991微甲08-12班期中考解答和評分標準

1. (10%) Given $g(2) = 4$, $f(2) = 2$ and $g'(x) = \sqrt{x^2+5}$, $f'(x) = \sqrt{x^3+1}$ for all $x > 0$, find the derivative of $g(f(x))$ at $x = 2$.

   Sol:

   $\frac{d}{dx}g(f(x))\Big|_{x=2} = g'(f(2))f'(2) = 9. \quad (10\%)$

2. (10%) Find $f'(x)$, where $f(x) = \int_{2-x}^{x^2} \frac{dt}{t^3+1}$.

   Sol:

   Let $u = x^2, v = 2-x$, then

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\int_v^u \frac{dt}{t^3+1} \\
&= \frac{d}{du}\int_0^u \frac{dt}{t^3+1} * \frac{du}{dx} + \frac{d}{dv}\int_v^0 \frac{dt}{t^3+1} * \frac{dv}{dx} \\
&= \frac{2x}{x^6+1} + \frac{1}{(2-x)^3+1}. \quad (10\%)
\end{aligned}
$$

3. (10%) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1,1)$ of the curve $x^3 + x^2y + 4y^2 = 6$.

   Sol:

   Let y=y(x), then take derivative of $x^3 + x^2y + 4y^2 = 6$ on both side over x variable , we get

$$
3x^2 + 2xy + x^2y' + 8yy' = 0 \qquad \text{(a)}
$$

   .Then substitute x=1,y=1 into this equation we obtain $3+2+y'+8y' = 0$,i.e,$y' = -5/9$. Taking derivative of $3x^2 + 2xy + x^2y' + 8yy' = 0$ again we have second derivative of y, it become

$$
6x + 2y + 2xy' + 2xy' + x^2y'' + 8(y')^2 + 8yy'' = 0 \qquad \text{(b)}
$$

   . When evaluating at $(1,1)$ point, and using $y'(1) = -5/9$ from above, we have $6 + 2 + 2(\frac{-5}{9}) + 2(\frac{-5}{9}) + y'' + 8(\frac{-5}{9})^2 + 8y'' = 0$. That is, $y''(1) = -\frac{668}{729}$.

   Correction standard for this problem:

   (1) Correct answer got whole points.Equations for (a) and (b) cost 4 points.

   (2) If you just lost one to two terms and you knew the calculation of chain rule along your calculation, it would cost you one to two points.

   (3) Methods that are different from above are OK. And your calculation will be graded by the same principles as (1) and (2).

4. (12%) A particle is moving along the curve $y = \sin^{-1}(\frac{x}{2})$. As the particle passes the point $(\sqrt{2}, \frac{\pi}{4})$, its y-coordinate increases at a rate of $2\,\mathrm{cm/s}$. How fast is the distance from the particle to the origin changing at this moment?

Sol:

Let $x(t)$ and $y(t)$ be the x-coordinate and y-coordinate of the particle at time t, respectively. And t_0 be the time point such that $x(t_0) = \sqrt{2}$ and $y(t_0) = \frac{\pi}{4}$. Then the distance from the particle to the origin at time t is $z(t) = \sqrt{x^2(t) + y^2(t)}$. By differentiating z with respect to t, we have

$$z'(t_0) = \frac{1}{2}(x^2(t_0) + y^2(t_0))^{\frac{-1}{2}}(2x(t_0)x'(t_0) + 2y(t_0)y'(t_0))$$

where $y'(t_0) = 2$ and

$x'(t_0) = \frac{d}{dt}x(t)\Big|_{t=t_0} = \frac{d}{dt}2\sin y(t)\Big|_{t=t_0} = (2\cos y(t))y'(t)\Big|_{t=t_0} = (2)(\frac{\sqrt{2}}{2})(2) = 2\sqrt{2}.$

With all the information above, one can clearly get the solution to this problem

$$z'(t_0) = \frac{4 + \frac{\pi}{2}}{\sqrt{2 + \frac{\pi^2}{16}}}.$$

Grading Policy:

(1) Two points for successfully specifying the objective function $z(t)$.

(2) An extra of eight points for correctly computing $z'(t)$.

(3) You get the final two points for getting the above $z'(t_0)$.

5. (10%) Find $g'(e^{-1})$, where $g(x)$ is the inverse function of

$$f(x) = e^{\frac{-1}{\sqrt{x^2-1}}}, \qquad 1 < x < \infty.$$

Sol:

step 1 $g(x) = \sqrt{1 + (\log x)^2}$ (6%)

step 2 $g'(x) = -\left[1 + (\log x)^2\right]^{\frac{-1}{2}}\left(\log x\right)^{-3}\left(\frac{1}{x}\right)$ (2%)

step 3 $g'(e^{-1}) = \frac{1}{\sqrt{2}}e$ (2%)

6. (12%) Find the maximal volume of a cylindrical can (with top and bottom) with a fixed surface area A.

Sol:

$V = \pi r^2 h \quad A = 2\pi rh + 2\pi r^2$ (2%)

$(0 \le r \le \sqrt{\frac{A}{2\pi}})$ (2%)

$\Rightarrow h = \frac{A - 2\pi r^2}{2\pi r}$

$\Rightarrow V = \pi r^2 \frac{A - 2\pi r^2}{2\pi r} = \frac{A}{2} r - \pi r^3$ (3%)

$\Rightarrow V' = \frac{A}{2} - 3\pi r^2$

critical number: $V' = 0 \Rightarrow 3\pi r^2 = \frac{A}{2} \Rightarrow r = \sqrt{\frac{A}{6\pi}}$ $(\because r \ge 0)$ (3%)

$\because V(0) = 0, \quad V(\sqrt{\frac{A}{6\pi}}) = \frac{A}{3}\sqrt{\frac{A}{6\pi}}, \quad V(\sqrt{\frac{A}{2\pi}}) = 0$

$\Rightarrow V_{max} = \frac{A}{3}\sqrt{\frac{A}{6\pi}}$ (2%)

7. (26%) Given $f(x) = (2x^2 + 3x)e^{-x}$, Answer the following and **show** all your work. Each blank is worth 3%.

(a) The function is increasing on the interval(s) ____________________
and decreasing on the interval(s) ____________________.

(b) The function has local maxima at ____________________
and local minima at ____________________.

(c) The function is concave upward on the interval(s) ____________________
and concave downward on the interval(s) ____________________.

(d) The function has inflections at ____________________.

(e) The function has asymptotes ____________________.

(f) Sketch the graph of $f(x)$. (2%)

Sol:

$f(x) = (2x^2 + 3x)e^{-x}$ then

$f'(x) = (4x + 3)e^{-x} - (2x^2 + 3x)e^{-x} = -(2x^2 - x - 3)e^{-x} = -(x + 1)(2x - 3)\ e^{-x}$

and $f''(x) = -(4x - 1)e^{-x} + (2x^2 - x - 3)e^{-x} = (2x^2 - 5x - 2)e^{-x}$.

(a) $f'(x) = 0$ as $x = -1$ or $\frac{3}{2}$, $f'(x) > 0$ on $(-1, \frac{3}{2})$

and $f'(x) < 0$ on $(-\infty, -1)$, $(\frac{3}{2}, \infty)$

$\Rightarrow f$ is increasing on $(-1, \frac{3}{2})$ and decreasing on $(-\infty, -1)$ and $(\frac{3}{2}, \infty)$.

(b) f has local maxima $9e^{-\frac{3}{2}}$as $x = \dfrac{3}{2}$ and local minima $-e$ as $x = -1$.

(c) $f''(x) = 0$ as $x = \dfrac{5-\sqrt{41}}{4}$ or $\dfrac{5+\sqrt{41}}{4}$,

$f''(x) > 0$ on $(-\infty, \dfrac{5-\sqrt{41}}{4}), (\dfrac{5+\sqrt{41}}{4}, \infty)$ and $f''(x) < 0$ on $(\dfrac{5-\sqrt{41}}{4}, \dfrac{5+\sqrt{41}}{4})$

$\Rightarrow f$ is concave upward on $(-\infty, \dfrac{5-\sqrt{41}}{4}), (\dfrac{5+\sqrt{41}}{4}, \infty)$

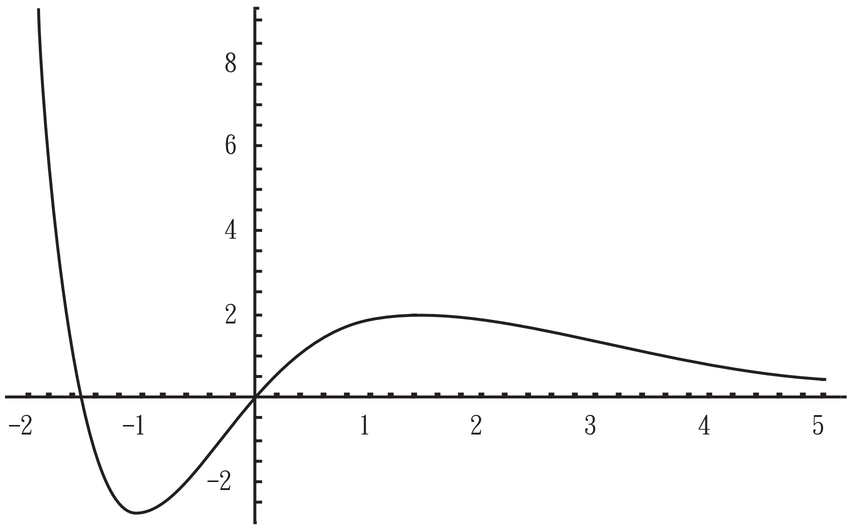
and concave downward on $(\dfrac{5-\sqrt{41}}{4}, \dfrac{5+\sqrt{41}}{4})$.

(d) $f''(x) = 0$ or $f''(x)$ doesn't exist and f'' changes sign at x

it follows that such $x = \dfrac{5+\sqrt{41}}{4}$ and $x = \dfrac{5-\sqrt{41}}{4}$ which become the inflection points of f.

(e) $\lim_{x\to\infty} f(x) = 0$ which implies f has $y = 0$ as an asymptote.

(f)



評分標準:

(a)∼(c)：每格3分。

(d)∼(f)：If there exists any mistakes in your answer, you get zero credit for that blank.

8. (10%) Find the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^7}{e^x}$.

(b) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^{x^2}$.

Sol:

(a) Using L'Hospital rule seven times, (2pts)

we get $\lim_{x\to\infty} \frac{x^7}{e^x} = \lim_{x\to\infty} \frac{7!}{e^x} = 0$ (3pts)

(b)

$$\begin{aligned}
\lim_{x\to\infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^{x^2} &= \exp(\lim_{x\to\infty} x^2 \ln(1 + \frac{3}{x} + \frac{5}{x^2})) \\
&= \exp(\lim_{x\to\infty} \frac{\ln(1 + \frac{3}{x} + \frac{5}{x^2})}{\frac{3}{x} + \frac{5}{x^2}} x^2(\frac{3}{x} + \frac{5}{x^2})) \\
&= \exp(\lim_{x\to\infty} \frac{\ln(1 + \frac{3}{x} + \frac{5}{x^2})}{\frac{3}{x} + \frac{5}{x^2}} \lim_{x\to\infty} x^2(\frac{3}{x} + \frac{5}{x^2})) \quad \text{(3pts)} \\
&= \exp(1 \cdot \infty) \quad \text{(2pts)} \\
&= \infty
\end{aligned}$$

Using L'Hospital rule is ok.