1. (12%) Evaluate the integral $\int_1^4 \int_{\sqrt[5]{y}}^{\sqrt{y}} \sqrt{x^3-1}\, dxdy + \int_4^{32} \int_{\sqrt[5]{y}}^{2} \sqrt{x^3-1}\, dxdy$.

   Sol:

$$
\begin{aligned}
&\int_1^4 \int_{\sqrt[5]{y}}^{\sqrt{y}} \sqrt{x^3-1}\, dxdy + \int_4^{32} \int_{\sqrt[5]{y}}^{2} \sqrt{x^3-1}\, dxdy \quad (8\%) \\
&= \int_1^2 \int_{x^2}^{x^5} \sqrt{x^3-1}\, dydx \\
&= \int_1^2 x^2(x^3-1)^{\frac{3}{2}}\, dx \\
&= \frac{2}{15}\sqrt{7^5} \\
&= \frac{98}{15}\sqrt{7} \quad (12\%)
\end{aligned}
$$

2. (12%) Evaluate the integral $\iint_R (x+y)^2\, dA$ where $R$ is the region bounded by the ellipse $x^2+xy+y^2=1$.

   Sol:

$$
\begin{aligned}
\iint_R (x+y)^2\, d\theta &= \iint_{u^2+v^2\le 1} (u+\frac{v}{\sqrt{3}})^2 \times \frac{2}{\sqrt{3}}\, dudv \quad (u = x+\frac{y}{2},\ v=\frac{\sqrt{3}}{2}) \quad (5\%) \\
&= \int_0^{2\pi}\int_0^1 (r\cos\theta + \frac{r\sin\theta}{\sqrt{3}})^2 \times \frac{2}{\sqrt{3}} r\, drd\theta \quad (u = r\cos\theta,\ v = r\sin\theta) \quad (9\%) \\
&= \frac{2\sqrt{3}\pi}{9} \quad (12\%)
\end{aligned}
$$

3. (12%) Let $C$ be the path consisting of the segments from $(0,0,0)$ to $(1,0,0)$, from $(1,0,0)$ to $(1,1,0)$, and from $(1,1,0)$ to $(1,1,1)$. Evaluate $\int_C (6xy^3+2z^2)\, dx + 9x^2y^2\, dy + (4xz+1)\, dz$.

   Sol:

   There are two ways to solve this problem. Let

$$
F(x,y,z) =< P(x,y,z), Q(x,y,z), R(x,y,z) >=< 6xy^3+2z^2, 9x^2y^2, 4xz+1 >
$$

   be a vector field on $\mathbb{R}^3$. According to the problem, we have to find the line integral

$$
I = \int_C F dr,
$$

   where $C$ is the path indicated in the problem.

## Solution 1

Let $t \in [0,1]$ and

$$\begin{cases} C_1 : r_1(t) =< t, 0, 0 > & \cdots \text{line segment from (0,0,0) to (1,0,0),} \\ C_2 : r_2(t) =< 1, t, 0 > & \cdots \text{line segment from (1,0,0) to (1,1,0),} \\ C_3 : r_3(t) =< 1, 1, t > & \cdots \text{line segment from (1,1,0) to (1,1,1).} \end{cases}$$

Note that

$$\begin{cases} r_1'(t) =< 1, 0, 0 >, \\ r_2'(t) =< 0, 1, 0 >, \\ r_3'(t) =< 0, 0, 1 >, \end{cases}$$

for $t \in [0,1]$.

Since $C = C_1 \cup C_2 \cup C_3$, we find that

$$\begin{aligned} I &= \int_{C_1} F dr + \int_{C_2} F dr + \int_{C_3} F dr \\ &= \int_0^1 F(r_1(t)) r_1'(t) dt + \int_0^1 F(r_2(t)) r_2'(t) dt + \int_0^1 F(r_3(t)) r_3'(t) dt \\ &= \int_0^1 < 0, 0, 1 >< 1, 0, 0 > dt + \int_0^1 < 6t^3, 9t^2, 1 >< 0, 1, 0 > dt + \int_0^1 < 2t^2, 9, 4t+1 >< 0, 0, 1 > dt \\ &= \int_0^1 0 dt + \int_0^1 9t^2 dt + \int_0^1 (4t+1) dt \\ &= 3t^3 \Big|_0^1 + (2t^2 + t) \Big|_0^1 = 3 + 3 = 6. \end{aligned}$$

## Solution 2

Since

$$\begin{aligned} \text{curl} F &= < (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}), (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}), (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) > \\ &= < (0 - 0), (4z - 4z), (18xy^2 - 18xy^2) > \\ &= < 0, 0, 0 >, \end{aligned}$$

and $P(x,y,z), Q(x,y,z), R(x,y,z)$ are smooth functions on $\mathbb{R}^3$, we find that $F$ is conservative. Thus the potential function $f(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ of $F$ exists. i.e.

$$\bigtriangledown f = F$$

Let

$$\bigtriangledown f(x,y,z) =< f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) > .$$

Since

$$f_z(x,y,z) = \frac{\partial f}{\partial z}(x,y,z) = R(x,y,z) = 4xz+1,$$

it follows that

$$f(x,y,z) = 2xz^2 + z + g(x,y)$$

for some function $g(x,y)$. Thus

$$f_y(x,y,z) = \frac{\partial g}{\partial y}(x,y) = Q(x,y,z) = 9x^2y^2.$$

This implies that

$$g(x,y) = 3x^2y^3 + h(x),$$

for some function $h(x)$ and

$$f(x,y,z) = 3x^2y^3 + 2xz^2 + z + h(x).$$

Finally since

$$f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) = 6xy^3 + 2z^2 + h'(x) = P(x,y,z) = 6xy^3 + 2z^2,$$

we find that

$$h'(x) = 0$$

Thus $h(x) = c$ for some $c \in \mathbb{R}$ is a constant function and

$$f(x,y,z) = 3x^2y^3 + 2xz^2 + z + c.$$

By the foundamental theorem for line integrals, we have

$$I = f(1,1,1) - f(0,0,0) = 6.$$

評分標準:

若是用 solution 1 , 則每一段 line integral 的配分爲 4 (有三段) , 在這 4 分中 , 線段的參數式 $r(t)$ 1 分 , $F(r(t))\ r'(t)$ 1 分 , 最後計算 2 分 。

若是用 solution 2 , 則說明 $curlF = 0 \ \Rightarrow$ 6 分 , 計算 $f(xyz) \ \Rightarrow$ 6 分 。

4. (12%) Let $R$ be the region bounded by $y$-axis, $y=1$, and $y=x^3$, and $\partial R$ be the positively oriented boundary curve of $R$. Evaluate $\displaystyle\int_{\partial R} x^2 y \sin(y^2)\,dx + \frac{2}{3}x^3y^2\cos(y^2)\,dy$.

Sol:

Let $C$ be a boundary of $R$. By the Green's theorem we have

$$\begin{aligned}
\int_C x^2y\sin(y^2)\,dx + \frac{2}{3}x^3y^2\cos(y^2)\,dy &= \iint_R ((\frac{2}{3}x^3y^2\cos(y^2))_x - (x^2y\sin(y^2))_y\,dA \\
&= \int_R -x^2\sin(y^2)\,dA \qquad (6\%) \\
&= \int_0^1\int_0^{y^{\frac{1}{3}}} -x^2\sin(y^2)\,dxdy \qquad (2\%) \\
&= \int_0^1 -\sin(y^2)\frac{y}{3}\,dy \\
&= \frac{\cos 1 - 1}{6} \qquad (4\%)
\end{aligned}$$

5. (12%) Find the surface area of the spherical band that lies on the sphere $x^2+y^2+z^2=3$ between the planes $z=\frac{\sqrt{3}}{2}$ and $z=-\frac{\sqrt{3}}{2}$.

Sol:

**Method 1.** Use spherical coord for the parametrization

$\mathbf{r}(\phi,\theta) = <\sqrt{3}\sin\phi\cos\theta, \sqrt{3}\sin\phi\sin\theta, \sqrt{3}\cos\phi> \quad (3\%)$

$D = \{(\phi,\theta) : \frac{\pi}{3} \le \phi \le \frac{2\pi}{3}\ ,\ 0\le\theta\le 2\pi\} \quad (2\%)$

$\mathbf{r}_\phi\times\mathbf{r}_\theta = <3\sin^2\phi\cos\theta, 3\sin^2\phi\sin\theta, 3\sin\phi\cos\phi> = 3\sin\phi <\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi>$

$|\mathbf{r}_\phi\times\mathbf{r}_\theta| = 3\sin\phi \quad (3\%)$

$$\begin{aligned}
\text{surface area} &= \iint_D |\mathbf{r}_\phi\times\mathbf{r}_\theta|\,dA \qquad (2\%) \\
&= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 3\sin\phi\,d\phi d\theta = 6\pi \qquad (2\%)
\end{aligned}$$

**Method 2.**

Use $z = +\sqrt{3-x^2-y^2}$

$\mathbf{r}(x,y) = <x, y, \sqrt{3-x^2-y^2}> \quad (3\%)$

$(x,y)\in R = \{(x,y) : (\frac{3}{2})^2 \le x^2+y^2 \le 3\ \} \quad (2\%)$

$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{3-x^2-y^2}}\ ,\ \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{3-x^2-y^2}}$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = (\frac{3}{3 - x^2 - y^2})^{\frac{1}{2}} \quad (3\%)$$

$$\begin{aligned}
\text{surface area} &= 2 \iint_R |\mathbf{r}_x \times \mathbf{r}_y| \, dA \quad (2\%) \\
&= 2 \iint_R \sqrt{\frac{3}{3 - x^2 - y^2}} \, dA, \quad \text{where } R = \{(x, y) : (\frac{3}{2})^2 \leq x^2 + y^2 \leq 3\} \\
&= 2\sqrt{3} \int_0^{2\pi} \int_{\frac{3}{2}}^{\sqrt{3}} \frac{r}{\sqrt{3 - r^2}} \, dr d\theta \\
&= 4\sqrt{3}\pi \Big[ -(3 - r^2)^{\frac{1}{2}} \Big|_{\frac{3}{2}}^{\sqrt{3}} \Big] \\
&= 4\sqrt{3} \times \frac{\sqrt{3}}{2}\pi = 6\pi \quad (2\%)
\end{aligned}$$

6. (14%) Let $S$ be the part of the upper sphere $x^2 + y^2 + z^2 = 1$ that lies inside the cylinder $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$.

(a) Parametrize the surface $S$.

(b) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $S$ is oriented upward.

Sol:

(a) $S : \gamma(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, with (3 pts)

$D =: \{(x, y) | x^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}\} = \{(r, \theta) | 0 \leq r \leq \sin\theta,\ 0 \leq \theta \leq \pi.\}$ (2 pts)

(b)

$$\begin{aligned}
\gamma_x &= (1, 0, \frac{-x}{\sqrt{1 - x^2 - y^2}}) \\
\gamma_y &= (0, 1, \frac{-y}{\sqrt{1 - x^2 - y^2}}) \\
\gamma_x \times \gamma_y &= (\frac{x}{\sqrt{1 - x^2 - y^2}}, \frac{y}{\sqrt{1 - x^2 - y^2}}, 1) \quad \text{(2 pts)}
\end{aligned}$$

$$\iint_S F \cdot dS = \iint_D F \cdot \gamma_x \times \gamma_y dA$$

$$= \iint_D (y, -x, \sqrt{1-x^2-y^2}) \cdot (\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1) dA$$

$$= \iint_D \sqrt{1-x^2-y^2} dA$$

$$= \int_0^\pi \int_0^{\sin\theta} r\sqrt{1-r^2} dr d\theta \qquad \text{(5 pts)}$$

$$= \frac{-1}{2} \int_0^\pi \frac{2}{3}(1-r^2)^{\frac{3}{2}} \Big|_0^{\sin\theta} d\theta$$

$$= \frac{-1}{3} \int_0^\pi (1-\sin^2\theta)^{\frac{3}{2}} - 1 d\theta$$

$$= \frac{1}{3}\left[\pi - \int_0^\pi |\cos^3\theta| d\theta\right]$$

$$= \frac{\pi}{3} - \frac{4}{9}. \qquad \text{(2 pts)}$$

7. (12%) Verify the Stokes' theorem $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ for $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$, where $S$ is the surface of the paraboloid $z = 2 - (x^2 + y^2)$ above the $xy$-plane.

Sol:

(i) With parametric equation $\mathbf{r}(\theta) = \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, 0 \rangle,\ 0 \le \theta < 2\pi$, (2 pts)

$\mathbf{r}'(\theta) = \langle -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta, 0 \rangle$, (1 pt)

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 2y, 3x, -z^2 \rangle$, $C = \{(x, y, 0) \mid x^2 + y^2 = 2\}$.

$$\text{LHS} = \int_0^{2\pi} \langle\ 2(\sqrt{2}\sin\theta),\ 3(\sqrt{2}\cos\theta),\ 0\ \rangle \cdot \mathbf{r}'(\theta) d\theta \quad \text{(1 pt)}$$

$$= \int_0^{2\pi} (-4\sin^2\theta + 6\cos^2\theta) d\theta \ = \ 2\pi. \quad \text{(1 pt)}$$

(ii) Parametric equation of S: $\mathbf{r}(x, y) = \langle\ x,\ y,\ 2 - x^2 - y^2\ \rangle$ on $D = \{(x, y) \mid x^2 + y^2 \le 2\}$, (2 pts) where $\mathbf{r_x} \times \mathbf{r_y} = \langle 2x, 2y, 1 \rangle$. (1 pt)

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x & \partial y & \partial z \\ 2y & 3x & -z^2 \end{vmatrix} = \langle 0, 0, (3-2) \rangle = \langle 0, 0, 1 \rangle. \quad \text{(2 pts)}$$

$$\text{RHS} = \iint_D \langle 0,0,1\rangle \cdot (\mathbf{r_x} \times \mathbf{r_y}) dA \quad (1 \text{ pt})$$
$$= \iint_D 1 dA = \text{ area of } D = 2\pi. \quad (1 \text{ pt})$$

(iii) Since

$$\text{LHS} = \oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \text{RHS},$$

the Stokes' theorem is verified.

Note: If you only evaluate the integral of LHS or RHS, or you "apply" the Stokes' theorem to get the value of the integral, at most 5 pts will be given.

8. (14%) Let $D$ be the ball $x^2+y^2+z^2 \le 1$ and $E$ be the ellipsoid region $x^2+2y^2+3z^2 \le 4$. Let $\mathbf{F}(x,y,z) = \left(\frac{x}{r^3}+x\right)\mathbf{i} + \left(\frac{y}{r^3}+y\right)\mathbf{j} + \left(\frac{z}{r^3}+z\right)\mathbf{k}$, where $r = \sqrt{x^2+y^2+z^2}$.

(a) Find the outward flux of $\mathbf{F}$ across the boundary of $D$.

(b) Find the outward flux of $\mathbf{F}$ across the boundary of $E$.

Sol:

(a) (法一) 參數化並求積分, 觀察單位球之外法向量即爲 $(x,y,z)$ ,

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{A} = \iint_{\partial D} \frac{x^2+y^2+z^2}{r^3} + x^2+y^2+z^2 \, dA = \iint_{\partial D} \frac{1}{r} + r^2 \, dA = 2 \cdot 4\pi \cdot 1^2 = 8\pi$$

(法二) 若要使用 divergence theorem , 由於 $(0,0,0)$ 爲 div$\mathbf{F}$ 的奇異點, 因此必須先取一個範圍挖掉原點。取 $B_\epsilon$ 爲以原點爲球心、半徑 $\epsilon$ 的球面, 由 divergence theorem 知

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{A} - \iint_{\partial B_\epsilon} \mathbf{F} \cdot d\mathbf{A} = \iiint_{D \setminus B_\epsilon} \text{div}\mathbf{F} \, dV$$

注意到左式後者之法向量爲朝向原點, 故爲負號, 經移項至等號右邊。另外計算 div$\mathbf{F} = 3$:

$$\begin{aligned}
\text{div}\mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{x}{r^3}+x\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}+y\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}+z\right) \\
&= \left(\frac{1}{r^3} + (-3)\frac{x}{r^4}\frac{x}{r} + 1\right) + \left(\frac{1}{r^3} + (-3)\frac{y}{r^4}\frac{y}{r} + 1\right) + \left(\frac{1}{r^3} + (-3)\frac{y}{r^4}\frac{y}{r} + 1\right) \\
&= \frac{3}{r^3} - \frac{3}{r^3} + 3 = 3
\end{aligned}$$

以及小球面上的單位法向量爲 $\frac{1}{\epsilon}(x,y,z)$ 。因此有

$$\iint_{\partial D} \mathbf{F}\cdot d\mathbf{A} = \iiint_{D\setminus B_\epsilon} 3dV + \iint_{\partial B_\epsilon} \left(\frac{1}{r}+r^2\right)\frac{1}{\epsilon}\ dA$$
$$= 3(\frac{4}{3}\pi 1^3 - \frac{4}{3}\pi\epsilon^3) + (\frac{1}{\epsilon}+\epsilon^2)\frac{1}{\epsilon}(4\pi\epsilon^2) = 8\pi \quad \text{(可取 } \epsilon\to 0 \text{ 再來計算)}$$

(b) 由於在橢球面上積分較爲困難, 剛好 $\text{div}\mathbf{F} = 3$ 形式簡單, 因此應用 divergence theorem 簡化計算, 橢球體積可直接利用公式 $\frac{4}{3}\pi abc$ , 可得

$$\iint_{\partial E} \mathbf{F}\cdot\ d\mathbf{A} = \iiint_{E\setminus D} \text{div}\mathbf{F}\ dV + \iint_{\partial D} \mathbf{F}\cdot\ d\mathbf{A}$$
$$= 3\left(\frac{4}{3}\pi 2\sqrt{2}\frac{2}{\sqrt{3}} - \frac{4}{3}\pi\right) + 8\pi$$
$$= \frac{16\sqrt{6}}{3}\pi + 4\pi$$

[評分標準]

(a) 小題 5 分: 列式佔 1 ∼ 2 分, 計算錯誤或者球面表面積記錯以 5 分倒扣 1∼ 2 分計。

(b) 小題 9 分: div**F** 的計算佔 2 分; 面積分參數化佔筆墨分 2 分; 使用變數變換計算 Jacobian 及橢球面積, 或者直接使用公式 $\frac{4}{3}\pi abc$ 佔 2 分; 正確使用 divergence theorem 佔 5 ∼ 6 分, 過程正確但有筆誤或者其他小錯誤將以 9 分到扣 0 ∼ 3 分計。未正確使用 divergence theorem, 其前述其他地方所得之分數不會超過 5 分。