93學年下學期微甲一組期末考

1. Find the volume of the solid enclosed by the ellipsoid $x^2+2xy+5y^2+4yz+4z^2 = 1$.

   Solution:

$$(x+y)^2 + (2y+z)^2 + \left(\sqrt{3}z\right)^2 = 1$$

.

$$u = x+y, \quad v = 2y+z, \quad w = \sqrt{3}z$$

.

$$\begin{aligned} x &= u - \frac{v}{2} + \frac{w}{2\sqrt{3}} \\ y &= \frac{v}{2} - \frac{w}{2\sqrt{3}} \\ z &= \frac{w}{\sqrt{3}} \end{aligned}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{vmatrix} = \frac{1}{2\sqrt{3}}$$

$$V = \iiint\limits_{E_{xyz}} dV \quad = \quad \iiint\limits_{E_{uvw}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV = \frac{1}{2\sqrt{3}} \cdot \frac{4}{3}\pi = \frac{2\pi}{3\sqrt{3}}$$

2. Let

$$\begin{aligned} \mathbf{F}(x,y,z) &= \left(2x + 3y^4 + 2xyze^{x^2} + \frac{2x}{1+x^2+z^2}\right)\mathbf{i} \\ &+ \left(12xy^3 + ze^{x^2}\right)\mathbf{j} + \left(ye^{x^2} + \frac{2z}{1+x^2+z^2}\right)\mathbf{k} \end{aligned}$$

   (a) Find a function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$.

   (b) Evaluate the line integral $\int_C \mathbf{F} \cdot dr$ along the curve C given by $r(t) = \left\langle \cos t, \sin t, \frac{t}{2\pi}\right\rangle$, $0 \le t \le 2\pi$.

3. Evaluate the line integral $\displaystyle\int_C \frac{-y\,dx + (x^2+y^2-x)\;dy}{(x-1)^2+y^2}$ along the curve C which is defined by the polar equation $r = \dfrac{1}{2} + \cos\theta$, $0 \le \theta \le 2\pi$, and is oriented by increasing θ angle. You must draw the curve C first.

4. Suppose that a thin spiral ramp (helicoid) H, given by the parametric equation $r(u,v) = \langle v\cos u, v\sin u, u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2$, has density function $\rho(x,y,z) = z\sqrt{x^2+y^2}$.

(a) Find the equation of the tangent plane to H at $(x, y, z) = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}, \frac{\pi}{3}\right)$.

(b) Find the mass of H.

Solution:

(a)

$$\frac{\partial r}{\partial u} = (-v\sin u, v\cos u, 1)$$

$$\frac{\partial r}{\partial v} = (\cos u, \sin u, 0)$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (-\sin u, \cos u, -v)$$

$$\left(\frac{\sqrt{3}}{2}, \frac{3}{2}, \frac{\pi}{3}\right) = r\left(\frac{\pi}{3}, \sqrt{3}\right)$$

$$\left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)\Big|_{\left(\frac{\pi}{3}, \sqrt{3}\right)} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}\right)$$

Equation of tangent plane:

$$\left(x - \frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(y - \frac{3}{2}\right)\cdot\frac{1}{2} + \left(z - \frac{\pi}{3}\right)\cdot(-\sqrt{3}) = 0$$

$$\Longleftrightarrow -\frac{\sqrt{3}}{2}x + \frac{y}{2} - \sqrt{3}z + \frac{\pi}{\sqrt{3}} = 0$$

(b) $\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \| = \sqrt{1 + v^2}$

The mass equals

$$\iint_S \rho\, d\sigma = \int_0^{2\pi}\int_0^2 uv\sqrt{1+v^2}\, dv\, du$$

$$= \int_0^{2\pi} u\left[\frac{1}{3}\left(1+v^2\right)^{\frac{3}{2}}\Big|_0^2\right] du$$

$$= \frac{1}{3}\int_0^{2\pi}\left(5^{\frac{3}{2}} - 1\right) u\, du = \frac{2\pi^2}{3}\left(5^{\frac{3}{2}} - 1\right)$$

5. Find the flux of $\mathbf{F}(x, y, z) = \ln\left(x^2 + y^2\right)\mathbf{i} - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)\mathbf{j} + z\sqrt{x^2+y^2}\mathbf{k}$ across the boundary surface Γ of $D = \{(x, y, z) :\ 1 \le x^2 + y^2 \le 2,\ -1 \le z \le 2\}$. Here Γ is oriented by the outward normal to D.