1}{y} = \left(-\frac{1}{x^2}\ln(1+x) + \frac{1}{x(1+x)}\right)$. So $f'(x) = (1+x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(1+x) + \frac{1}{x(1+x)}\right)$.

(iii) $f'(0) = \lim_{h\to 0}\dfrac{f(h)-f(0)}{h-0} = \lim_{h\to 0} f'(c) = \lim_{c\to 0} f'(c)$ where c is between h and 0 by the Mean Value Theorem. For $x \neq 0$, we have

$$\begin{aligned}f'(x) &= (1+x)^{\frac{1}{x}}\left[-\frac{1}{x^2}(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots) + \frac{1}{x} - \frac{1}{x+1}\right]\\ &= (1+x)^{\frac{1}{x}}\left(-\frac{1}{x+1} + \frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \cdots\right).\end{aligned}$$

Therefore $f'(0) = \lim_{c\to 0} f'(c) = e(-1 + \dfrac{1}{2} + 0) = -\dfrac{e}{2}$.

(iv)

$$f''(x) = (1+x)^{\frac{1}{x}}\left(-\frac{1}{x+1} + \frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \cdots\right)^2 + (1+x)^{\frac{1}{x}}\left(\frac{1}{(x+1)^2} - \frac{1}{3} + \frac{x}{2} - \cdots\right)$$

As above, $f''(0) = e(-1 + \dfrac{1}{2})^2 + e(1 - \dfrac{1}{3}) = \dfrac{11}{12}e$

Therefore

$$\begin{aligned}c_0 &= f(0) = e\\ c_1 &= f'(0) = -\frac{e}{2}\\ c_2 &= \frac{f''(0)}{2} = \frac{11}{24}e\end{aligned}$$

or $f(x) = e - \dfrac{e}{2}x + \dfrac{11e}{24}x^2 - \cdots$.