1. (10%) Find the values of real number p for which the series is divergent, conditionally convergent, or absolutely convergent.

(a) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^p}$.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{1}{n^p}\right)\cdot \ln n$.

Sol:

(a) Since

$$\frac{d}{dx}\left(\frac{1}{x(\ln x)^p}\right) = -\frac{(\ln x)^p + x\cdot p(\ln x)^{p-1}\frac{1}{x}}{x^2(\ln x)^{2p}} = -\frac{(\ln x)^{p-1}(\ln x + p)}{x^2(\ln x)^{2p}} < 0 \quad \text{for } x > e^{-p}.$$

Hence $\forall p \in \mathbb{R}$, we have $\dfrac{1}{n(\ln n)^p}$ is decreasing for n large enough. $(n > e^{-p})$

If $p > 0$, then $\displaystyle\lim_{n\to\infty}\frac{1}{n(\ln n)^p} = 0$. And if $p < 0$, then by L'hospital Law, we have

$$\lim_{n\to\infty}\frac{1}{n(\ln n)^p} = \lim_{n\to\infty}\frac{(\ln n)^{-p}}{n} = \lim_{n\to\infty}\frac{-p(\ln n)^{-p-1}\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{-p(\ln n)^{-p-1}}{n}$$

$$= \cdots = \lim_{n\to\infty}\frac{(-1)^{[-p]+1}(p)(p+1)(p+2)\cdots(p+[-p])(\ln n)^{-p-([-p]+1)}}{n} = 0.$$

Hence by alternating series test , $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(\ln n)^p}$ is convergent for all $p \in \mathbb{R}$.

Now consider $a_n \equiv \left|(-1)^n\dfrac{1}{n(\ln n)^p}\right| = \dfrac{1}{n(\ln n)^p}$. Suppose $p \neq 1$, since we have proven a_n is decreasing for n large enough, thus "integral test" is available here

$$\int_2^{\infty}\frac{1}{x(\ln x)^p}dx = \int_{\ln 2}^{\infty} u^{-p}du\Big|_{u=\ln x} = \lim_{b\to\infty}\left(\frac{u^{1-p}}{1-p}\right)_{\ln 2}^{b}$$

$$= \lim_{b\to\infty}\frac{b^{1-p}-(\ln 2)^{1-p}}{1-p} = \begin{cases} \infty & \text{for } p<1 \\ \dfrac{(\ln 2)^{1-p}}{p-1} & \text{for } p>1 \end{cases}$$

$$\Rightarrow \begin{cases} \displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p} & \text{, is convergnet for } p>1 \\ \displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p} & \text{, is divergnet for } p><1 \end{cases}.$$

Suppose $p = 1$, we also use integral test, since

$$\int_2^{\infty}\frac{1}{x(\ln x)}dx = \int_{\ln 2}^{\infty}\frac{1}{u}du\Big|_{u=\ln x} = \lim_{b\to\infty}(\ln b - \ln(\ln 2)) = \infty.$$

So we get the answer is

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(\ln n)^p} \quad \text{is} \quad \begin{cases} \text{absolutely convergent for } p>1 \ \ (2.5 \text{ pts}) \\ \text{conditionally convergent for } p\le 1 \ \ (2.5 \text{ pts}). \end{cases}$$

(b) For $p \le 0$, since $\lim_{n\to\infty}\sin(\frac{1}{n^p})\ln n \neq 0$, thus $\sum_{n=1}^{\infty}(-1)^n\sin(\frac{1}{n^p})\ln n$ is divergent.
Now suppose $p>0$, by limit comparison test, since

$$\lim_{n\to\infty}\frac{\sin(\frac{1}{n^p})\ln n}{\frac{1}{n^p}\ln n} = 1 \ \forall p \in \mathbb{R}^+ = (0,\infty)$$

Thus we can examine $\sum_{n=1}^{\infty}\frac{1}{n^p}\ln n$ to get the answer.

For $0<p\le 1$,by comparison , since $\frac{\ln n}{n^p} \ge \frac{1}{n^p}$ for $n>3$, and $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent for $0<p\le 1$, so is $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$.

For $p>1$ then

$$\frac{\ln n}{n^p} = \frac{1}{n^{\frac{p+1}{2}}}\cdot\frac{\ln n}{n^{\frac{p-1}{2}}}.$$

Since $\lim_{n\to\infty}\frac{\ln n}{n^{\frac{p-1}{2}}} = 0$, thus $\frac{\ln n}{n^{\frac{p-1}{2}}} < 1$ for n large enough, that is $\frac{\ln n}{n^p} < n^{-\frac{p+1}{2}}$ for n large enough, since $\frac{p+1}{2} > 1$, thus by compare with $\sum_{n=1}^{\infty}n^{-\frac{p+1}{2}}$, we get $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$ converges for $p>1$.

So we get the answer is

$$\sum_{n=1}^{\infty}(-1)^n\sin(\frac{1}{n^p})\ln n \quad \text{is} \quad \begin{cases} \text{divergent for } p\le 0 \ \ (1 \text{ pt}) \\ \text{conditionally convergent for } 0<p\le 1 \ \ (2 \text{ pts}) \\ \text{absolutely convergent for } p>1 \ \ (2 \text{ pts}) \end{cases}$$

評分標準:

1. 照詳解上的配分方式給分.
2. alternating series test 前提條件不算分, 有寫有加分.
3. 寫答案未寫理由者不予給分.

2. (8%) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\ln\left(1+\tan^{-1}\frac{1}{n}\right)$ is absolutely convergent, conditionally convergent, or divergent.

Sol:

Let $f(x) = \ln(1 + \tan^{-1}\dfrac{1}{x})$, then

$$f'(x) = \frac{1}{1 + (\tan^{-1}(\frac{1}{x}))^2}\frac{-1}{x^2+1} < 0\ ,\ \forall x > 0.$$

That is , $a \equiv \ln(1 + \tan^{-1}(\dfrac{1}{n}))$ is decreasing , and

$$\lim_{n\to\infty} \ln(1 + \tan^{-1}\frac{1}{n}) = 0.$$

Hance by alternating series test , we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n \ln(1 + \tan^{-1}\frac{1}{n})$ converves.

(8 pts) Now consider $b_n = \ln(1 + \tan^{-1}\dfrac{1}{n}) > 0$, since by Taylor expansion

$$\ln(1 + \tan^{-1}\frac{1}{n}) = \left(\frac{1}{n} - \frac{1}{3}(\frac{1}{n})^2 + \cdots\right) - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{3}(\frac{1}{n})^2 + \cdots\right)^2 + \cdots$$

Hance compare with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{\ln(1 + \tan^{-1}\frac{1}{n})}{\frac{1}{n}} = \lim_{t\to 0^+}\frac{\ln(1 + \tan^{-1} t)}{t} = \lim_{t\to 0^+}\frac{\frac{1}{1+\tan^{-1} t}\frac{1}{1+t^2}}{1} = 1$$

So we get $\displaystyle\sum_{n=1}^{\infty}\ln(1 + \tan^{-1}\frac{1}{n})$ diverges , that is

$$\sum_{n=1}^{\infty}(-1)^n \ln(1 + \tan^{-1}\frac{1}{n}) \text{ is conditionally convergent}$$

評分標準:

1. 照詳解上的配分方式給分.
2. alternating series test 前提條件不算分, 有寫有加分.
3. 寫答案未寫理由者不予給分.

3. (15%) The Fibonacci sequence $\{f_n\}$ is defined as $f_1 = f_2 = 1,\ f_n = f_{n-1} + f_{n-2}\ \forall n \geq 3$. Let $a_n = \dfrac{f_{n+1}}{f_n}$.

   (a) Show that $a_n = 1 + \dfrac{1}{a_{n-1}}$ and $1 \leq a_n \leq 2$ for all n.

   (b) Show that $\{a_{2n+1}\}$ is a monotonic sequence and is convergent.

   (c) Find $\displaystyle\lim_{n\to\infty} a_{2n+1}$.

   (d) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{2}\right)x^n$.

Sol:

(a) $a_n = \dfrac{f_{n+1}}{f_n} = \dfrac{f_n + f_{n-1}}{f_n} = 1 + \dfrac{1}{\frac{f_n}{f_{n-1}}} = 1 + \dfrac{1}{a_{n-1}}$ (2 pts)

$1 \le a_1 = \dfrac{1}{1} = 1 \le 2,$

assume that $1 \le a_k \le 2,$

for $n = k+1,\ \dfrac{1}{2} \le \dfrac{1}{a_k} \le 1 \Rightarrow 1 \le 1 + \dfrac{1}{2} \le 1 + \dfrac{1}{a_k} \le 1 + 1 = 2 \Rightarrow 1 \le a_{k+1} \le 2$

Hence, $1 \le a_n \le 2$ for all n by induction. (2 pts)

(b) $a_1 = 1 \le \dfrac{3}{2} = a_3,$

assume that $a_{2k-1} \le a_{2k+1} \Rightarrow 1 + \dfrac{1}{a_{2k-1}} \ge 1 + \dfrac{1}{a_{2k+1}} \Rightarrow a_{2k} \ge a_{2k+2}$

$\Rightarrow 1 + \dfrac{1}{a_{2k}} \le 1 + \dfrac{1}{a_{2k+2}} \Rightarrow a_{2k+1} \le a_{2k+3}$ (3 pts)

Hence, $\{a_{2n+1}\}$ is increasing by inducion and bounded by (a) $\Rightarrow$ convergent. (1 pt)

(c) Assume $\lim\limits_{n\to\infty} a_{2n+1} = \alpha$

$a_{2n+1} = 1 + \dfrac{1}{a_{2n}} = 1 + \dfrac{1}{1 + \frac{1}{a_{2n-1}}} = \dfrac{2a_{2n-1}+1}{a_{2n-1}+1}$ (2 pts)

$\lim\limits_{n\to\infty} a_{2n+1} = \lim\limits_{n\to\infty} \dfrac{2a_{2n-1}+1}{a_{2n-1}+1}$

$\Rightarrow \alpha = \dfrac{2\alpha+1}{\alpha+1} \Rightarrow \alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha = \dfrac{1+\sqrt{5}}{2}$ (1 pt)

(d) $\sum\limits_{n=1}^{\infty} f_n \sin(\dfrac{n\pi}{2})x^n = \sum\limits_{n=0}^{\infty} (-1)^n f_{2n+1} x^{2n+1}$ (1 pt)

$a_{2n+2} = 1 + \dfrac{1}{a_{2n+1}} \Rightarrow \lim\limits_{n\to\infty} a_{2n+2} = 1 + \dfrac{1}{\lim_{n\to\infty} a_{2n+1}} = 1 + \dfrac{1}{\alpha} = \dfrac{1+\sqrt{5}}{2}$ (1 pt)

By ratio test, $\lim\limits_{n\to\infty} \left| \dfrac{f_{2n+3}}{f_{2n+1}} x^2 \right| < 1 \Leftrightarrow \left| \lim\limits_{n\to\infty} \dfrac{f_{2n+3}}{f_{2n+2}} \lim\limits_{n\to\infty} \dfrac{f_{2n+2}}{f_{2n+1}} x^2 \right| < 1$ (1 pt)

$\Leftrightarrow \left| \lim\limits_{n\to\infty} a_{2n+2} \lim\limits_{n\to\infty} a_{2n+1} x^2 \right| < 1 \Leftrightarrow \left| (\dfrac{1+\sqrt{5}}{2})^2 x^2 \right| < 1 \Leftrightarrow |x| < \dfrac{2}{1+\sqrt{5}} = \dfrac{\sqrt{5}-1}{2}$

Hence, $R = \dfrac{\sqrt{5}-1}{2}$ (1 pt)

4. (8%) Find the Maclaurin series for $\ln(x + \sqrt{1+x^2})$. (You must write down the nth term.)

Sol:

$f(x) = \ln(x + \sqrt{1+x^2})$

$\Rightarrow f'(x) = \dfrac{(1 + \frac{x}{\sqrt{1+x^2}})}{(x + \sqrt{1+x^2})} = \dfrac{1}{\sqrt{1+x^2}}$ (2 pts)

$$\Rightarrow f'(x) = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{2k} \quad (2 \text{ pts}) \quad \forall |x| < 1 \quad (1 \text{ pt})$$

$$\Rightarrow f(x) = C + \int f'(x)dx = C + \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1} \quad \forall |x| < 1 \quad (2 \text{ pts})$$

$f(0) = \ln 1 = 0 \Rightarrow C = 0 \quad (1 \text{ pt})$

$$\Rightarrow f(x) = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1} \left(= \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!\, x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \right) \quad \forall |x| < 1$$

5. (15%) Let $\mathbf{r}(t) = \langle t, \sqrt{2}\ln\cos t, \tan t - t\rangle$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, and P be the point $\mathbf{r}\left(\frac{\pi}{4}\right)$.

(a) Find the length of the arc $\mathbf{r}(t)$, $0 \le t \le \frac{\pi}{4}$.

(b) Find the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$ and the binormal vector $\mathbf{B}$ at P.

(c) Find the curvature κ at P.

(d) Find the center of the osculating circle (the circle of curvature) at P.

Sol: $\mathbf{r}'(t) = (1, -\sqrt{2}\tan t, \tan^2 t) \Rightarrow |\mathbf{r}'(t)| = \sec^2 t$

(a) $\int_0^{\frac{\pi}{4}} \sec^2 t\, dt \quad (1 \text{ pt}) \quad = \tan t\Big|_0^{\frac{\pi}{4}} = 1 \quad (2 \text{ pts})$

(b) $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad (1 \text{ pt})$

$\mathbf{T}_P = (\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}) \quad (1 \text{ pt})$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}(t)|} \quad \mathbf{T}'(t) = (-\sin 2t,\ -\sqrt{2}\cos 2t,\ \sin 2t) \quad (1 \text{ pt})$

$\mathbf{N}_P = (-\frac{\sqrt{2}}{2},\ 0\ , \frac{\sqrt{2}}{2}) \quad (1 \text{ pt})$

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \quad (1 \text{ pt})$

$\mathbf{B}_P = (-\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}) \quad (1 \text{ pt})$

(c) $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad (2 \text{ pts}) \quad \kappa_P = \frac{\sqrt{2}}{2} \quad (1 \text{ pt})$

(d) The radius $r = \frac{1}{\kappa}$. The center $C_P = P + r\mathbf{N} \quad (2 \text{ pts})$

$C_P = (\frac{\pi}{4} - 1, -\sqrt{2}\ln\sqrt{2}, 2 - \frac{\pi}{4}) \quad (1 \text{ pt})$

6. (12%) Let $f(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$.

(a) Find $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Let $L(x,y)$ be the linear approximation of f at $(0,0)$. Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}}$ exist?

(c) Find the directional derivative of f at $(0,0)$ in the direction $\langle 1, m\rangle$.

(d) Is $f(x,y)$ differentiable at $(0,0)$?

Sol:

(a) By definition of partial derivative, $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0$.

Similarly $f_y(0,0) = 0$.

(b) By definition, $L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y$.

Since $f_x(0,0) = f_y(0,0) = 0$, we have $L(x,y) = 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{|x^{\frac{1}{3}}y^{\frac{2}{3}}|}{\sqrt{x^2+y^2}}$$

Along $x = y$,

$$\lim_{(x,y)\to(0,0)} \frac{|x^{\frac{1}{3}}y^{\frac{2}{3}}|}{\sqrt{x^2+y^2}} = \lim_{x\to} \frac{|x|}{\sqrt{2}|x|} = \frac{1}{\sqrt{2}}.$$

Along $x = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{|x^{\frac{1}{3}}y^{\frac{2}{3}}|}{\sqrt{x^2+y^2}} = 0.$$

Therefore, the limit does not exist.

(c) First normalize the vector $< 1, m >$ to unit vector $\mathbf{u} = \dfrac{1}{\sqrt{1+m^2}} < 1, m >$. By definition of the directional derivative, we have

$$D_{\mathbf{u}}f(0,0) = \lim_{t\to 0} \frac{f(0+\frac{t}{\sqrt{1+m^2}}, 0+\frac{tm}{\sqrt{1+m^2}}) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{m^{\frac{2}{3}}}{\sqrt{1+m^2}} = \frac{m^{\frac{2}{3}}}{\sqrt{1+m^2}}.$$

(d) By (c), $D_{\mathbf{u}}f(0,0) = \dfrac{m^{\frac{2}{3}}}{\sqrt{1+m^2}}$.

By (a), $\bigtriangledown f(0,0)\cdot\mathbf{u} = 0$.

Therefore, $\bigtriangledown f(0,0)\cdot\mathbf{u} \neq D_{\mathbf{u}}f(0,0)$ in general. Hence f is not differentiable at $(0,0)$.

Grading Policy :

Question (a) worth 2 pts. Only correct answer would get 2 pts.

Question (b) worth 5 pts. You would get 1 pt if you write down the linear approximation $L(x,y)$ of f at $(0,0)$. Another 4 pts depends on your answer to the limit

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}}.$$

Question (c) worth 3 pts. You would receive 2 pts if you did not normalized the vector $< 1, m >$ and the answer you give is $m^{\frac{2}{3}}$.

Question (d) worth 2 pts. If you only answer NO, you would get 1 pt. The other 1pt depends on your explnation .

7. (12%) Let $f(x,y) = xye^{-xy^2}$.

   (a) Find the gradient of f.

   (b) Find the directional derivative of f at the point $(1,1)$ in the direction $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$.

   (c) Find the tangent plane of $z = f(x,y)$ at the point $(1, 1, \frac{1}{e})$.

   (d) Let $z = f(x,y)$ and $x = u^2 + 3v,\ y = uv - 3v$. Find $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(2,-1)}$.

Sol:

(a) $\bigtriangledown f(x,y) = e^{-xy^2}(y - xy^3, x - 2x^2y^2)$.

(b) Since f is differentiable on $\mathbb{R}^2$, we have

$$D_{<\frac{2}{\sqrt{2}},\frac{1}{\sqrt{5}}>} f(1,1) = \bigtriangledown f(1,1)\cdot < \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{5}} >= \frac{-1}{\sqrt{5}e}.$$

(c) The tangent plane of $z = f(x,y)$ at the point $(1, 1, \frac{1}{e})$ is

$$(z - \frac{1}{e}) = f_x(1,1)(x-1) + f_y(1,1)(y-1) = -(y-1).$$

(d) By chain rule,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (y - xy^3)e^{-xy^2}\cdot 3 + (x - 2x^2y^2)e^{-xy^2}\cdot(u-3).$$

When $(u,v) = (2,-1)$, $(x,y) = (1,1)$. Therefore,

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(2,-1)} = 0\cdot 3 + \frac{-1}{e}\cdot(-1) = \frac{1}{e}.$$

Grading Policy :

Question (a) worth 3 pts. You would get 2 pts if you only give correct formula to either f_x or f_y.

Question (b) worth 3 pts. You would get 1 pt if you know

$$D_{<\frac{2}{\sqrt{2}},\frac{1}{\sqrt{5}}>}f(1,1) = \nabla f(1,1)\cdot < \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{5}} > .$$

Another 2 pts depends on your calculation.

Question (c) worth 3 pts. You would get 1 pt if you know

$$(z - \frac{1}{e}) = f_x(1,1)(x-1) + f_y(1,1)(y-1) = -(y-1).$$

Another 2 pts depends on your calculation.

Question (d) worth 3 pts. You would get 1 pt if you know $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$.

Another 2 pts depends on your calculation.

8. (10%) Find the local maximum and minimum values and the saddle points, if exist, of

$$f(x,y) = x^3 + x^2 + \frac{3}{2}x^2y + 2xy + 2y^2 + \frac{1}{2}y.$$

Sol:

$f_x = 3x^2 + 2x + 3xy + 2y$

$f_y = \frac{3}{2}x^2 + 2x + 4y + \frac{1}{2}$ (2 pts)

critical points: $\begin{cases} 3x^2 + 2x + 3xy + 2y & = 0 \Rightarrow (3x+2)(x+y) = 0 \Rightarrow x = -\frac{2}{3} \text{ or } x = -y \\ \frac{3}{2}x^2 + 2x + 4y + \frac{1}{2} & = 0 \end{cases}$

$x = -\frac{2}{3} \Rightarrow \frac{3}{2}\frac{4}{9} - 2\frac{2}{3} + 4y + \frac{1}{2} = 0 \Rightarrow y = \frac{1}{24}$

$x = -y \Rightarrow \frac{3}{2}y^2 - 2y + 4y + \frac{1}{2} = 0 \Rightarrow 3y^2 + 4y + 1 = 0 \Rightarrow y = -1 \text{ or } y = -\frac{1}{3}$

So critical points is $(-\frac{2}{3}, \frac{1}{24}), (1,-1), (\frac{1}{3}, -\frac{1}{3})$ (2 pts)

$f_{xx} = 6x + 2 + 3y$

$f_{yy} = 4$

$f_{xy} = 3x + 2$

$D = f_{xx}f_{yy} - f_{xy}^2 = 24x + 8 + 12y - (3x+2)^2$ (3 pts)

$D(-\frac{2}{3}, \frac{1}{24}) = -16 + 8 + \frac{1}{2} - 0 < 0 \Rightarrow (-\frac{2}{3}, \frac{1}{24})$ is saddle point

$D(1, -1) = 24 + 8 - 12 - 25 < 0 \Rightarrow (1, -1)$ is saddle point

$D(\frac{1}{3}, -\frac{1}{3}) = 8 + 8 - 4 - 9 > 0, f_x x = 2 + 2 - 1 > 0$

$\Rightarrow (\frac{1}{3}, -\frac{1}{3})$ is local minimum with minimum value

$f(\frac{1}{3}, -\frac{1}{3}) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \frac{3}{2}\left(\frac{1}{3}\right)^2 \frac{1}{3} - 2\frac{1}{3}\frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + \frac{1}{2}\frac{1}{3} = -\frac{2}{27}$ (3 pts)

9. (10%) Let Γ be the ellipse with center at the origin that is the intersection of the plane $x + y + 2z = 0$ and the surface $x^2 + 2y^2 + 4z^2 = 35$.

(a) Find the lengths of the major and the minor axes (長軸與短軸) of Γ.

(b) Find the area of the region enclosed by Γ.

Sol:

(a) The length of the major axis: $\sqrt{105}$. The length of the minor axis: $2\sqrt{14}$

(b) The area of the region enclosed by Γ: $\frac{7}{2}\sqrt{30}\pi$

Let $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x + y + 2z$, $h(x, y, z) = x^2 + 2y^2 + 4z^2 - 35$

Applying the method of Lagrange multipliers, we need to solve

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \cdots (*) \\ g = 0 \\ h = 0 \end{cases} \quad \text{or} \quad \begin{cases} 2x = \lambda + 2\mu x \cdots (1) \\ 2y = \lambda + 4\mu y \cdots (2) \\ 2z = 2\lambda + 8\mu z \cdots (3) \\ x + y + 2z = 0 \cdots (4) \\ x^2 + 2y^2 + 4z^2 - 35 = 0 \cdots (5) \end{cases}$$

$$\Rightarrow \begin{cases} 2x(1-\mu) = \lambda \cdots (1)' \\ 2y(1-2\mu) = \lambda \cdots (2)' \\ 2z(1-4\mu) = 2\lambda \cdots (3)' \end{cases}$$

Utilizing (4) and (5) and considering $(1) \cdot x + (2) \cdot y + (3) \cdot z$ we have $x^2 + y^2 + z^2 = 35\mu$

Case 1: $\lambda = 0$

If $\mu = 1 \Rightarrow y = z = 0, \therefore x = 0$ by (1), but contradicts (5). Similarly, $\mu = \frac{1}{2} \Rightarrow x = z = 0$ and $\mu = \frac{1}{4} \Rightarrow x = y = 0$ both lead to contradiction.
If $\mu \neq 1, \frac{1}{2}, \frac{1}{4} \Rightarrow x = y = z = 0$, also a contradiction.

$\therefore \lambda \neq 0$

Case 2: $\lambda \neq 0$
Substitute $x = \dfrac{\lambda/2}{1-\mu}, y = \dfrac{\lambda/2}{1-2\mu}, z = \dfrac{\lambda}{1-4\mu}$ into (4):
$\lambda\{\dfrac{1}{2(1-\mu)} + \dfrac{1}{2(1-2\mu)} + \dfrac{2}{1-4\mu}\} = 0$
Arranging the numerator, we have $20\mu^2 - 23\mu + 6 = 0$

$\Rightarrow (5\mu - 2)(4\mu - 3) = 0$
$\Rightarrow \mu = \dfrac{2}{5}$ or $\dfrac{3}{4}$. Therefore $x^2 + y^2 + z^2 = 14$ or $\dfrac{105}{4}$.
The length of the major axis is $2 \times \sqrt{\dfrac{105}{4}} = \sqrt{105}$ and the length of the minor axis is $2\sqrt{14}$.
The area is $\pi \cdot \dfrac{\sqrt{105}}{2} \cdot \sqrt{14} = \dfrac{7}{2}\sqrt{30}\pi$

- 2 points for (1) to (3) each (6 points in total).

If only ($*$) is present (i.e., without (1) to (3) and without solving):

* If the scalar function f is reasonably defined $\Rightarrow$ 3 points

* If f is a scalar function but not correctly defined $\Rightarrow$ 2 points

* If f is not even a scalar function (i.e., in the form of constraint) $\Rightarrow$ 1 point

Note: Using g to eliminate one variable in f and h is acceptable, but then the two functions should only have two dimensions.

- If both lengths of the axes are correct, 3 points will be credited. Half the values (i.e., $\dfrac{\sqrt{105}}{2}$ and $\sqrt{14}$) are also regarded as correct answers.

* If only one of them is correct $\Rightarrow$ 2 points

* If both are incorrect but the two values of μ have been solved $\Rightarrow$ 1 point

- 1 point for the area.