992微甲08-12班期末考解答和評分標準

1. (10%) Find the total electric charge over the region

$$R = \{(x,y):\ \ -1 \le x+y \le 1 \text{ and } -1 \le x-y \le 1\}$$

with charge density (per unit area) $\rho(x,y) = |x| + |y|$. (*Hint:* Use symmetry.)

Sol:

By symmetry,

$$\iint_R \rho(x,y)\,dA = 4 \cdot \int_0^1 \int_0^{1-x} (x+y)\,dydx. \qquad \text{(6 pts)}$$

Compute the integral in the right hand side,

$$\int_0^1 \int_0^{1-x} (x+y)\,dydx = \int_0^1 (yx + \frac{1}{2}y^2)\Big|_0^{1-x} dx = \int_0^1 (\frac{1}{2} - \frac{1}{2}x^2)\,dx = \frac{1}{3}. \qquad \text{(4 pts)}$$

Therefore,

$$\iint_R \rho(x,y)\,dA = \frac{4}{3}.$$

2. (10%) Evaluate the integral $\displaystyle\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dydx$.

Sol:

Let $x = r\cos\theta$, $y = r\sin\theta$. (6 pts)

Then

$$\begin{aligned}\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dydx &= \int_0^2 \int_0^{\pi} e^{-r^2} r\,d\theta dr \qquad \text{(2 pts)}\\ &= \pi \cdot \int_0^2 e^{-r^2} r\,dr \\ &= \pi \cdot \Big(\frac{-1}{2}e^{-r^2}\Big)\Big|_0^2 \\ &= \frac{\pi}{2}(1 - e^{-4}). \qquad \text{(2 pts)}\end{aligned}$$

3. (10%) Find $\displaystyle\iiint_E xyz\,dV$, where

$$E = \{(x,y,z):\ x \ge 0,\ y \ge 0,\ z \ge 0 \text{ and } 36x^2 + 16y^2 + 9z^2 \le 144\}.$$

Sol:

Let $x = \frac{1}{6}r\sin\phi\cos\theta,\ y = \frac{1}{4}r\sin\phi\sin\theta,\ z = \frac{1}{3}r\cos\phi$, where $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$

then we have $\dfrac{r^2\sin\phi}{72} = \| \dfrac{\partial(\mathbf{x},\mathbf{y},\mathbf{z})}{\partial(\mathbf{r},\phi,\theta)} \|$ (4%)

$$\Rightarrow \iiint_E xyz\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{12} \frac{1}{72}r^3(\sin\phi)^2\cos\phi\sin\theta\cos\theta\frac{1}{72}r^2\sin\phi\,dr d\phi d\theta \quad (3\%)$$

$$= \frac{1}{72^2}\int_0^{12} r^5\,dr \int_0^{\frac{\pi}{2}}(\sin\phi)^3\cos\phi\,d\phi \int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{72^2}\times\frac{12^6}{6}\times\frac{1}{4}\times\frac{1}{2} = 12 \quad (3\%)$$

4. (10%) Evaluate $\iint\limits_R \sin\left(\dfrac{2y-x}{2y+x}\right)dA$, where R is the region enclosed by $2y+x=1$, $2y+x=2$, $2y-x=0$ and $2y+5x=0$.

Sol:

Let $v = 2y+x,\ u = 2y-x$. Hence $x = \dfrac{(v-u)}{2}$, $y = \dfrac{(u+v)}{4}$.

Then the range $2y+x=1,\ 2y+x=2$ imply $v=1, v=2$, $2y-x=0, 2y+5x=0$ imply $u=0,\ u=\dfrac{3}{2v}$.

So the integral become

$$\int\int \sin\left(\frac{2y-x}{2y+x}\right)dxdy = \int_1^2\int_0^{\frac{3}{2v}} \sin\left(\frac{u}{v}\right)\frac{1}{4}\,dudv \quad \text{(a)}$$

the $\dfrac{1}{4}$ is the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ from changing of coordinate.

$$\int_1^2\int_0^{\frac{3}{2v}} \sin\left(\frac{u}{v}\right)\frac{1}{4}\,dudv = \frac{1}{4}\int_1^2 -v\cos\frac{u}{v}\Big|_0^{\frac{3}{2v}}\,dv \quad (1)$$

$$= \frac{1}{4}\int_1^2 -v\cos\frac{3}{2}+v\,dv \quad (2)$$

$$= \frac{1}{4}\frac{v^2}{2}\left(1-\cos\frac{3}{2}\right)\Big|_1^2 \quad (3)$$

$$= \frac{3}{8}\left(1-\cos\frac{3}{2}\right) \quad (4)$$

Correction rule:

(1)write down the complete integral(a) without answer you got 5 points. Missing some part will cost 1 to 2 points.

(2)write down the integral (1) you got 2 points.

(3)Complete answer cost the remained 3 point.

5. (10%) Let $\mathbf{F} = \cos y\mathbf{i} + (z^2 - x\sin y)\mathbf{j} + 2(y+1)z\mathbf{k}$. Find a scalar function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$ and then evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment form $(1,0,0)$ to $(2,2\pi,1)$.

Sol:

$$\frac{\partial f}{\partial x} = \cos(y), \ \frac{\partial f}{\partial y} = z^2 - x\sin(y), \ \frac{\partial f}{\partial z} = 2(y+1).$$

We have

$$f(x,y,z) = x\cos(y) + yz^2 + z^2. \quad \text{(6pts)}$$

$$\int_C f \cdot dr = f(2, 2\pi, 1) - f(1,0,0) = 2 + 2\pi. \quad \text{(4pts)}$$

6. (8%) Find the line integral $\int_C 6y^2 dx + 4x^3\, dy$, where C is the arc of the parabola $y = 1 - x^2$ from $(1,0)$ to $(0,1)$ and then to $(-1,0)$.

Sol:

(Solution 1)

Let $x = t,\ y = 1 - t^2,\ t: 1 \to -1.$

$$\int_C 6y^2 dx + 4x^3 dy = \int_C 6(1-t^2)^2 dt + 4t^3(-2t)dt....\text{(4pts)}$$

The answer is $\frac{-16}{5}$. (4pts)

(Solution 2)

Let $C_1 := \{y = 0 | x : 1 \to -1\}$. Let D be the region bounded by C and C_1. Using Green's theorem,

$$\int_C 6y^2 dx + 4x^3 dy = \int_D (12x^2 - 12y) dA + \int_{C_1} 6y^2 dx + 4x^3 dy. \quad \text{(2pts)}$$

$$\int_D (12x^2 - 12y) dA = \frac{-16}{5}. \quad \text{(4pts)}$$

$$\int_{C_1} 6y^2 dx + 4x^3 dy = 0. \quad \text{(2pts)}$$

7. (10%) Evaluate the line integral $\oint_C (x^2 - y)\, dx + (1 + y^2)\, dy$, where C is the loop of the four leaved rose $r = \cos 2\theta,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, oriented counterclockwise.

Sol:

Let D be the region enclosed by the curve C. By Green's Theorem,

$$\oint_C (x^2-y)dx+(1+y^2)\,dy = \iint_D 1\,dA \qquad (4\text{pts})$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\,drd\theta \qquad (4\text{pts})$$

$$= \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\,d\theta$$

$$= \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 4\theta)\,d\theta$$

$$= \frac{\pi}{8} \qquad (2\text{pts})$$

8. (10%) Find the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = x^2y\mathbf{i} + xy^2\mathbf{j} + 2xyz\mathbf{k}$ and $S = \{(x,y,z):\ x^2+y^2+z^2 = 2, x\geq 0,\ y\geq 0,\ z\geq 1\}$ with the normal pointing upwards.

Sol:

$S_1 = \{(x,y,z)|x^2+z^2\leq 2,\ x\geq 0,\ y=0,\ z\geq 1\}$

$S_2 = \{(x,y,z)|y^2+z^2\leq 2,\ x=0,\ y\geq 0,\ z\geq 1\}$

$S_3 = \{(x,y,z)|x^2+y^2\leq 1,\ z=1\}$

E is the area enclosed by $S,\ S_1,\ S_2,\ S_3$

$$\int_E div\mathbf{F}dV = \int_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S} + \int_{\mathbf{S}_1} \mathbf{F}\cdot d\mathbf{S}_1 + \int_{\mathbf{S}_2} \mathbf{F}\cdot d\mathbf{S}_2 + \int_{\mathbf{S}_3} \mathbf{F}\cdot d\mathbf{S}_3 \qquad (2\text{ pts})$$

On S_1, $\because y=0$, then $\mathbf{F}=0 \ \Rightarrow \ \int_{\mathbf{S}_1} \mathbf{F}\cdot d\mathbf{S}_1 = 0$ (1 pt)

On S_2, $\because x=0$, then $\mathbf{F}=0 \ \Rightarrow \ \int_{\mathbf{S}_2} \mathbf{F}\cdot d\mathbf{S}_2 = 0$ (1 pt)

$$\int_{\mathbf{S}_3} \mathbf{F}\cdot d\mathbf{S}_3 = \int_{S_3} -2xy\,dS_3 = -\int_0^{\frac{\pi}{2}}\int_0^1 2r^3\sin\theta\cos\theta\,drd\theta = -\frac{1}{4} \qquad (2\text{ pts})$$

$$\int_E div\mathbf{F}\,dV = \int_E 6xy\,dV = \int_0^{\frac{\pi}{2}}\int_0^1\int_1^{\sqrt{2-r^2}} 6r^3\sin\theta\cos\theta\,dzdrd\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta \int_0^1 r^3(\sqrt{2-r^2}-1)\,dr$$

$$\because \int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta = 1 \text{ and } \int_0^1 r^3\sqrt{2-r^2}\,dr = \frac{1}{2}\int_0^1 u\sqrt{2-u}\,du = \frac{8\sqrt{2}-7}{15}$$

$$\text{then } \int_E div\mathbf{F}\,dV = \frac{8\sqrt{2}-7}{5} - \frac{3}{4} \qquad (4\text{ pts})$$

$$\Rightarrow \int_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S} = \frac{8\sqrt{2}-7}{5} - \frac{1}{2}$$

9. (10%) (a) Find curl $\mathbf{v}$, where $\mathbf{v} = -y^3\mathbf{i} + x^3\mathbf{j} + e^{z^2}\mathbf{k}$ and evaluate $\iint_S \text{curl } \mathbf{v} \cdot d\mathbf{S}$, where S is the portion of the surface of $z = x^3 + y^3 - 3xy$ within the cylinder $x^2 + y^2 = a^2$ with upward normal. (b) Use Stokes' Theorem to evaluate $\int_C \mathbf{v} \cdot d\mathbf{r}$, where C is the boundary of S oriented counterclockwise when viewed from above.

Sol:

(a) curl $\mathbf{v} = \nabla \times \mathbf{v} = (0, 0, 3x^2 + 3y^2)$ (2 pts)

$g(x, y) = (x, y, x^3 + y^3 - 3xy)$

$$\begin{cases} g_x = (1, 0, 3x^2 - 3y) \\ g_y = (0, 1, 3y^2 - 3x) \end{cases}$$

$g_x \times g_y = (-3x^2 + 3y, 3x - 3y^2, 1)$ (2 pts)

$$\begin{aligned} \iint_S \text{curl } \mathbf{v} \cdot d\mathbf{S} &= \iint_D \text{curl } \mathbf{v} \cdot (g_x \times g_y)\, dA \quad (D : \{(x, y) | x^2 + y^2 \le a^2\}) \qquad \text{(2pts)} \\ &= \iint_D 3(x^2 + y^2)\, dxdy \\ &= 3 \int_0^{2\pi} \int_0^a r^2 \cdot r\, dr d\theta \\ &= 6\pi \cdot \frac{1}{4} r^4 \Big|_0^a \\ &= \frac{3}{2}\pi a^4 \quad \text{(2pts)} \end{aligned}$$

(b) $\int_C \mathbf{v} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{v} \cdot d\mathbf{S} = \frac{3}{2}\pi a^4$ (2 pts)

10. (12%) Let S_1 be the upper semi-sphere $x^2+y^2+z^2 = 1$ with $z \ge 0$, S_2 the unit disk $x^2+y^2 \le 1$ on xy-plane and V the region enclosed by $S_1 \cup S_2$. Endow $S_1 \cup S_2$ with outward normal. Let $\mathbf{F} = xz^2\mathbf{i} + (yx^2 + e^z)\mathbf{j} + (y^2z + \cos(x^2 + y^2))\mathbf{k}$. (a) Find div$\mathbf{F}$ and evaluate $\iiint_V \text{div}\mathbf{F}\, dV$.

(b) Use Divergence Theorem to evaluate $\iint_{S_1 \cup S_2} \mathbf{F} \cdot d\mathbf{S}$ and then find $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

Sol:

(a) i.

$$\text{div}\mathbf{F} = z^2 + x^2 + y^2.$$

ii.

$$\iiint_V \text{div}\mathbf{F} dV = \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^4 \sin\phi d\phi d\theta d\rho = \frac{2}{5}\pi.$$

(b) i. By the divergence theorem, we have

$$\iint\limits_{S_1 \cup S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_V \mathrm{div}\mathbf{F} dV = \frac{2}{5}\pi.$$

ii.

$$\begin{aligned}\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_{S_1 \cup S_2} \mathbf{F} \cdot d\mathbf{S} - \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S} \\ &= \frac{2}{5}\pi - \iint\limits_{S_2} \mathbf{F} \cdot (0,0,-1) dS \\ &= \frac{2}{5}\pi - (-\int_0^1 \int_0^{2\pi} r \cos r^2 d\theta dr) \\ &= (\frac{2}{5} + \sin 1)\pi.\end{aligned}$$

Grading policy:

1. $\mathrm{div}\mathbf{F}$. (2 pts)
2. $\iiint\limits_V \mathrm{div}\mathbf{F} dV$. (4 pts)
3. Apply the divergence theorem (2 pts). Here you will get full points even if making a mistake in calculation of $\iiint\limits_V \mathrm{div}\mathbf{F} dV$.
4. $\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S}$ (4 pts) or $\iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}$ (2 pts. Note the normal vector is $(0,0,-1)$). That is, you could receive partial credit even without the result of (a).