

1. 求下列極值, 如極限不存在, 請說明.

(a) (5分)  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$

(b) (6分)  $\lim_{x \rightarrow -\infty} \sqrt{x^2 - 2x} + x$

2. (a) (8分) 考慮曲線  $y^3 + y^2x + x + y = 0$ . 試求  $\frac{dy}{dx}$  及  $\frac{d^2y}{dx^2}$  在點(0, 0) 的值.

**Solution.**

$$3y^2y' + 2yy'x + y^2 + 1 + y' = 0 \quad \therefore y' = -1$$

$$6y'^2 + 3y^2y'' + 2y'^2x + 2yy'y'' + 2xyy'' + 2yy'y'' + y'' = 0 \quad \therefore 6 + y'' = 0, y'' = -6$$

(b) (7分) 王先生在距離高4公尺之電線桿9公尺處放天燈, 該天燈以  $\frac{1}{3}$  m/sec 之固定速度垂直上升. 當天燈達到距地面16公尺時, 請問電線桿之天燈影以多大速度縮短?

**Solution.**

$$y - 4 = 9 \tan \theta$$

$$y' = 9 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{1}{3} = 9 \left(\frac{5}{3}\right)^2 \cdot \frac{d\theta}{dt}$$

$$\therefore \theta' = \frac{1}{75} \text{ rad/sec}$$

又  $x = 4 \cot \theta$ ,

$$\frac{dx}{dt} = -4 \csc^2 \theta \frac{d\theta}{dt} = -4 \left(\frac{5}{4}\right)^2 \cdot \frac{1}{75} = -\frac{25}{4} \times \frac{1}{75} = -\frac{1}{12} \text{ m/sec.}$$

3. (15分) 試繪  $f(x) = \frac{x}{x^2+9}$  之圖形. 請討論對稱性. 遞增. 遞減. 極值. 凹性. 反曲點. 漸進線.

**Solution.** 奇函數, 過原點.  $y = 0$  為水平漸進線.

$$f'(x) = \frac{9 - x^2}{(x^2 + 9)^2} = 0, x = \pm 3.$$

local min. is  $f(-3) = -\frac{1}{6}$ , local max. is  $f(3) = \frac{1}{6}$ .

$$f''(x) = \frac{2x(x^2 - 27)}{(x^2 + 9)^3} = 0, x = 0, \pm 3\sqrt{3}.$$

4. (10分) 試求  $\frac{d}{dx} \int_x^{\tan x} \frac{1}{u^4+1} du$ .

5. (a) (5分) 證明  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n} = 1$

**Solution.** By definition of integral,

$$\sum_{i=1}^n \sin\left(\frac{i\pi}{2n}\right) \frac{\pi}{2n} \rightarrow \int_0^{\frac{\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{\pi}{2}} = 1 \text{ as } n \rightarrow \infty.$$

(b) (6分) 求  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 - \left(\frac{i}{n}\right)^2} \frac{1}{n}$

**Solution.** 爲  $\sqrt{1-x^2}$  在  $[0, 1]$  之 Riemann sum, 故此極限爲  $\frac{\pi}{4}$ .

6. (a) (4分) 試證當  $0 < x < \frac{\pi}{2}$  時,  $\sin x < x$ .

(b) (7分)  $f(x) = \frac{\tan x}{x}$ , 證明若  $0 < x_1 < x_2 < \frac{\pi}{2}$  時,  $f(x_1) < f(x_2)$ .

**Solution.**  $f'(x) = \frac{x \sec^2 x - \tan x}{x^2}$ .

欲證:  $x \sec^2 x > \tan x$ , 即  $x > \sin x \cos x = \frac{1}{2} \sin 2x$ .

令  $g(x) = x - \frac{1}{2} \sin 2x$ ,  $g(0) = 0$ ,  $g'(x) = 1 - \cos 2x > 0$ .

7. 求下列積分

(a) (6分)  $\int \frac{x^7}{\sqrt{x^4+1}} dx$ .

(b) (6分)  $\int_{-\pi}^{\pi} \frac{x^3 \cos x}{1+x^6} dx$ .

8. (15分) 試求內接於單位圓的最大等腰三角形面積.