1. (12%) Solve the differential equation $(x^2+1)\frac{dy}{dx}+4xy=x$ with the initial condition $y(2)=1$.

2. (12%) (a) Solve the differential equation $\frac{dy}{dt}=\lambda y(y-1)$, $0<y<1$ and $\lambda>0$ is a constant with the initial condition $y(0)=\frac{1}{2}$.
   (b) Evaluate $\lim_{t\to\infty} y(t)$.

3. (12%) Evaluate $\int_{-\infty}^{\infty} e^{-x^2+4x}dx$. (You can use $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$.)

4. (12%) In a Poisson process, $P(k,t)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$ indicates the probability of k occurrences of a specific event in the time interval $[0,t]$. Let W denote the time of the second occurrence (counted from the beginning of the process).
   (a) Find $P(W>t)$.
   (b) Find the probability density function $f_W(t)$ of W.

5. (16%) Let X be a random variable with the probability density function $f(x)=\frac{A}{x^2}$, $1\le x\le e$, where A is a constant.
   (a) (6%) Find A.
   (b) (5%) Find $\mathrm{E}(X)$.
   (c) (5%) Find $\mathrm{Var}(X)$.

6. (16%) Rolling a fair dice, we define two random variables

$$X=\begin{cases}1 & \text{if the outcome is even}\\ 0 & \text{if the outcome is odd}\end{cases}, \qquad Y=\begin{cases}1 & \text{if the outcome is in } \{1,2,3\},\\ 0 & \text{if the outcome is in } \{4,5,6\}.\end{cases}$$

   Let $Z=X+Y$
   (a) (6%) Are X and Y independent?
   (b) (5%) Find $\mathrm{E}(Z)$.
   (c) (5%) Find $\mathrm{Var}(Z)$.

7. (10%) Let X, Y be two independent random variables both with the probability density $f(t)=\lambda e^{-\lambda t}$, $t\ge 0$. Find the probability density function $f_Z(t)$ of the random variable $Z=X+Y$.

8. (10%) Let X be the random variable with the probability density function $f_X(t)=\frac{1}{\sqrt{\pi}}e^{-t^2}$. Find the probability density function $f_W(t)$ of the random variable $W=X^2$.