1. [a](3分) Graph the curves $r = 3\sin\theta$ and $r = 1 + \sin\theta$.
sol.
(None.)

[b](7分) Find the area of the region that lies inside the curve $r = 3\sin\theta$ and outside the curve $r = 1 + \sin\theta$.
sol.
Two curves intersect at

$$\begin{aligned} & 3\sin\theta = 1 + \sin\theta \\ & \Rightarrow 2\sin\theta = 1 \\ & \Rightarrow \sin\theta = \frac{1}{2} \\ & \Rightarrow \theta = \frac{\pi}{6} \text{ and } \frac{5\pi}{6} \end{aligned}$$

Thus

$$
\begin{aligned}
A &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}[(3\sin\theta)^2 - (1+\sin\theta)^2]d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta)d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(8\sin^2\theta - 1 - 2\sin\theta)d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [8(\frac{1-\cos 2\theta}{2}) - 1 - 2\sin\theta]d\theta \\
&= \frac{1}{2}(3\theta - 2\sin(2\theta) + 2\cos\theta)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
&= \frac{1}{2}3(\frac{4\pi}{6}) - \sin(\frac{5\pi}{3}) + \sin(\frac{\pi}{3}) + \cos(\frac{5\pi}{3}) - \sin(\frac{\pi}{3}) \\
&= \pi + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \\
&= \pi
\end{aligned}
$$

2. (15分) For the curve given by $\overrightarrow{r}(t) = (t^{-1}, 2\ln t, 2t)$.Find

[a] the unit tangent vector $\overrightarrow{T}$

sol.

$\overrightarrow{r}'(t) = (-t^2, \frac{2}{t}, 2)$, $|\overrightarrow{r}'(t)| = \sqrt{(-t^2)^2 + (\frac{2}{t})^2 + (2)^2} = t^{-2} + 2$. Thus $\overrightarrow{T} = \frac{\overrightarrow{r}(t)}{|\overrightarrow{r}(t)|} = \frac{(-t^{-2}, \frac{2}{t}, 2)}{t^{-2}+2} = \frac{(-1, 2t, 2t^2)}{1+2t^2}$.

[b] the unit normal vector $\overrightarrow{N}$

sol.

$$
\begin{aligned}
\overrightarrow{T}' &= \frac{(1+2t^2)(0,2,4t) - 4t(-1,2t,2t^2)}{(1+2t^2)^2} \\
&= \frac{(4t, 2-4t^2, 4t)}{(1+2t^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
|\overrightarrow{T}'| &= \frac{\sqrt{(4t)^2 + (2-4t^2)^2 + (4t)^2}}{(1+2t^2)^2} \\
&= \frac{\sqrt{4 + 16t^2 + 16t^4}}{(1+2t^2)^2} \\
&= \frac{2\sqrt{(1+2t^2)^2}}{(1+2t^2)^2} \\
&= \frac{2}{1+2t^2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\overrightarrow{N} &= \frac{\overrightarrow{T}'}{|\overrightarrow{T}'|} \\
&= \frac{(4t, 2-4t^2, 4t)}{2(1+2t^2)}
\end{aligned}
$$

[c] the curvature K

sol.

$$
\begin{aligned}
\overrightarrow{K} &= \frac{|\overrightarrow{T}'|}{|\overrightarrow{r}'|} \\
&= \frac{2/(1+2t^2)}{t^{-2}+2} \\
&= \frac{2t^2}{(1+2t^2)^2}
\end{aligned}
$$

[d] the osculating plane at the point $P(1, 0, 2)$

sol.

$\overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N}|_{t=1} \Rightarrow$

$\overrightarrow{B} = \begin{vmatrix} i & j & k \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}).$

Then the osculating plane is

$$\begin{aligned}&\overrightarrow{B}\cdot(x-1,y-0,z-2)=0\\&\Rightarrow 2(x-1)+2y-(z-2)\\&\Rightarrow 2x-2+2y-z+2=0\\&\Rightarrow z=2x+2y\end{aligned}$$

3. (10分) 定義 $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & 若\ (x,y)\neq(0,0)\\ 0 & 若\ (x,y)=(0,0)\end{cases}$

[a] $f(x,y)$ 在平面上那些點是連續的?

sol.

All points on the plane.

[b] 求所有方向 $\overrightarrow{u}=\alpha\overrightarrow{i}+\beta\overrightarrow{j}$, $\alpha^2+\beta^2=1$, 使得方向導數 $D_{\overrightarrow{u}}f(0,0)$ 存在。問 $D_{\overrightarrow{u}}f(0,0)=f_x(0,0)\alpha+f_y(0,0)\beta$?

sol.

All $\overrightarrow{u}$ with $D_{\overrightarrow{u}}f(0,0)=\alpha^2\beta$. No!

[c] $f(x,y)$ 在 $(0,0)$ 點可微分否?

sol.

Not.

4. (15分) 求 $f(x,y)=(\frac{1}{2}-x^2+y^2)e^{1-(x^2+y^2)}$ 的所有 critical points 並加以分類。並求 $f(x,y)$ 在 $x^2+y^2\leq 1$ 的最大最小值。

sol.

saddle point: $(0,0)$

local maximum points: $(0,\sqrt{\frac{1}{2}})$, $(0,-\sqrt{\frac{1}{2}})$

local minimum points: $(\sqrt{\frac{3}{2}},0)$, $(-\sqrt{\frac{3}{2}},0)$

max $=\sqrt{e}$, min $=-\frac{1}{2}$.

5. (10分) 有一金屬平板上的溫度分佈是 $T(x,y)=-e^{-2y}\cos x$. 有一昆蟲在平板上爬行, 其在每一點的運動方向是沿著溫度增加最大的方向前進。若此昆蟲通過點 $(x=\frac{\pi}{4},y=0)$, 求一函數 $f(x)$ 使得圖形 $y=f(x)$ 恰是此昆蟲的運動軌跡。

sol.

$f'(x)=\frac{T_y}{T_x}=2\frac{\cos x}{\sin x}$, so $f(x)=2\ln(\sqrt{2}\sin x)$.

6. (10分) 給 $w=x^2y^2+yz-z^3, x^2+y^2+z^2=6$, 求 $(\frac{\partial w}{\partial y})_z$ 及 $\frac{\partial^2 w}{\partial y\partial z}$ 在 $(w,x,y,z)=(4,2,1,-1)$ 的值。

sol.
$(\frac{\partial w}{\partial y})_z = 5$, $\frac{\partial^2 w}{\partial y \partial z} = 5$.

7. (10分) Let $f(x) = \int_1^x e^{y^2} dy$. Find the average valus of f on the interval $[0, 1]$.
sol.

$$
\begin{aligned}
\overline{f} &= \frac{1}{1-0}\int_0^1 f(x)dx \\
&= \int_0^1 \int_1^x e^{y^2} dydx \\
&= -\int_0^1 \int_x^1 e^{y^2} dydx \\
&= -\int_0^1 \int_0^y e^{y^2} dxdy \text{ (by Fubini Thm)} \\
&= -\int_0^1 ye^{y^2} dy \\
&= -\frac{1}{2}e^{y^2}\Big|_0^1 \\
&= \frac{1}{2}(1-e)
\end{aligned}
$$

8. (10分) Let D be the lamina enclosed by x-axis, $y = \sin x$, and $0 \le x \le \pi$. The density at (x, y) is given by $\rho(x, y) = y$. Find the coordinates of the center of mass and the moment of inertial about the y-axis.
sol.
由對稱性知 $\overline{x} = \frac{\pi}{2}$.
$m = \int_0^\pi \int_0^{\sin x} ydydx = \frac{\pi}{4}$.
$M_x = \int_0^\pi \int_0^{\sin x} y^2 dydx = \frac{4}{9}$.

$\therefore \overline{y} = \frac{M_x}{m} = \frac{16}{9\pi}$

$$
\begin{aligned}
I_y &= \int_0^{\pi} \int_0^{\sin x} x^2 y dy dx \\
&= \int_0^{\pi} \frac{1}{2} x^2 \sin^2 x dx \\
&= \frac{1}{2} \int_0^{\pi} x^2 \cdot \frac{1-\cos 2x}{2} dx \\
&= \frac{1}{4} \int_0^{\pi} x^2 dx - \frac{1}{4} \int_0^{\pi} x^2 \cos 2x dx \\
&= \frac{\pi^3}{12} - \frac{1}{8} \int_0^{\pi} x^2 d(\sin 2x)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\pi} x^2 d(\sin 2x) &= (x^2 \sin 2x)|_0^{\pi} - \int_0^{\pi} (\sin 2x) 2x dx \\
&= 0 + \int_0^{\pi} x d \cos 2x \\
&= (x \cos 2x)|_0^{\pi} - \int_0^{\pi} (\cos 2x) dx \\
&= \pi - (\frac{1}{2} \sin 2x)|_0^{\pi} \\
&= \pi - 0 = \pi
\end{aligned}
$$

$I_y = \frac{\pi^3}{12} - \frac{1}{8}\pi.$

9. $(10$分$)$ 找圓柱體 $x^2+y^2 \le \psi$ 與橢球體 $4x^2+4y^2+z^2 \le b\psi$ 共同部分的體積。

sol.

$2 = \pm\sqrt{b\psi - 4x^2 - 4y^2}$, $D = \{(x, y) : x^2 + y^2 \leq \psi\}$

$$\begin{aligned}
V &= \iint_D \sqrt{b\psi - 4x^2 - 4y^2} - (-\sqrt{b\psi - 4x^2 - 4y^2})dA \\
&= 2\iint_D \sqrt{b\psi - 4x^2 - 4y^2}dA \\
&= 2\int_0^{2\pi}\int_0^2 2\sqrt{1b - r^2}rdrd\theta \\
&= \psi\int_0^{2\pi}\int_0^2 r(1b - r^2)^{\frac{1}{2}}drd\theta \\
&= \psi \cdot 2\pi \cdot \left[-\frac{1}{3}(1b - r^2)^{\frac{3}{2}}\right]\Big|_0^2 \\
&= \frac{8\pi}{3}(b\psi - 24\sqrt{3})
\end{aligned}$$

10. (10分) Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$.

sol.

$f(x, y) = xyz$, $9x^2 + 36y^2 + 4z^2 = 36, 18x + 8zz_x = 0$ and $72y + 8zz_y = 0$.

$f_x = yz + xyz_x$, $f_y = xz + xyz_y$

$f_x = 0 \Rightarrow yz^2 + xyzz_x = 0 \Rightarrow y\frac{36-9x^2-36y^2}{4} + xy\frac{(-18x)}{8} = 0$

$f_y = 0 \Rightarrow xz^2 + xyzz_y = 0 \Rightarrow x\frac{36-9x^2-36y^2}{4} + xy\frac{(-72x)}{8} = 0$

$\Rightarrow x^2 + 2y^2 = 2$, $x^2 + 8y^2 = 4$

$\Rightarrow x^2 = \frac{4}{3}$, $y^2 = \frac{1}{3}$, $z^2 = \frac{9}{3} \Rightarrow xyz = \frac{2}{\sqrt{3}}$.