992微甲08-12班期中考解答和評分標準

1. (8% total) (a) Write down the general terms of the Maclaurin series of $f(x) = \ln\sqrt{1+x^2}$. (5%)

   (b) Find the values of $f^{(10)}(0)$. (3%)

   Sol:

   (a) $f(x) = fracx^2 2 - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^10}{10} \cdots$

   (b) $f(x) = \frac{1}{2}\ln(1+x^2) = \frac{1}{2}\Big\{\frac{x^2}{1} - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - ...\Big\}$, so $f^{(10)}(0) = 10! \cdot \frac{1}{10} = 9!$.

   評分標準:

   (a) 小題5分，(b) 小題3分，基本上要全對才給分 。

   若 (a) 小題錯，則 (b) 小題0分 (唯一的例外是真的把函數微分10次) 。

   可以背 $\ln(1+x)$ 的泰勒展開式。

   若 (b) 小題中忘記 10!，扣2分。

   若寫出正確答案，但答案紙上沒有關鍵步驟的計算，而充滿無謂的計算，零分 。(有異議者可以找老師或助教討論)

2. (10%) Find the curvature of the curve $\mathbf{r}(t) = 3t\mathbf{i} + \sin(3t)\mathbf{j} + \cos(3t)\mathbf{k}$ at $t = \pi$.

   Sol:
   $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{1}{2}$. Note that $\kappa$ is a scalar, not a vector.

   Or one can use another formulae: $\mathbf{T} = \frac{r'}{|\mathbf{r}'|}$, $\kappa = \frac{\mathbf{T}'}{|\mathbf{r}'|}$.

   用正確的公式可得5分，答案對再給5分 。

   使用第二組公式必須兩個都對才給分。

3. (10%) Find the first three terms of the Maclaurin series of $\big((1-x^2)(1+x)\big)^{-\frac{1}{4}}$.

   Sol:

Using the binomial theorem,

$$
\begin{aligned}
(1-x^2)^{\frac{-1}{4}} &= 1+\frac{1}{4}x^2 .....(3 \text{ pts}),\\
(1+x)^{\frac{-1}{4}} &= 1-\frac{1}{4}x+\frac{5}{32}x^2 .....(3 \text{ pts}),\\
\to ((1-x^2)(1+x))^{\frac{-1}{4}} &= 1-\frac{1}{4}x+\frac{13}{32}x^2 ....(4 \text{ pts}).
\end{aligned}
$$

or

$$
\begin{aligned}
((1-x^2)(1+x))^{\frac{-1}{4}} &= (1+x-x^2-x^3)^{\frac{-1}{4}}\\
&= 1-\frac{1}{4}(x-x^2-x^3)+\frac{5}{32}(x-x^2-x^3)^2 .....(6 \text{ pts}),\\
\to ((1-x^2)(1+x))^{\frac{-1}{4}} &= 1-\frac{1}{4}x+\frac{5}{32}x^2 .....(4 \text{ pts}).
\end{aligned}
$$

4. (10%) Find the tangent plane to the surface $x^2+y^2+z^2=6xyz-3$ at the point $(-1,1,-1)$.

Sol:

Set $f(x,y,z)=x^2+y^2+z^2-6xyz+3$,

$\nabla f(x,y,z)=(2x-6yz, 2y-6xz, 2z-6xy)$ (6 pts).

$\nabla f(-1,1,-1)=(4,-4,4)$ (2 pts).

The tangent plane is

$$4(x+1)-4(y-1)+4(z+1)=0, \to x-y+z=-3 \quad (2 \text{ pts}).$$

5. (10% total) Determine the convergence of the following series (5% each)

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$.

(b) $\displaystyle\sum_{n=1}^{\infty}\sqrt{\sin\frac{1}{n^3}}$.

Sol:

(a)

$$a_n=\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}.\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{(\sqrt{n+1}+\sqrt{n})^2}=\frac{1}{2n+1+\sqrt{n(n+1)}}$$

The above equation is $\boxed{\text{2 points}}$

By Comparison Test : $a_n > b_n$

$$a_n = \frac{1}{2n+1+\sqrt{n(n+1)}} > b_n = \frac{1}{4n+3}$$

The series $\sum_{n=0}^{\infty} b_n$ is *divergent* and for all n , then $\sum_{n=0}^{\infty} a_n$ is also divergent.

Remark:

1.The rationalizing denominators of question (a) get **zero points**.

2.The answers are correct, but the derivative process is incorrect, they get **zero points**.

(b) We use the Limit Comparison Test withm$a_n = \sqrt{sin(\frac{1}{n^3})}$ , $b_n = \sqrt{\frac{1}{n^3}}$

and obtain

$$\lim_{x\to\infty} \frac{a_n}{b_n} = \lim_{x\to\infty} \frac{\sqrt{sin(\frac{1}{n^3})}}{\sqrt{\frac{1}{n^3}}} = \lim_{x\to\infty} \sqrt{\frac{sin(\frac{1}{n^3})}{\frac{1}{n^3}}} = 1 > 0$$

The series $\sum_{n=0}^{\infty} b_n$ is *convergent* because it is a $p-series$ with p=3/2.

Since this limit exists and $\sum_{n=0}^{\infty} b_n$ is a convergent series, the given series converges by the Limit Comparison Test.

Remark:

1.The answers are correct, but the derivative process is incorrect, they get **zero points**.

2.If $\sum_{n=0}^{\infty} a_n = 0$,we cant not conclude that $\sum_{n=0}^{\infty} a_n$ is convergent.

6. (10%) Given $a > 1$, let $C_a$ be the curve $y = x^a$, $x > 0$. Find the limit of $\frac{xy}{x^3-y^3}$ as $(x, y)$ tends towards $(0, 0)$ along the curve $C_a$. *Warning!* The limit asked depends on $a$.

Sol:

Let $y = x^a$,, then $\frac{xy}{x^3-y^3} = \frac{x^{a+1}}{x^3-x^{3a}}$. (2 pts)

(i) If $a+1 = 3$ then $a = 2$, $\lim_{x\to 0} \frac{x^{a+1}}{x^3-x^{3a}} = \lim_{x\to 0} \frac{x^3}{x^3-x^6} = 1$. (2 pts)

(ii) If $a+1>3$ then $a>2$, $\lim_{x\to 0}\frac{x^{a+1}}{x^3-x^{3a}}=\lim_{x\to 0}\frac{x^{a-2}}{1-x^{3a-3}}=0.$

Since when $x\to 0$, $x^{a-2}\to 0$ and $x^{3a-3}\to 0$. (3 pts)

(iii) If $a+1<3$ then $a<2$, $\lim_{x\to 0}\frac{x^{a+1}}{x^3-x^{3a}}=\lim_{x\to 0}\frac{1}{x^{2-a}-x^{2a-1}}\to\infty.$

Since $2-a>0$ and $2a-1>0$, $x^{2-a}-x^{2a-1}\to 0$ as $x\to 0$. (3 pts)

Hence, if $y=x^a$ the limit is $\begin{cases}\text{doesn't exist}, & \text{if } 1<a<2.\\ 1, & \text{if } a=2.\\ 0, & \text{if } a>2.\end{cases}$

7. (10%) Let $z=e^{\frac{1}{x^2+1}}\cos\pi y$.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x=0$ and $y=\frac{1}{2}$.

(b) Let $x=\frac{u}{u^2+v^3+1}$ and $y=\frac{v}{u+v^4+1}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $u=0$ and $v=1$.

Sol:

(a) $\frac{\partial z}{\partial x}=e^{\frac{1}{x^2+1}}\frac{-2x}{(x^2+1)^2}\cos\pi y=0$ as $x=0, y=\frac{1}{2}$ (2%)

$\frac{\partial z}{\partial y}=-\pi e^{\frac{1}{x^2+1}}\sin\pi y=-\pi e$ as $x=0, y=\frac{1}{2}$ (2%)

(b) $x=0, y=\frac{1}{2}$ as $u=0, v=1$

$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$, $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

$\frac{\partial x}{\partial u}=\frac{-u^2+v^3+1}{(u^2+v^3+1)^2}=\frac{1}{2}$ as $u=0, v=1$ (1%)

$\frac{\partial y}{\partial u}=\frac{-v}{(u+v^4+1)^2}=-\frac{1}{4}$ as $u=0, v=1$ (1%)

$\frac{\partial x}{\partial v}=\frac{-3uv^2}{(u^2+v^3+1)^2}=0$ as $u=0, v=1$ (1%)

$\frac{\partial y}{\partial v}=\frac{u-3v^4+1}{(u+v^4+1)^2}=-\frac{1}{2}$ as $u=0, v=1$ (1%)

As $u=0, v=1$

$\frac{\partial z}{\partial u}=-e\pi(-\frac{1}{4})=\frac{e\pi}{4}$ (1%)

$\frac{\partial z}{\partial v}=-e\pi(-\frac{1}{2})=\frac{e\pi}{2}$ (1%)

8. (10%) Find the critical points of $f(x,y) = x^3 + y^2 - 2xy + 7x - 8y + 2$. Which of them give rise to maximum values? Minimum values? Saddle points?

Sol:

Let $\nabla f(x,y) = (0,0)$, then

$$\begin{cases} f_x(x,y) & = 3x^2 - 2y + 7 & = 0, \\ f_y(x,y) & = 2y - 2x - 8 & = 0. \end{cases} \qquad (3\%)$$

So we can get $(-\frac{1}{3}, \frac{11}{3})$ and $(1,5)$ are critical points. (2%)

Since $f_{xx}(x,y) = 6x,\ f_{xy}(x,y) = f_{yx}(x,y) = -2,\ f_{yy}(x,y) = 2$ and

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -2 \\ -2 & 2 \end{vmatrix} = 12x - 4 = \begin{cases} < 0, & x = -\frac{1}{3} \\ > 0, & x = 1 \end{cases} \qquad (3\%)$$

Moreover, $f_{xx}(1,5) = 6 > 0$, $(-\frac{1}{3}, \frac{11}{3})$ and $f(1,5)$ are saddle point and local minimum value, respectively (2%).

9. (10%) In what direction does $f(x,y,z) = \frac{z^2}{xy}$ attain the maximum rate of change at the point $(1,2,4)$? What is this maximum rate of change?

Sol:

(1) $(f_x, f_y, f_z) = (\frac{-z^2}{x^2y}, \frac{-z^2}{xy^2}, \frac{2z}{xy})$. (4 pts)

(2) $\nabla f\Big|_{(1,2,4)} = (-8,-4,4)$, therefore the direction is $t(-2,-1,1)$, $t > 0$. (4 pts)

(3) The maximun rate of change is $\sqrt{8^2 + 4^2 + (-4)^2} = 4\sqrt{6}$. (2 pts)

10. (12%) The temperature at the point $(x,y,z)$ is given by $T(x,y,z) = 6xy + 8xz$.

Use Lagrange multipliers to find the highest temperature and lowest temperature on the sphere $x^2 + y^2 + z^2 = 50$.

Sol:

Part 1 (3 pts)

$T(x,y,z) = 6xy + 8xz,\ S(x,y,z) = x^2 + y^2 + z^2 - 50$

$\nabla T = (6y + 8z, 6x, 8x), \nabla S = (2x, 2y, 2z)$

$\nabla T = \lambda \nabla S$

Part 2 (1 pt)

$6y + 8z = \lambda 2x - (1)$

$6x = \lambda 2y - (2)$

$8x = \lambda 2z - (3)$

$x^2 + y^2 + z^2 = 50 - (4)$

Part 3 (4 pts)

If $\lambda = 0$, then $(x, y, z) = (0, 4\sqrt{2}, -3\sqrt{2})$ or $(0, -4\sqrt{2}, 3\sqrt{2})$

If $\lambda \neq 0$, by (2),(3), $y = \dfrac{3x}{\lambda}, z = \dfrac{4x}{\lambda}$

If $x = 0$, then $(x, y, z) = (0, 0, 0)$, by (4),$\rightarrow\leftarrow$

If$x \neq 0$, by(1), $\lambda = 5$ or $-5$

Part 4 (2 pts)

If $\lambda = 5$, by (4), $x = 5$ or $-5$ then $(x, y, z) = (5, 3, 4)$ or $(-5, -3, -4)$

If $\lambda = -5$, by (4), $x = 5$ or $-5$ then $(x, y, z) = (5, -3, -4)$ or $(-5, 3, 4)$

Part 5 (2 pts)

Since $T(x, y, z) = 0$ if $(x, y, z) = (0, 4\sqrt{2}, -3\sqrt{2}), (0, -4\sqrt{2}, 3\sqrt{2})$,

$T(x, y, z) = 250$ if $(x, y, z) = (5, 3, 4), (-5, -3, -4)$,

$T(x, y, z) = -250$ if $(x, y, z) = (-5, -3, -4), (-5, 3, 4)$,

then $T(x, y, z)$ has maximum 250 and minimum $-250$.