982微積分乙01-05班期中考解答與評分標準

1. (10%) 設 $f(x,y)=2x^2-3xy-2y^2$ 。

   (a) 求 $f(x,y)$ 在點 $(1,2)$ 處沿 $(3,4)$ 方向之方向導數 。

   (b) 在點 $(1,2)$ 處 $f(x,y)$ 沿哪一方向，方向導數最大 ?

   Sol:

   (a) $\frac{\partial f}{\partial u} \restriction_{(a,b)} = \nabla f(a,b)\cdot \overrightarrow{u} = (4x-3y,-3x-4y)\restriction_{(1,2)} \cdot(\frac{3}{5},\frac{4}{5}) = (-2,-11)\cdot(\frac{3}{5},\frac{4}{5}) = -10$
   $\nabla f(a,b)$ for 2 points, u for 2 points, the inner product for 1 point

   (b) by text book's theorem, we have the answer $\nabla f(a,b)=(-2,-11)$ this for 5 points;
   but if you know this, you have an error computation, then you get 3 points;
   if you know this, you have no computation, then you get 1 points.

2. (10%) 求曲面 $\ln(1+x^2-y^2)+\sin^2 z=0$ 在點 $(1,1,0)$ 處之切面方程 。

   Sol:

   $\frac{\partial f}{\partial x}=\frac{2x}{1+x^2-y^2}$ (2pts)

   $\frac{\partial f}{\partial y}=\frac{-2y}{1+x^2-y^2}$ (2pts)

   $\frac{\partial f}{\partial z}=2\sin z\cos z$ (2pts)

   $(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})\Big|_{(1,1,0)}=(-2,-2,0)$ (2pts)

   $2(x-1)-2(y-1)+0(z-0)=0\Rightarrow x-y=0$ (2pts)

3. (10%) 求函數 $f(x,y)=x^2+xy-y^2-2x+y+5$ 之候選點 。並決定其是否極大 、極小或鞍點 。

   Sol:

   Let $\begin{cases} f_x(x,y)=2x+y-2=0 \\ f_y(x,y)=x-2y+1=0 \end{cases}$ (each 2 points)

   $\Rightarrow$the only candidate(critical point) is $(\frac{3}{5},\frac{4}{5})$ (1 point)

   $D(\frac{3}{5},\frac{4}{5})=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix}(\frac{3}{5},\frac{4}{5})=\begin{vmatrix} 2 & 1 \\ 1 & -2\end{vmatrix}=-5$ (1 point)

$D < 0 \Rightarrow (\frac{3}{5}, \frac{4}{5})$ is a saddle point. (4 points)

4. (15%) 用 Lagrange 乘子法求 $f(x,y) = x^2 + xy + y^2$ 在 $x^2 + y^2 = 1$ 上之最大值及最小值 。

Sol:

Step 1. (3 points)

Let $g(x,y) = x^2 + y^2 - 1$, and solve the following equation

$$\begin{cases} \bigtriangledown f(x,y) = \lambda \bigtriangledown g(x,y) \\ g(x,y) = 0 \end{cases}.$$

We have

$$\begin{cases} 2x + y = 2\lambda x \\ x + 2y = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}. \tag{1}$$

Step 2. (6 points)

From (1), we get

$$\begin{cases} x^2 = y^2 \\ x^2 + y^2 = 1 \end{cases}.$$

Hence,

$$(x,y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \vee (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \vee (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \vee (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}). \tag{2}$$

Step 3. (6 points)

In fact, $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{3}{2}$, and $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$.

Therefore, max $= \frac{3}{2}$ and min $= \frac{1}{2}$.

5. (10%) 求曲線 $r = 1 + \sin\theta$ 包圍之面積 。

Sol:

$$\begin{aligned} \text{area} &= \int_0^{2\pi} \int_0^{1+\sin(\theta)} r dr d\theta \quad \text{(the region of integral is 3 pts, } rdrd\theta \text{ is 2 pts)} \\ &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2}(1+\sin(\theta))^2 d\theta = \frac{1}{2}(\theta - \cos(\theta) + \frac{1}{2}\theta - \frac{-\sin(\theta)}{4})\Big|_0^{2\pi} \quad \text{(3 pts)} \\ &= \frac{3\pi}{2} \quad \text{(2 pts)} \end{aligned}$$

6. (15%) 求 $\int_0^1 \int_x^1 y^2 e^{xy}\, dy\, dx$ 之值。

Sol:

$$\begin{aligned}\int_0^1 \int_x^1 y^2 e^{xy} dydx &= \int_0^1 \int_0^y y^2 e^{xy} dxdy \qquad \text{(10 points)}\\ &= \int_0^1 ye^{xy}\Big|_0^y dy \qquad \text{(2points)}\\ &= \int_0^1 (ye^{y^2} - y)dy\\ &= (\frac{1}{2}e^{y^2} - \frac{1}{2}y^2)\Big|_0^1 \quad \text{(2points)}\\ &= \frac{1}{2}e - 1 \qquad \text{(1point)}\end{aligned}$$

7. (15%) 計算 $\iint\limits_{\Omega} \left(\sqrt{xy} + \sqrt{\frac{2y}{x}}\right) dA$，其中 Ω 爲 $xy = 1$，$xy = 9$，$y = x$，$y = 2x$ 所圍成，而在 $x > 0$，$y > 0$ 部分之區域。

Sol:

Set $u = xy, v = \frac{y}{x} \Rightarrow x = \sqrt{\frac{u}{v}}, y = \sqrt{uv}$. (3 pts)

And we have boundary with $xy = 1, xy = 9, y = x, y = 2x$.

$\Rightarrow 1 \le u \le 9, 1 \le v \le 2$. (2 pts)

And the determinant of Jacobian matrix is $\frac{1}{2v}$

(5 pts, 2 pts if there's computational error.)

$$\begin{aligned}\Rightarrow \iint_{\Omega} (\sqrt{xy} + \sqrt{\frac{2y}{x}})dxdy &= \int_1^2 \int_1^9 (\sqrt{u} + \sqrt{2v})(\frac{1}{2v})dudv \quad \text{(1pt)}\\ &= \int_1^2 \int_1^9 (\frac{\sqrt{u}}{2v} + \frac{1}{\sqrt{2v}})dudv\\ &= \int_1^2 (\frac{u^{\frac{3}{2}}}{3v} + \frac{u}{\sqrt{2v}}\Big|_1^9)dv\\ &= \int_1^2 (\frac{26}{3v} + \frac{4\sqrt{2}}{\sqrt{v}})dv\\ &= (\frac{26}{3}\ln v + 8\sqrt{2v})\Big|_1^2\\ &= \frac{26}{3}\ln 2 + 16 - 8\sqrt{2}\end{aligned}$$

(2pts for correct integral process, 2pts for the answer.)

8. (15%) 計算 $\iiint\limits_{\Omega} x\,dV$ ，其中 Ω 爲 $x=0$ ，$y=0$ ，$z=0$ 及 $x+\frac{y}{2}+\frac{z}{3}=1$ 所圍成之區域。

Sol:

Fix $x,\ y,\ z$ is valued from 0 to $3(1-x-\frac{y}{2})$.

Fix $x,\ y$ is valued from 0 to $2(1-x)$.

x is valued from 0 to 1.

Thus,

$$\begin{aligned}
\int\int\int x\,dV &= \int_0^1\int_0^{2(1-x)}\int_0^{3(1-x-\frac{y}{2})} x\,dzdydx \\
&= \int_0^1\int_0^{2(1-x)} xz\Big|_{z=0}^{z=3(1-x-\frac{y}{2})} dydx \\
&= \int_0^1\int_0^{2(1-x)} 3x-3x^2-\frac{3}{2}xy\,dydx \\
&= \int_0^1 3xy-3x^2y-\frac{3}{4}xy^2\Big|_{y=0}^{y=2(1-x)} dx \\
&= \int_0^1 3x^3-6x^2+3x\,dx \\
&= \frac{3}{4}x^4-2x^3+\frac{3}{2}x^2\Big|_{x=0}^{x=1} \\
&= \frac{1}{4}
\end{aligned}$$

Write down the exact integral domain : 8 points.

Write down the exact integral value : 7 points.

If your integral domain is not exact, there is no point for you. (sorry)

If your integral domain is exact but the integral value is not true, give you some points acorddng to the wrong extent.