1. (14%) Determine whether the following limits exist. If the limit exists, evaluate it. If the limit does not exist, explain why?

(a) $\lim_{x\to 0}\left(\sin^{-1}x\right)\left(\sin\frac{1}{x}\right) =$ ______________________________.

(b) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) =$ ______________________________.

Sol:

(a) Since $\left|\sin\frac{1}{x}\right| \le 1,\ \forall x \ne 0$, (2%)

we have

$-\left|\sin^{-1}x\right| \le (\sin^{-1}x)(\sin\frac{1}{x}) \le \left|\sin^{-1}x\right|,\ \forall x \ne 0.$ (2%)

(missing absolute values: 1%)

Also, $\lim_{x\to 0}\left|\sin^{-1}x\right| = \sin^{-1}(0) = 0.$ (2%)

By the squeeze theorem, the limit exists and equals 0. (1%)

(b) Apply l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1}\frac{x\ln x-(x-1)}{(x-1)\ln x}\\ &= \lim_{x\to 1}\frac{\ln x}{\ln x+1-\frac{1}{x}}\\ &= \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}\end{aligned}$$

Note: Correctly using l'Hôpital's rule (2%), differentiation calculation (2%), the limit value (3%).

2. (12%) Let $f(x) = \begin{cases} \frac{\sin^2 ax}{x}, & x > 0 \\ |2x+1|-|2x-1|+b\cos x, & x \le 0, \end{cases}$ where $a,\ b$ are constants.

(a) For what values of $a$ and $b$ is $f(x)$ continuous at $x = 0$?

Answer: $a$ : ____________________, $b$ : ____________________.

(b) For what values of $a$ and $b$ is $f(x)$ differentiable at $x = 0$?

Answer: $a$ : ____________________, $b$ : ____________________.

Sol:

(a) $f$ is continuous at $x=0$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \qquad (1\%)$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\sin^2 ax}{x} = \lim_{x\to 0^-} |2x+1| - |2x-1| + b\cos x = 1 - 1 + b\cos 0 \qquad (1\%)$$

If $a \neq 0$

$$\Rightarrow \lim_{x\to 0^+} a^2 x \left(\frac{\sin ax}{ax}\right)^2 = \lim_{x\to 0^-} (2x+1) - (1-2x) + b\cos x = b \qquad (2\%)$$

$$\Rightarrow 0 \cdot 1^2 = b = b \ \Rightarrow\ b = 0$$

If $a = 0 \ \Rightarrow\ f(x) = 0 \ \text{ for } \ x > 0 \qquad (1\%)$

$$\Rightarrow \lim_{x\to 0^+} 0 = b \ \Rightarrow\ b = 0$$

$$\Rightarrow\ a \in \mathbb{R}, \quad b = 0 \qquad (1\%)$$

(b) $f$ is differentiable at 0

$\Rightarrow$ $f$ is continuous at 0 and, by (a), $b = 0$, $f(0) = 0$ (1%)

(or you may try another way to get this credit)

And by definition

$$\Rightarrow \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} \qquad (1\%)$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\sin^2 ax}{x^2} = \lim_{x\to 0^-} \frac{|2x+1| - |2x-1|}{x} \qquad (1\%)$$

If $a \neq 0$

$$\Rightarrow \lim_{x\to 0^+} a^2 \left(\frac{\sin ax}{ax}\right)^2 = \lim_{x\to 0^-} \frac{4x}{x} \qquad (1\%)$$

$$\Rightarrow a^2 \cdot 1^2 = 4 \ \Rightarrow\ a = \pm 2 \qquad (1\%)$$

If $a = 0 \ \Rightarrow\ f(x) = 0 \ \text{ for } \ x > 0 \qquad (1\%)$

$$\Rightarrow \lim_{x\to 0^+} 0 = 4 \quad \rightarrow\leftarrow$$

$$\Rightarrow\ a = \pm 2, \quad b = 0$$

3. (8%) The equation of the tangent line of the curve $\sin^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(xy)$ at $\left(\sqrt{2}, \frac{\sqrt{6}}{2}\right)$

is ________________________________________.

Sol:

$\sin^{-1}(\frac{y}{x}) = \tan^{-1} xy$

Implicit differentiation$\Rightarrow$ $\frac{1}{\sqrt{1-(\frac{y}{x})^2}} \cdot \frac{x \cdot y' - y \cdot 1}{x^2} = \frac{1}{1+(xy)^2} \cdot (1 \cdot y + x \cdot y')$

Let $(x, y) = (\sqrt{2}, \frac{\sqrt{6}}{2})$ $\Rightarrow$ $\frac{1}{\sqrt{1-(\frac{\sqrt{3}}{2})^2}} \cdot \frac{\sqrt{2}y' - \frac{\sqrt{6}}{2}}{2} = \frac{1}{1+(\sqrt{3})^2} \cdot (\frac{\sqrt{6}}{2} + \sqrt{2}y')$

$\Rightarrow$ $y' = \frac{5\sqrt{3}}{6}$ =slope at $(\sqrt{2}, \frac{\sqrt{6}}{2})$

$\Rightarrow$ The tangent line : $y - \frac{\sqrt{6}}{2} = \frac{5\sqrt{3}}{6}(x - \sqrt{2})$ (or $5\sqrt{3}x - 6y = 2\sqrt{6}$)

5 points for the implicit differentiation.

2 points for the computation of $y' = \frac{5\sqrt{3}}{6}$.

1 point for the equation of the tangent line.

4. (8%) Let $f$ be a continuous function satisfying

$$\int_0^x f(t)\, dt + \int_0^{x^3} e^{-t} f(\sqrt[3]{t})\, dt = xe^{2x} + \pi^x + x^\pi \text{ for all } x > 0.$$

The explicit formula for $f(x)$ is ________________________________.

Sol:

$\int_0^x f(t)dt + \int_0^{x^3} e^{-t} f(\sqrt[3]{t})dt = xe^{2x} + \pi^x + x^\pi$, $x > 0$

Differentiate both side $\Rightarrow$ $f(x) + e^{-x^3} f(\sqrt[3]{x^3}) \cdot 3x^2 = e^{2x} + xe^{2x} \cdot 2 + \pi^x \cdot \ln x + \pi x^{\pi-1}$

$\Rightarrow f(x)(1 + 3x^2 e^{-x^3}) = (1+2x)e^{2x} + \pi^x \ln x + \pi x^{\pi-1}$

$\Rightarrow f(x) = \frac{(1+2x)e^{2x} + \pi^x \ln x + \pi x^{\pi-1}}{1+3x^2 e^{-x^3}}$

1 point for $\frac{d}{dx}\int_0^x f(t)dt = f(x)$.

2 points for $\frac{d}{dx}\int_0^{x^3} e^{-t} f(\sqrt[3]{t})dt = e^{-x^3} f(\sqrt[3]{x^3}) \cdot 3x^2 = 3x^2 e^{-x^3} f(x)$.

1 point for $\frac{d}{dx} xe^{2x} = (1+2x)e^{2x}$.

3 points for $\frac{d}{dx}(\pi^x + x^\pi) = \pi^x \ln x + \pi x^{\pi-1}$.

1 point for the final answer $f(x) = \frac{(1+2x)e^{2x} + \pi^x \ln x + \pi x^{\pi-1}}{1+3x^2 e^{-x^3}}$.

5. (12%) (a) $\int \frac{\cos(\ln x)}{x}\, dx =$ ________________________________.

(b) Let $f(x) = \begin{cases} \dfrac{1}{1+x^2}, & -1 \le x \le 0 \\ \sec^2 x, & 0 < x \le 1. \end{cases}$

Then $\displaystyle\int_{-1}^{1} f(x)\,dx =$ ____________________________________.

Sol:

(a) Let $u = \ln x \Rightarrow du = \dfrac{1}{x}\,dx$, (2%)

then $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \int \cos u du = \sin u + C = \sin(\ln x) + C$, $C$ is a constant. (2%)

(b) Since $\displaystyle\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)\,dx$

and

$$\int_{-1}^{0} f(x)dx = \int_{-1}^{0} \frac{1}{1+x^2}dx = \tan^{-1} x\Big|_{-1}^{0} (2\%) = \tan^{-1} 0 - \tan^{-1}(-1) = 0 - \frac{-\pi}{4} = \frac{\pi}{4} \quad (2\%)$$

and

$$\int_{0}^{1} f(x)dx = \int_{0}^{1} \sec^2 x dx = \tan x\Big|_{0}^{1} (2\%) = \tan 1 - \tan 0 = \tan 1 - 0 = \tan 1. \quad (2\%)$$

Therefore, $\displaystyle\int_{-1}^{1} f(x)dx = \frac{\pi}{4} + \tan 1.$

6. (14%) Two different methods are applied to estimate $\ln(1.2)$ as follows.

(a) The linear approximation of $\ln x$ at $x = 1$ is ____________________________.

Use this to estimate $\ln(1.2)$.

Answer: $\ln(1.2) \approx$ ____________________________ by linear approximation.

(b) Apply Mean Value Theorem to show that $\dfrac{1}{1+x} < \dfrac{\ln(1+x)}{x} < 1$ for $x > 0$.

(c) Use the result in (b) to find an interval in which $\ln(1.2)$ is located. Then use the midpoint of this interval to estimate $\ln(1.2)$.

Answer: $\ln(1.2) \approx$ ____________________________ by mean value theorem.

Sol:

(a) (1) $L[\ln(x)] = \ln(1) + \dfrac{d}{dx}\ln(x)\Big|_{x=1} \cdot (x-1) = x - 1$ (3pts)

(2) $L[\ln(1.2)] = 0.2$ (3pts)

(b) $\ln(y+1)$ is continuous on $[0, x]$, differentiable on $(0.x)$.

By mean value theorm,

there exists a constant $c$ in $(0, x)$ such that $\dfrac{\ln(1+x)-\ln(1)}{x} = \dfrac{1}{1+c} < 1.$

Also, $\dfrac{1}{1+c} > \dfrac{1}{1+x}$, so $\dfrac{1}{1+x} < \dfrac{\ln(1+x)}{x} < 1$ (5pts)

(c) Using (b), $\dfrac{1}{1.2} < \dfrac{\ln(1.2)}{0.2} < 1$, $\dfrac{1}{6} < \ln(1.2) < \dfrac{1}{5}$.

Take midpoint, $\ln(1.2) \approx \dfrac{11}{60}$ (3pts)

7. (12%) Let $L$ be any line through the point $(3, 24)$.

(a) The equation of the line $L$ that cuts off the least area from the first quadrant is ________________________.

The least area is ________________________.

(b) The equation of the line $L$ which cuts off shotrest segment by the first quadrant is ________________________.

The length of the shortest segment is ________________________.

Sol:

(a) Line equation: $y - 24 = -8(x-3)$ (3 points)

the least area is 144 (3 points)

(b) Line equation: $y - 24 = -2(x-3)$ (3 points)

the shortest segment length is $15\sqrt{5}$ (3 points)

Let $y - 24 = m(x-3)$ be a line $L$ through $(3, 24)$ with slope $m < 0$

(No need to consider vertical line $m = \pm\infty$ and horizontal line $m = 0$)

x-intercept of L: $3 - \dfrac{24}{m}$

y-intercept of L: $24 - 3m$

(a) Area $A(m) = \dfrac{1}{2}(3 - \dfrac{24}{m})(24-3m) = \dfrac{9}{2}(16 - m - \dfrac{64}{m}) = \dfrac{9}{2}(16 - f(m))$

where $f(m) = m + \dfrac{64}{m}$, $f'(m) = 1 - \dfrac{64}{m^2} = 0$, $m^2 = 64$, $m = \pm 8$

Since $m < 0$ and $f'(m) > 0$ when $m < -8$ and $f'(m) < 0$ when $-8 < m < 0$,

$f$ has an absolute maximum at $m = 8$, $f(-8) = -16$

$\therefore$ The least area $= \dfrac{9}{2}(16 - (-16)) = 144$

$L : y - 24 = -8(x-3)$ or $8x + y = 48$

(b) Squared segment length

$L(m) = (3 - \frac{24}{m})^2 + (24 - 3m)^2 = 9[65 - \frac{16}{m} - 16m + \frac{64}{m^2} + m^2] = 9[65 - g(m)]$
where $g(m) = 16m + \frac{16}{m} - m^2 - \frac{64}{m^2}$,

$$g'(m) = 16 - 2m - \frac{16}{m^2} + \frac{128}{m^3} = \frac{16m^3 - 2m^4 - 16m + 128}{m^3} = \frac{-2}{m^3}[(m^3+8)(m-8)],$$

$m < 0$

Since $g'(m) > 0$ when $m < -2$ and $g'(m) < 0$ when $-2 < m < 0$, $m = -2$

The length of the shortest segment $= \sqrt{(3+12)^2 + (24+6)^2} = 15\sqrt{5}$

$L : y - 24 = -2(x - 3)$ or $y + 2x = 30$

• Four answers, 3 points each. But if the answer is correct but the procedure is wrong, no points will be credited.

* For the area in part (a), 2 points will be credited if your answer differs from the correct answer only by a simple factor (e.g. if your answer is 288).

* If your answer of any line equation is in the form of $\frac{y-a}{x-b} = m$, 1 point will be deducted.

• Partial credits will be given only if the answers are all wrong:

Parameterization of x-intercept and y-intercept: 1 point each, not given twice for (a) and (b).

Formulation of area (in terms of a single parameter) in (a): 1 point.

Formulation of segment length (in terms of a single parameter) in (b): 1 point.

8. (20%) Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Find the horizontal asymptotes if exist. Ans: ____________________.

(b) $f'(x) =$ ____________________.

(c) $y = f(x)$ is increasing on interval(s) ____________________.

$y = f(x)$ is decreasing on interval(s) ____________________.

(d) $f''(x) =$ ____________________.

(e) $y = f(x)$ is concave upward on interval(s) ____________________.

$y = f(x)$ is concave downward on interval(s) ________________________.

(Hint: $f''(x) = 0 \Leftrightarrow x = -1$)

(f) If the extreme values exist,

$f(x)$ has a local maximum: ________________ at $x =$ ________________.

$f(x)$ has a local minimum: ________________ at $x =$ ________________.

(g) $f(x)$ has inflection point(s): ________________________.

(h) Sketch the graph of $y = f(x)$.

Sol:

(a) $\lim_{x\to\infty} |x|^x = \infty^\infty = \infty$,........................................(1 point)

$\lim_{x\to-\infty} |x|^x = \lim_{x\to-\infty} e^{x\ln|x|} = e^{\lim_{x\to-\infty} x\ln|x|}$

$= e^{-\infty\cdot\infty} = e^{-\infty} = 0$,........................................(1 point)

Therefore, $y = 0$ is a horizontal asymptotes of $f(x)$.

(b) When $x \neq 0$, $y = |x|^x$

$\ln y = \ln|x|^x = x\ln|x|$

$\frac{y'}{y} = \ln|x| + \frac{x}{|x|}\frac{d}{dx}|x| = \ln|x| + 1$

$\Rightarrow y' = f'(x) = |x|^x(\ln|x| + 1)$ for $x \neq 0$

When $x = 0$,

$$\begin{aligned}\lim_{x\to0} f(x) &= \lim_{x\to0} |x|^x = \lim_{x\to0} e^{x\ln|x|} \\ &= e^{\lim_{x\to0} x\ln|x|} = e^{\lim_{x\to0} \frac{\ln|x|}{\frac{1}{x}}} \\ &= e^{\lim_{x\to0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{\lim_{x\to0} -x} \\ &= e^0 = 1 = f(0).\end{aligned}$$

Therefore, $f(x)$ is continuous at 0.

Since $f'(x) = |x|^x(\ln|x| + 1) \to -\infty$ as $x \to 0$,

So, $f'(0)$ doesn't exist.

Thus, $f'(x) = |x|^x(\ln|x| + 1)$, when $x \neq 0$, ..................(2 point)

$f'(x) = -\infty$, when $x = 0$. ................................(1 point)

(c) $f'(x) > 0 \Rightarrow x > e^{-1}$ or $x < -e^{-1}$

$\Rightarrow f(x)$ is increasing on $(-\infty, -e^{-1}) \cup (e^{-1}, \infty)$...............(1 point)

$f'(x) < 0 \Rightarrow -e^{-1} < x < e^{-1}$

$\Rightarrow f(x)$ is decreasing on $(-e^{-1}, e^{-1})$........................(1 point)

(d) $f'(x) = |x|^x(\ln|x| + 1)$ for $x \neq 0$

$\Rightarrow f''(x) = |x|^x(\ln|x| + 1)^2 + \dfrac{1}{x}|x|^x$ for $x \neq 0$..................(2 point)

(e) $f''(x) > 0 \Rightarrow x > 0$ or $x < -1$

$\Rightarrow f(x)$ is concave upward on $(-\infty, -1) \cup (0, \infty)$...............(2 point)

$f''(x) < 0 \Rightarrow -1 < x < 0$

$\Rightarrow f(x)$ is concave downward on $(-1, 0)$.......................(1 point)

(f) From (c),

$f(x)$ has a local maximum $f(-\dfrac{1}{e}) = e^{\frac{1}{e}}$ at $x = -\dfrac{1}{e}$............(2 point)

$f(x)$ has a local minimum $f(\dfrac{1}{e}) = e^{-\frac{1}{e}}$ at $x = \dfrac{1}{e}$...............(2 point)

(g) From (e),

$f(x)$ has inflection points $(-1, 1)$, $(0, 1)$.......................(1 point)

(h) 1. 畫出漸近線$y = 0$與$\lim\limits_{x\to\infty} |x|^x = \infty$........................(1 point)

2. 標出極值點與函數在$(0, 1)$的鉛直切線........................(1 point)

3. 正確的畫出函數的上凹與下凹.............................(1 point)