Ch11.Ch14 參數式. 極座標

1. [a] 給予函數 $f(x,y)=\begin{cases}\frac{2x^2y}{x^4+y^2} & 若\ (x,y)\neq(0,0)\\ 0 & 若\ (x,y)=(0,0)\end{cases}$

令 $\varphi(t)=(t,at)$, $\psi(t)=(t,t^2)$, a 爲任意非零參數, 問下列能否成立?

(i) $\lim_{t\to0}f(\varphi(t))=\lim_{t\to0}f(\psi(t))$, (ii) f 在 $(0,0)$ 連續.

sol.

(i)

$\lim_{t\to0}f(\varphi(t))=\lim_{t\to0}\frac{2t^2\cdot at}{t^4+a^2t^2}=\lim_{t\to0}\frac{2at}{a^2+t^2}=0$

$\lim_{t\to0}f(\psi(t))=\lim_{t\to0}\frac{2t^2\cdot t^2}{t^4+t^4}=\lim_{t\to0}1=1$

$\lim_{t\to0}f(\varphi(t))\neq\lim_{t\to0}f(\psi(t))$

(ii) 若 f 在 $(0,0)$ 連續, 則 $\lim_{(x,y)\to(0,0}f(x,y)=f(0,0)$, 但由此, 對前述 φ,ψ

$\lim_{t\to0}f(\varphi(t))=0=f(0,0)=\lim_{t\to0}f(\psi(t))=1\Longrightarrow 0=1$, 予盾。故 f 在 $(0,0)$ 不連續。

[b] 給予函數 $f(x,y)=\begin{cases}\frac{|x|}{y^2}e^{-\frac{|x|}{y^2}} & 若\ y\neq0\\ 0 & 若\ y=0\end{cases}$

問極限 $\lim_{(x,y)\to(0,0)}f(x,y)$ 存在否?

sol.

$$f(x,y)=\begin{cases}\frac{|x|}{y^2}e^{-\frac{|x|}{y^2}} & 若\ y\neq 0\\ 0 & 若\ y=0\end{cases}$$

考慮 $\varphi(t)=(t^2,t)$, $\lim_{t\to 0} f(\varphi(t))=\lim_{t\to 0} f(t^2,t)=\lim_{t\to 0}\frac{|t^2|}{t^2}e^{-\frac{|t^2|}{t^2}}=\lim_{t\to 0} e^{-1}=e^{-1}$.

考慮 $\psi(t)=(t^3,t), f(\psi(t))=\lim_{t\to 0}\frac{|t^3|}{t^2}e^{-\frac{|t^3|}{t^2}}$,

依 $t>0, t\neq 0$ 或 $t<0$, $\lim_{t\to 0} f(\psi(t))=\lim_{t\to 0}\pm te^{t}=0$.

至此, $\lim_{t\to 0} f(\varphi(t))\neq \lim_{t\to 0} f(\psi(t))$, 故 $\lim_{(x,y)\to(0,0)} f(x,y)$不存在。

2. 一質點在圓螺旋線 (circular helix) $\overrightarrow{r}(t)$ 運動,

$$\overrightarrow{r}(t)=(a\cos\omega t, a\sin\omega t, b\omega t)$$

[a] 問此點之速率 (speed) 是否爲常數?

sol.

velocity vector$=\frac{d\overrightarrow{r}(t)}{dt}=(-a\omega\sin\omega t, a\omega\cos\omega t, b\omega)$

$|\overrightarrow{r}|=\sqrt{a^2\omega^2\sin^2\omega t+a^2\omega^2\cos^2\omega t+b^2\omega^2}$

speed$=|\overrightarrow{r}|=\sqrt{a^2+b^2}\omega$, constant.

[b] 求此質點之速度向量 (velocity vector) 與 z 軸之交角

sol.

$\overrightarrow{z}$之單位向量 $\overrightarrow{k}=(0,0,1)$, $\overrightarrow{r}$ 與 $\overrightarrow{k}$ 的夾角爲 θ,

則 $\cos\theta=\frac{\overrightarrow{r}\cdot\overrightarrow{k}}{|\overrightarrow{r}||\overrightarrow{k}|}=\frac{b\omega}{(a^2+b^2)^{\frac{1}{2}}\omega}=\frac{b}{(a^2+b^2)^{\frac{1}{2}}}$,

$\theta = \cos^{-1}\left(\frac{b}{(a^2+b^2)^{\frac{1}{2}}}\right)$, 一個常數。

[c] 利用 (b) 或取曲線 $\overrightarrow{r}(t)$ 上的二個特殊點, 解答下列問題: 向量函數 (如 $\overrightarrow{r}(t)$) 能否有單變數實數值函數 $f : [a, b] \rightarrow \mathbb{R}$ 之 Mean Value Theorem ?(假定 f 具有你所要的良好性質)

sol.

以 $\overrightarrow{r}(t) = (a\cos\omega t, a\sin\omega t, b\omega t)$爲例, 取 $t_1 = 0, t_2 = \frac{2\pi}{\omega}$, 則

$\overrightarrow{r}(t_1) = (a, 0, 0)$, $\overrightarrow{r}(t_2) = (a, 0, 2\pi b)$, 由此 $\overrightarrow{r}(t_2) - \overrightarrow{r}(t_1) = (0, 0, 2\pi b)$, 此向量之方向是垂直於 x, y 平面 (向上, 若$b > 0$)

若 $\overrightarrow{r}$ 這個向量函數有類同單變數實數值函數之 Mean Value Theorem, 則有 ξ 介於 t_1與 t_2 之間使得 $\overrightarrow{r}(t_2) - \overrightarrow{r}(t_1) = (t_2 - t_1)\overrightarrow{r}'(\xi)$, 即 $(0, 0, 2\pi b) = \frac{2\pi}{\omega}(-a\omega\sin\omega\xi, a\omega\cos\omega\xi, b\omega)$, 於是 $\sin\omega\xi = 0$, $\cos\omega\xi = 0$, 此爲不可能之事。故 (c) 之答案是否定的。

[d] 求圓螺旋線 (在任何一點) 的曲率

sol.

質點之 unit velocity vector= $\overrightarrow{T} = \frac{\overrightarrow{v}}{|\overrightarrow{v}|}$, 以$s$爲圓螺線之 parameter,

$$\begin{aligned}\overrightarrow{T} &= \frac{d\overrightarrow{r}}{dt} / \frac{ds}{dt} = \frac{d\overrightarrow{r}}{ds} \\ &= \frac{d}{ds}(a\cos\omega t, a\sin\omega t, b\omega t) \\ &= \frac{dt}{ds}\cdot\frac{d}{dt}(-a\omega\sin\omega t, a\omega\cos\omega t, b\omega)\end{aligned}$$

要決定$\frac{dt}{ds}$,

$$\begin{aligned}1 &= (\frac{dt}{ds})^2\cdot(a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t + b^2\omega^2) \\ &= (\frac{dt}{ds})^2\cdot(a^2+b^2)\omega^2\end{aligned}$$

$\frac{dt}{ds} = \frac{1}{\sqrt{a^2+b^2}\omega}$, 由此 $\overrightarrow{T} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin\omega t, a\cos\omega t, b\omega)$

$$\begin{aligned}\frac{d\overrightarrow{T}}{ds} &= \frac{1}{\sqrt{a^2+b^2}\frac{d}{ds}(-a\sin\omega t, a\cos\omega t, b\omega)} \\ &= \frac{1}{\sqrt{a^2+b^2}\frac{dt}{ds}\frac{d}{dt}(-a\sin\omega t, a\cos\omega t, b\omega)}\end{aligned}$$

曲率 $K = \left|\frac{d\overrightarrow{T}}{ds}\right| = \frac{1}{\sqrt{a^2+b^2}}\left|\frac{dt}{ds}\right|\cdot\sqrt{a^2\omega^2\cos^2\omega t + a^2\omega^2\sin^2\omega t}$

至此圓螺旋線形在任何一點之曲率爲 $\frac{a}{a^2+b^2}$.

3. Let $\mathbf{r}(t) = (e^t - t, 2\sqrt{6}e^{\frac{t}{2}}, \sqrt{3}t)$, $-1 \le t \le 1$,

[a] Find the length of the curve;

sol.

$\mathbf{r}'(t) = (e^t - 1, \sqrt{6}e^{\frac{t}{2}}, \sqrt{3})$,
$||\mathbf{r}'(t)|| = ((e^t - 1)^2 + (\sqrt{6}e^{\frac{t}{2}})^2 + (\sqrt{3})^2)^{\frac{1}{2}} = (e^{2t} + 4e^t + 4)^{\frac{1}{2}} = e^t + 2$.
$\mathbf{L} = \int_{-1}^{1} ||\mathbf{r}'(t)||dt = \int_{-1}^{1}(e^y + 2)dt = e - e^{-1} + 4$.

[b] Find the curvature of $\mathbf{r}(t)$ at $t = 0$;

sol.
$\mathbf{r}''(t) = (e^t, \frac{1}{2}\sqrt{6}e^{\frac{t}{2}}, 0)$,
$\mathbf{r}'(0) = (0, \sqrt{6}, \sqrt{3})$, $\mathbf{r}''(0) = (1, \frac{\sqrt{6}}{2}, 0)$, $\mathbf{r}'(0)\times\mathbf{r}''(0) = (\frac{-3\sqrt{2}}{2}, \sqrt{3}, -\sqrt{6})$.
So $k(0) = \frac{||\mathbf{r}'(0)\times\mathbf{r}''(0)||}{||\mathbf{r}'(0)||^3} = \frac{1}{2}$.

4. Let $\mathbf{r}(t) = (t^2, -\sin t + t\cos t, \cos t + t\sin t)$, $t > 0$, find an equation of the osculating plane of the curve $\mathbf{r}(t)$ at the point $(\pi^2, -\pi, -1)$.

sol.
$\mathbf{r}'(t) = (2t, -t\sin t, t\cos t)$,
$||\mathbf{r}'(t)|| = ((2t)^2 + (-t\sin t)^2 + (t\cos t)^2)^{\frac{1}{2}} = \sqrt{5}|t| = \sqrt{5}t$ (by $t > 0$).
$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{1}{\sqrt{5}}(2, -\sin t, \cos t)$,
$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}(0, -\cos t, -\sin t)$, $||\mathbf{T}'(t)|| = \frac{1}{\sqrt{5}}$,
$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = (0, -\cos t, -\sin t)$,
$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{5}}(1, 2\sin t, -2\cos t)$.
So the osculating plane of the curve $\mathbf{r}(t)$ at the point $(\pi^2, -\pi, -1)$ has normal vector $\mathbf{B}(\pi) = \frac{1}{\sqrt{5}(1,0,2)}$, so

an equation is $1(x-\pi^2)+0(y+\pi)+2(z+1)=0$ or $x+2z=\pi^2-2$.

5. Let $f(x,y,z)=\begin{cases}\frac{xy+yz^3}{x^2+z^6} & \text{if } (x,y,z)\neq(0,0,0)\\ 0 & \text{if } (x,y,z)=(0,0,0)\end{cases}$, determine the set of points at which f is continuous.

sol.

The function $f(x,y,z)$ is continuous for $(x,y,z)\neq(0,0,0)$ since it is equal a rational function there.

For $(x,y,z)\to(0,0,0)$ along curve $\mathbf{r}(t)=(kt^3,kt^3,t)$, we see that $f(x,y,z)=f(kt^3,kt^3,t)=\frac{(kt^3)^2+kt^3t^3}{(kt^3)^2+t^6}=\frac{(k^2+k)}{k^2+1}$.

So $f(x,y,z)\to\frac{(k^2+k)}{k^2+1}$ as $(x,y,z)\to(0,0,0)$ along $\mathbf{r}(t)=(kt^3,kt^3,t)$.

Therefore, different path leads to different limit values, hence the limit of $f(x,y,z)$ at $(0,0,0)$ dose not exist, so $f(x,y,z)$ is not continuous at $(0,0,0)$.

Ch15 偏導數

6. $\frac{16}{3}(x^3+y^2)+xyz^2+8z=0$. 求在曲面上 $(-1,-1,0)$ 一點之 z_x, z_y, z_{xx}, z_{xy} 及過此點之切平面。(或只求z_x, z_{xx}, z_{xy}及切平面)

sol.

z_x:

$16x^2 + yz^2 + 2xyzz_x + 8z_x = 0$
$16 + 8z_x = 0, \therefore z_x = -2$
z_y:
$\frac{16}{3}2y + xz^2 + 2xyzz_y + 8z_y = 0$
$\frac{32}{3} = 8z_y \therefore z_y = \frac{4}{3}$
切平面:
$p(x+1) + q(y+1) + (-1) \cdot z = 0$
$-2(x+1) + \frac{4}{3}(y+1) - z = 0$
z_{xx} :
$32x + 2yzz_x + 2yzz_x + 2xyz_x^2 + 2xyzz_{xx} + 8z_{xx} = 0$
$32x + 2xyz_x^2 + 8z_{xx} = 0$
$-32 + 2(4) + 8z_{xx} = 0$
$-24 + 8z_{xx} = 0$
$\therefore z_{xx} = 3$
z_{xy}:
$z^2 + 2yzz_y + 2xzz_x + 2xyz_yz_x + 2xyzz_{xy} + 8_{xy} = 0$
$2z_yz_x + 8z_{xy} = 0$
$\therefore z_{xy} = \frac{-z_xz_y}{4} = -\frac{1}{4}(-2)(\frac{4}{3}) = \frac{2}{3}.$

7. 求 z_x, z_y, 切平面, 並求$(-\frac{9}{10}, -\frac{11}{10})$ 所在之z之估計值。

sol.

$z = 0 + -2(0.1) + \frac{4}{3}(-0.1)$
$= -0.2 - \frac{4}{30}$
$= -\frac{2}{10} - \frac{4}{30} = -\frac{10}{30} = -\frac{1}{3}.$

8. Find the absolute maximum and minimum values of

$f(x,y) = 4x + 6y - x^2 - y^2$ in $x^2 + y^2 \le 1$.

sol.

At a critical point, $f_x = 4 - 2x = 0, f_y = 6 - 2y = 0 \Longrightarrow x = 2, y = 3$. So there are no critical points in $x^2+y^2 < 1$. Need only look at boundary $x^2+y^2 = 1$.

method 1

Use $x = \cos\theta, y = \sin\theta$

$g(\theta) = 4\cos\theta + 6\sin\theta - 1$

$g'(\theta) = -4\sin\theta + 6\cos\theta = 0$ if $\tan\theta = \frac{3}{2}$

$\Longrightarrow \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{9}{4} = \frac{13}{4}$

$\Longrightarrow \cos^2\theta = \frac{4}{13}, \sin^2\theta = 1 - \cos^2\theta = \frac{9}{13}$. So 2 possibilities, $(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})$,$(-\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$. The first gives $\frac{26}{\sqrt{13}} - 1 = 2\sqrt{13} - 1 \longleftarrow$ *maximum.*

The second gives $-2\sqrt{13} - 1 \longleftarrow$ *minimum.*

method 2.

Lagrange multipliers:

$$4 - 2x = 2\lambda x \tag{1}$$
$$6 - 2y = 2\lambda y \tag{2}$$
$$x^2 + y^2 = 1 \tag{3}$$

(1),(2)$\Longrightarrow$ $2(1+\lambda)x = 4,\ 2(1+\lambda)y = 6 \Longrightarrow 3(1+\lambda)x = 2(1+\lambda)y \Longrightarrow x = \frac{2}{3}y$. Then (3)$\Longrightarrow$ $\frac{4}{9}y^2 + y^2 = 1 \Longrightarrow y^2 = \frac{9}{13} \Longrightarrow y = \pm\frac{3}{\sqrt{13}}$. So get $(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})$ or $(-\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$. Finish as in method 1.

9. Find the maximum value of $f(x, y, z) = xy + xz + yz$ subject to the constraints $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z = 1$.

sol.

If the maximum occurs in the interior of the set, then the equation $y + z = \lambda, x + z = \lambda, x + y = \lambda, x + y + z = 1$ must hold for some λ. Then

$$y + z = x + z \Longrightarrow x = y$$
$$x + z = x + y \Longrightarrow y = z$$

So $x = y = z = \frac{1}{3} \Longrightarrow f(x, y, z) = \frac{3}{9} = \frac{1}{3}$.

The other possibility is that the maximum occurs where one of x, y, z is 0.

Suppose $x = 0$. Then $y + z = 1$ and $f(0, y, z) = yz = y(1 - y)$. The maximum of this in $0 \leq y \leq 1$ is $\frac{1}{4}$. Similarly, if $y = 0$ or $z = 0$ we get a maximum of $\frac{1}{4}$. Hence the maximum value must be $\frac{1}{3}$.

10. The plane $2x + y + z = 10$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the highest point on the ellipse.

sol.

We want to maximize $f(x, y, z) = z$ subject to $x^2 + y^2 - z = 0$, $2x + y + z - 10 = 0$.

By the method of Lagrange multipliers, we get

$$0 = \lambda 2x + 2\mu \tag{1}$$
$$0 = \lambda 2y + \mu \tag{2}$$
$$1 = -\lambda + \mu \tag{3}$$
$$x^2 + y^2 - z = 0 \tag{4}$$
$$2x + y + z - 10 = 0 \tag{5}$$

(1),(2)$\Longrightarrow \lambda x = 2\lambda y \Longrightarrow \lambda = 0$ or $x = 2y$.
If $\lambda = 0$, then (1) $\Longrightarrow \mu = 0$ which contradicts (3).
So $x = 2y$. Then (4)$\Longrightarrow z = x^2 + y^2 = 5y^2$.
(5)$\Longrightarrow 4y + y + 5y^2 - 10 = 0 \Longrightarrow y^2 + y - 2 = 0 \Longrightarrow$
$(y+2)(y-1) = 0 \Longrightarrow y = -2, 1$.
$y = -2 \Longrightarrow (x, y, z) = (-4, -2, 20) \longleftarrow$ *highest point*.
$y = 1 \Longrightarrow (x, y, z) = (2, 1, 5) \longleftarrow$ *lowest point*.

11. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at he point $(1, 1, 2)$.

sol.

Let $f(x, y) = 3x^2 + 2y^2 + z^2 - 9$ and $g(x) = 3x^2 + 2y^2 + z^2 - 9$. Then $f(1, 1, 2) = g(1, 1, 2) = 0$ and the ellipsoid and the sphere are the level surfaces of f and g. Thus $\nabla f(1, 1, 2)$ and $\nabla g(1, 1, 2)$ are orthogonal to the ellipsoid and the sphere $(1, 1, 2)$.
$\nabla f(x, y, z) = \langle 6x, 4y, 2z \rangle$ and $\nabla g(x, y, z) = \langle 2x -$

$8, 2y-6, 2z-8\rangle$. Hence $\nabla f(1,1,2) = \langle 6,4,4\rangle$ is parallel $\nabla g(1,1,2) = \langle -6,-4,-4\rangle$. This implies that the ellipsoid $3x^2+2y^2+z^2=9$ and the sphere $x^2+y^2+z^2-8x-6y-8z+24=0$ are tengent to each other at he point $(1,1,2)$.

12. If $f(x,y)=0$ define y as a function of x, show that

$$\frac{d^2y}{dx^2} = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}$$

sol.

Differentiate $f(x,y)=0$ w.r.t x. We have

$$f_x + f_y\frac{dy}{dx} = 0 \tag{1}$$

Hence if $f_y \neq 0$, $\frac{dy}{dx} = \frac{-f_x}{f_y}$. Differentiate (1) w.r.t x. We have

$$f_{xx} + f_{xy}\frac{dy}{dx} + f_{yx}\frac{dy}{dx} + f_{yy}(\frac{dy}{dx})^2 + f_y\frac{d^2y}{dx^2} = 0$$

We assume that f_{xy} and f_{yx} are both continuous, and hence $f_{xy} = f_{yx}$. Also, from the equation (1), we have

$$f_{xx} + f_{xy}\frac{-f_x}{f_y} + f_{yx}\frac{-f_x}{f_y} + f_{yy}(\frac{-f_x}{f_y})^2 + f_y\frac{d^2y}{dx^2} = 0$$

Thus

$$\frac{d^2y}{dx^2} = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}$$

13. Let $f(x, y) = x^2(x^2+y^2)^{-\frac{1}{2}}e^{\sin(x^2y)}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.
(1) Let $\overrightarrow{u} = \langle\cos\theta, \sin\theta\rangle$ and find $D_{\overrightarrow{u}}f(0, 0)$.
(2) Prove that f is continuous at $(0, 0)$.
sol.

$$\begin{aligned} D_{\overrightarrow{u}}f(0,0) &= \lim_{h\to 0}\frac{f(h\cos\theta, h\sin\theta)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{h(\cos\theta)^2e^{\sin(h^3\cos^2\theta\sin\theta)}}{h} = (\cos\theta)^2 \end{aligned}$$

$|x^2(x^2+y^2)^{-\frac{1}{2}}e^{\sin(x^2y)}| \leq |x|e^1$.
Since $\lim\limits_{(x,y)\to(0,0)}|x| = 0$, it follows that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$.

Ch16 重積分

14. $D = \{(x, y)|x^2+y^2 \leq 4\}$, 試求 $\iint_D \sqrt{4-x^2-y^2}dA$.
sol.

令 $u = 4 - r^2$

$$原式 = \int_0^{2\pi}\int_0^2 \sqrt{4-r^2} r dr d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^4 \sqrt{u} du d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2}{3}u^{\frac{3}{2}}\Big|_0^4 d\theta$$
$$= \frac{8}{3}\int_0^{2\pi} d\theta = \frac{16}{3}\pi$$

15. 求$\int_0^1\int_x^1 \cos(y^2) dy dx.$

sol.

變換積分順序,

$$原式 = \int_0^1\int_0^4 \cos y^2 dx dy$$
$$= \int_0^1 y\cos y^2 dy$$
$$= \frac{\sin y^2}{2}\Big|_0^1$$
$$= \frac{\sin 1}{2}$$

16. 求$\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} dy dx.$

sol.

令 $u = x^2 + y^2 + 1$

$$\begin{aligned}
原式 &= \int_0^1 \frac{1}{2}x \int_{1+x^2}^{2+x^2} \frac{du}{\sqrt{u}}dx \\
&= \int_0^1 \frac{x}{2}\sqrt{2+x^2}dx - \int_0^1 \frac{x}{2}\sqrt{1+x^2}dx \\
&= \frac{1}{3}(2+x^2)^{\frac{3}{2}}\Big|_0^1 - \frac{1}{3}(1+x^2)^{\frac{3}{2}}\Big|_0^1 \\
&= \frac{1}{3}[3\sqrt{3} - 2\sqrt{2} - 2\sqrt{2} + 1] \\
&= \frac{1}{3}[3\sqrt{3} - 4\sqrt{2} + 1]
\end{aligned}$$

17. 試求 $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy$

sol.

$x = \sqrt{y} \Longrightarrow y = x^2$

$$
\begin{aligned}
\text{原式} &= \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dydx \\
&= \int_0^1 \frac{x^4 e^{x^2}}{2x^3} dx \\
&= \frac{1}{2} \int_0^1 xe^{x^2} dx \\
&= \frac{1}{4} e^{x^2} \Big|_0^1 \\
&= \frac{1}{4}(e-1)
\end{aligned}
$$

18. (a) 試求 $\iint_D \frac{1}{(x^2+y^2)^{\frac{n}{2}}} dA$, n 爲整數. $D : \{(x,y), a \leq \sqrt{x^2+y^2} \leq b\}$

sol.

$$
\begin{aligned}
\text{原式} &= \int_0^{2\pi} \int_a^b \frac{1}{r^n} r dr d\theta \\
&= 2\pi \int_a^b \frac{1}{r^{n-1}} dr
\end{aligned}
$$

(b) 當 $a \longrightarrow 0^+$, 試求n值使積分的極限存在。

sol.

(缺)

19. 試求曲面 $z = \sqrt{x^2+y^2}$ 與柱面 $x^2+y^2 = 2x$ 及 xy 平面所圍之體積。

sol.

$(x-1)^2 + y^2 = 1$, $r = 2\cos\theta$

$$\text{原式} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} rrdrd\theta$$
$$= 2\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 drd\theta$$

20. $\rho(x,y) = xy^2$, $D = \{(r,\theta), 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\}$,
試求D的質心。

sol.

(缺)

21. Consider the iterated integral $\int_0^8 \left(\int_{y^{\frac{1}{3}}}^2 \sqrt{1+x^4}dx \right) dy$.
Sketch the region of integration and then evaluate it by reversing the order of integration.

sol.

$\int_0^2 dx \int_0^{x^3} dy\sqrt{1+x^4} = \int_0^2 x^3\sqrt{1+x^4}dx = \frac{2}{3}\cdot\frac{1}{4}(1+x^4)^{\frac{3}{2}}\Big|_0^2 = \frac{1}{6}\{17^{\frac{3}{2}} - 1\}$

22. Consider the iterated integral $\int_0^2 \left(\int_{y^2}^4 y\sin(x^2)dx \right) dy$.
Sketch the region of integration and then evaluate it by reversing the order of integration.

sol.

$\int_0^4 dx \int_0^{\sqrt{x}} dy y \sin(x^2) = \int_0^4 \frac{x}{2} \sin(x^2) dx = \frac{1}{2}\frac{1}{2}(-\cos(x^2))|_0^4 = \frac{1}{4}(1 - \cos 16)$

23. By making an appropriate change of variables to evaluate the integral $\iint_{\mathbb{R}} \frac{dA}{1+9x^2+9y^2}$, where $\mathbb{R}$ is the region bounded by the ellipse $9x^2 + 9y^2 = 4$.

sol.

$x = r\cos\theta, y = r\sin\theta$

$\int_0^{2\pi} d\theta \int_0^{\frac{2}{3}} dr \frac{r}{1+9r^2} = 2\pi \frac{\ln(1+9r^2)}{18}\Big|_0^{\frac{2}{3}} = \frac{\pi}{9}\ln 5$