93學年度上學期微甲1組期中考考題

1. (25 points)

(a) Find $\dfrac{d}{dx}\sqrt[3]{\sec x+\tan x}$ and $\dfrac{d}{dx}\dfrac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})}$.

(b) An astroid has an equation of the form $x^{2/3}+y^{2/3}=a^{2/3}$, where a is a positive constant. Find $\dfrac{dy}{dx}$ and show that the length of the portion of any tangent line to the astroid cut off by the coordinate axes is constant.

Solution. i. $x^{2/3}+y^{2/3}=a^{2/3}\Rightarrow \dfrac{2}{3}x^{-1/3}+\dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx}=0$.
$\Rightarrow \dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}=-\left(\dfrac{y}{x}\right)^{1/3}$.

ii. Let (b,c) be on the curve, that is, $b^{2/3}+c^{2/3}=a^{2/3}$. At (b,c) the slope of the tangent line is $-\left(\dfrac{c}{b}\right)^{1/3}$ and an equation of the tangent line is

$$y-c=-\left(\frac{c}{b}\right)^{1/3}(x-b) \text{ or } y=-\left(\frac{c}{b}\right)^{1/3}x+(c+b^{2/3}c^{1/3})$$

Setting $y=0$, we find that the x-intercept is

$$b^{1/3}c^{2/3}+b=b^{1/3}(c^{2/3}+b^{2/3}).$$

Setting $x=0$, we find that the y-intercept is

$$c+b^{2/3}c^{1/3}=c^{1/3}(c^{2/3}+b^{2/3}).$$

So the length of the tangent line between these two points is

$$\begin{aligned}&\sqrt{\left[b^{1/3}(c^{2/3}+b^{2/3})\right]^2+\left[c^{1/3}(c^{2/3}+b^{2/3})\right]^2}\\&=\sqrt{b^{2/3}\left(a^{2/3}\right)^2+c^{2/3}\left(a^{2/3}\right)^2}\\&=\sqrt{\left(b^{2/3}+c^{2/3}\right)a^{4/3}}=\sqrt{a^{2/3}a^{4/3}}=\sqrt{a^2}=a=\text{constant}\end{aligned}$$

2. (20 points) Evaluate the following limits.

(a) $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$,

(b) $\displaystyle\lim_{x\to\infty}x\int_{2x}^{4x}\frac{1}{\sqrt[3]{t^6+100}}dt$.

3. (20 points) Let

$$F(x) = \int_{x^3}^{-1} (x^3 - t) f(\sqrt[3]{t}) dt, \qquad x < 0,$$

where $f(x)$ is a continuous function. Suppose $F(x)$ is an antiderivative of x^4, find $F(x)$ and $f(x)$.

4. (15 points) Among the tangent lines of the graph $y = x^2 - \frac{3}{2}$, find the one nearest to the origin.
(在圖形 $y = x^2 - \frac{3}{2}$ 所有的切線之中, 找出一條線與原點的距離最近。)

5. (20 points) Graph the function $\dfrac{(x+1)^3}{(x-1)^2}$ that reveal all the important aspects of the curve such as symmetry, the intervals of increase and decrease, extreme values, intervals of concavity, inflection points, and asymptotes.