982微積分甲01-05班期末考解答與評分標準

1. (10%) Let $f(x,y) = \dfrac{x^n + y^n}{2},\ x > 0,\ y > 0,\ n > 2,\ n \in \mathbb{N}$.

   (a) Apply the method of Lagrange multipliers to find the extreme values of the function $f(x,y)$ on the line $x + y = C$, where $C > 0$. (No credits if the method is not used.)

   (b) Use part (a) to prove the inequality $\dfrac{x^n + y^n}{2} \geq (\dfrac{x+y}{2})^n,\ x > 0,\ y > 0,\ n \in \mathbb{N}$.

Sol:

(a) Apply lagrange multiplier:

$\dfrac{nx^{n-1}}{2} = \lambda,\ \dfrac{ny^{n-1}}{2} = \lambda$

$x + y = C$

Slove it to get $(x,y) = (\dfrac{c}{2}, \dfrac{c}{2})$

Because $x + y = C$, $x \geqslant 0$, $y \geqslant 0$ is bounded and closed line segment

and $f(x,y)$is differentiable on this line segment.

It must contain maximum and minimun value.

Compare $(\dfrac{c}{2}, \dfrac{c}{2})$ with end points $(c,0)$, $(0,c)$

$f(\dfrac{c}{2}, \dfrac{c}{2}) = (\dfrac{c}{2})^n$ is minimum on $x + y = C$, $x \geqslant 0$, $y \geqslant 0$

$f(0,c) = f(c,0) = \dfrac{c^n}{2}$ is maximum on $x + y = C$, $x \geqslant 0$, $y \geqslant 0$

thus $f(\dfrac{c}{2}, \dfrac{c}{2}) = (\dfrac{c}{2})^n$ is minimum on $x + y = C$, $x > 0$, $y > 0$

(b) $n = 1$, $\dfrac{x+y}{2} \geqslant \dfrac{x+y}{2}$

$n = 2$, $\dfrac{x^2+y^2}{2} - (\dfrac{x+y}{2})^2 = \dfrac{(x-y)^2}{4} \geqslant 0$, so $\dfrac{x^2+y^2}{2} \geqslant (\dfrac{x+y}{2})^2$

$n > 2$, when $x + y = C$, $C > 0$. By (a), $\dfrac{x^n+y^n}{2} \geqslant (\dfrac{c}{2})^n = (\dfrac{x+y}{2})^n$

thus $\dfrac{x^n+y^n}{2} \geqslant (\dfrac{x+y}{2})^n$, $x > 0$, $y > 0$, $n \in \mathbb{N}$

評分標準:

Use lagrange method to find possible extreme values. (5 pts)

Verify $f(\dfrac{c}{2}, \dfrac{c}{2})$ is the extreme value(minimum). (3 pts)

prove inequality $\dfrac{x^n+y^n}{2} \geqslant (\dfrac{x+y}{2})^n$. (2 pts)

2. (8%) Evaluate $\displaystyle\int_0^1 \int_{\frac{y}{2}}^{y} y^3 e^{x^5} dx dy + \int_1^2 \int_{\frac{y}{2}}^{1} y^3 e^{x^5} dx dy$.

Sol:

Change the integral order (2 pts)

the domain of integral:$x \le y \le 2x,\ 0 \le x \le 1$

$$\int_0^1 \int_x^{2x} y^3 e^{x^5}\, dydx \quad (2 \text{ pts})$$
$$= \int_0^1 \frac{1}{4} y^4 e^{x^5} \Big|_{y=x}^{y=2x} dx$$
$$= \int_0^1 \frac{15}{4} x^4 e^{x^5}\, dx \quad (2 \text{ pts})$$
$$= \frac{3}{4} e^{x^5} \Big|_{x=0}^{x=1}$$
$$= \frac{3}{4}(e-1) \quad (2 \text{ pts})$$

3. (8%) Evaluate $\displaystyle\iint_D \frac{x^2}{\sqrt{x^2+y^2}}\, dA$, where $D = \{(x,y) \in \mathbb{R}^2 | 1 \le x^2+y^2 \le 4,\ y \ge x\}$.

Sol:

Using polar coordinate (1 pt)

$D = \{(r,\theta) | 1 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{5\pi}{4}\}$ (2 pts)

$$\int_{\pi/4}^{5\pi/4} \int_1^2 \frac{r^2 \cos^2\theta}{r} r drd\theta \quad (1 \text{ pt})$$
$$= \int_{\pi/4}^{5\pi/4} \frac{1}{3} r^3 \Big|_{r=1}^{r=2} \cos^2\theta d\theta \quad (1 \text{ pt})$$
$$= \frac{7}{3}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{\theta=\pi/4}^{\theta=5\pi/4} \quad (2 \text{ pts})$$
$$= \frac{7\pi}{6} \quad (1 \text{ pt})$$

4. (14%) (a) Evaluate $I_1 = \displaystyle\iint_{R_1} e^{-(x^2+xy+y^2)}\, dA$, where $R_1 = \{(x,y) | x^2+xy+y^2 \le 1\}$.

   (b) Evaluate $I_2 = \displaystyle\iint_{R_2} x^2y^2\, dA$, where R_2 is the region bounded by $xy = 1,\ xy = 2,\ y = x,\ y = 4x,$ and $x > 0,\ y > 0$.

Sol:

(a) Method1:

$x^2+xy+y^2 = (x+\frac{1}{2}y)^2 + \frac{3}{4}y^2$

Let $u = x + \frac{1}{2}y, v = \frac{\sqrt{3}y}{2},\quad x = u - \frac{v}{\sqrt{3}}, y = \frac{2v}{\sqrt{3}}$ (1 pt)

Preimage of R_1 is $R_1' = \{(u,v) | u^2+v^2 \le 1\}$ (1 pt)

$$J = \begin{vmatrix} 1 & \frac{-1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{vmatrix} = \frac{2}{\sqrt{3}} \quad \text{(2 pts)}$$

$$\begin{aligned} I_1 = \iint_{R_1} e^{-(x^2+xy+y^2)}\, dA &= \iint_{R_1'} e^{-(u^2+v^2)} \frac{2}{\sqrt{3}}\, dudv \\ &= \frac{2}{\sqrt{3}} \int_0^{2\pi} \int_0^1 e^{-r^2} r\, drd\theta \\ &= \frac{2}{\sqrt{3}} \int_0^{2\pi} \frac{1}{2}(1-\frac{1}{e})\, d\theta \\ &= \frac{2}{\sqrt{3}}\pi(1-\frac{1}{e}) \quad \text{(3 pts)} \end{aligned}$$

Method2:

Let $u = \frac{(x+y)}{\sqrt{2}}, v = \frac{-x+y}{\sqrt{2}}$ (1 pt)

$x^2+xy+y^2 = \frac{3}{2}u^2 + \frac{1}{2}v^2 \le 1$ (1 pt)

$$\begin{aligned} \iint_{R_1} e^{-(x^2+xy+y^2)}\, dA &= \iint_{\frac{3}{2}u^2+\frac{1}{2}v^2\le 1} e^{-(\frac{3}{2}u^2+\frac{1}{2}v^2)}\, dudv \\ &= \frac{1}{\sqrt{3}} \iint_{\eta^2+v^2\le 2} e^{-\frac{\eta^2+v^2}{2}}\, d\eta dv \quad \text{(2 pts)} \\ &= \frac{1}{\sqrt{3}} \int_0^{2\pi} \int_0^{\sqrt{2}} e^{-\frac{r^2}{2}} r\, drd\theta \\ &= \frac{2}{\sqrt{3}}\pi(1-\frac{1}{e}) \quad \text{(3 pts)} \end{aligned}$$

(b) Let $u = xy, v = \frac{y}{x}$

$x = \sqrt{\frac{u}{v}},\ y = \sqrt{uv}$ (1 pt)

Preimage of R_2 is $R_2' = \{(u,v) | 1 \le u \le 2, 1 \le v \le 4\}$ (1 pt)

$$J = \begin{vmatrix} \frac{1}{2}u^{\frac{-1}{2}}v^{\frac{-1}{2}} & \frac{-1}{2}u^{\frac{1}{2}}v^{\frac{-3}{2}} \\ \frac{1}{2}u^{\frac{-1}{2}}v^{\frac{-1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{\frac{-1}{2}} \end{vmatrix} = \frac{1}{2v} \quad \text{(2 pts)}$$

$$\begin{aligned} I_2 = \iint_{R_2} x^2y^2 dA &= \iint_{R_2} u^2 \frac{1}{2v}\, dudv \\ &= \frac{1}{2}\int_1^4 \int_1^2 u^2 v^{-1}\, dudv \\ &= \frac{1}{2}\int_1^4 \frac{7}{3}v^{-1} dv = \frac{7}{3}\ln 2 \quad \text{(3 pts)} \end{aligned}$$

5. (18%) Let $\mathbf{F}(x,y) = \left\langle \frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right\rangle$, $(x,y) \ne (0,0)$, and $H = \{(x,y)|y>0\}$ be the upper half plane.

(a) Compute $J_1 = \int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle with counterclockwise orientation. Is $\mathbf{F}$ conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$?

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any piecewise-smooth simple closed curve C in H.

(c) Suppose that Γ is a piecewise-smooth simple curve in H with initial point $P_1 = (r_1 \cos\theta_1, r_1 \sin\theta_1)$ and terminal point $P_2 = (r_2 \cos\theta_2, r_2 \sin\theta_2)$, $r_j > 0$, $0 < \theta_j < \pi$, $j = 1, 2$. Evaluate $J_2 = \int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ in terms of r_1, r_2, θ_1, and θ_2. (Hint. Try a path with constant r in one piece and constant θ in another piece.)

Sol:

(a) Let $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$, $0 \le \theta < 2\pi$, be the parametric equation for $\mathcal{C}$.

$$J_1 = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(\theta) d\theta = \int_0^{2\pi} (\cos^2\theta + \sin^2\theta) d\theta = 2\pi \neq 0$$

Therefore, $\mathbf{F}$ is NOT conservative on $\mathbb{R}^2 - \{(0,0)\}$.

(b) Let $\mathcal{C}$ be a piecewise-smooth simple closed curve in $\mathbf{H}$ and D be the region bounded by $\mathcal{C}$. Give $\mathcal{C}$ a counterclockwise orientation so that $\partial D = \mathcal{C}$. By Green's theorem, since $D \subseteq \mathbf{H}$, on which $\mathbf{F}$ is well-defined, and $\mathbf{H}$ is simply connected,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x+y}{x^2+y^2} \right) - \frac{\partial}{\partial y}\left(\frac{x-y}{x^2+y^2} \right) \right] dA = \iint_D 0 dA = 0$$

(c) It follows from (b) that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $\mathbf{H}$. Thus, to compute $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$, let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be a path connecting P_1 and P_2.

$$\mathcal{C}_1 : \mathbf{r}_1(t) = (r_1 \cos t, r_1 \sin t),\ t \text{ goes from } \theta_1 \text{ to } \theta_2$$

$$\mathcal{C}_2 : \mathbf{r}_2(t) = (t \cos\theta_2, t \sin\theta_2),\ t \text{ goes from } r_1 \text{ to } r_2$$

Then it's easy to see that $\mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) = 1$ and $\mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) = \frac{1}{t}$.

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1 \cup \mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\theta_1}^{\theta_2} 1 dt + \int_{r_1}^{r_2} \frac{1}{t} dt = \theta_2 - \theta_1 + \ln\left(\frac{r_2}{r_1}\right) = J_2$$

6. (12%) Evaluate $I = \int_C \frac{y^2}{\sqrt{R^2 + x^2}} dx + \left[4x + 2y \ln(x + \sqrt{R^2 + x^2}) \right] dy$, where C is the upper semicircle $x^2 + y^2 = R^2$, $y \ge 0$, $R > 0$, and is traversed from $A(-R, 0)$ to $B(R, 0)$. (Hint. Apply Green's Theorem.)

Sol:

We want to find I, where I is

$I = \int_C \frac{y^2}{\sqrt{R^2+x^2}}dx + [4x + 2y\log(x+\sqrt{R^2+x^2})]dy = \int_C Pdx + Qdy$

Worth 2 pts:

Before we do anything, we must deal with our curve C. In order to use Green's theorem we must make sure C satisfies all of the assumptions for the theorem, that is that C is a positively oriented, piecewise smooth, and simple closed curve. However, C does not satisfy these assumptions; hence we must make it so a line segment, call L, runs from $B(R, 0)$ to $A(-R, 0)$. Now we see that CUL is closed and piecewise smooth with clockwise orientation. Clockwise orientation is not positive orientation, so we will have a negative sign out in front.

Worth 4 pts:

Now note that $I = I + J$

J being:

$J = \int_L Pdx + Qdy$

Note that for J, $dy = 0 \Rightarrow = \int_L 0dx + Q * 0$

Hence $J = 0$

Worth 2 pts:

Now we can deal with the domain bounded by CUL, which we call D. By Green's Theorem:

$$\begin{aligned} I &= I + J \\ &= \int_{CUL} Pdx + Qdy \\ &= -\iint_D (Qx - Py)dA \end{aligned}$$

Worth 2 pts:

Finding Qx and Py: $Qx = 4 + \frac{2y}{\sqrt{R^2+x^2}}$, $Py = \frac{2y}{\sqrt{R^2+x^2}}$

Worth 2 pts:

Plugging the above into I gives: $I = -\iint_D 4dA = -2\pi R^2$

7. (12%) Evaluate $\iint_S \text{curl}\mathbf{G}\cdot d\mathbf{S}$, where $\mathbf{G}(x,y,z) = x^2yz\mathbf{i} + yz^2\mathbf{j} + z^3e^{xy}\mathbf{k}$, S is the part of the

sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

Sol:

Let $\mathcal{C}$ be the boundary of the surface S.

Then $\mathcal{C}$ can be parameterized as $(x, y, z) = (2\cos\theta, 2\sin\theta, 1),\ 0 \le \theta \le 2\pi$. (4 pts)

$G(x, y, z) = (8\sin\theta, 2\sin\theta, e^{4\sin\theta\cos\theta}),\ dr = (dx, dy, dz) = (-2\sin\theta, 2\cos\theta, 0)$.

By Stoke's theorem,

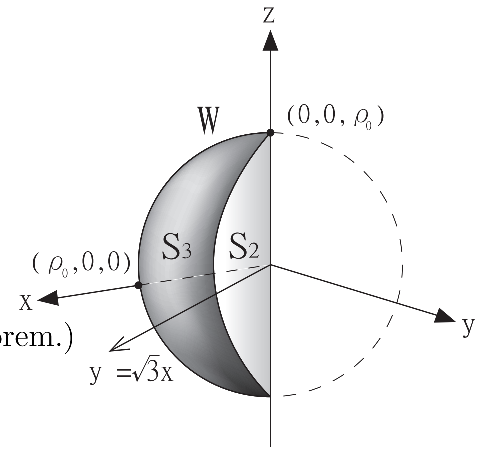
$$\begin{aligned}\iint_S \text{curl}\mathbf{G}\cdot d\mathbf{S} &= \int_{\mathcal{C}} \mathbf{G}\cdot d\mathbf{r} \quad \text{(4 pts)}\\ &= \int_0^{2\pi} -16\sin^2\theta\cos^2\theta + 4\sin\theta\cos\theta d\theta\\ &= \int_0^{2\pi} -4\sin^2 2\theta + 2\sin 2\theta d\theta\\ &= -4\pi \quad \text{(4 pts)}\end{aligned}$$

8. (18%) Let B be the ball centered at the origin with radius $\rho_0 > 0$, and W be the smaller wedge cut from B by two planes $y = 0$ and $y = \sqrt{3}x$. The boundary of W consists of 3 surfaces S_1, S_2, and S_3: $\partial W = S_1 \cup S_2 \cup S_3$, and is given with outward orientation. Here S_1 and S_2 are semidisks on $y = 0$ and $y = \sqrt{3}x$, respectively, and S_3 is on the boundary of B, a sphere of radius ρ_0. See the figure. Let $\mathbf{H}(x, y, z) = xz\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$.

(a) Find a parametric representation for the surface S_3.

(b) Compute $\displaystyle\iint_{S_3} \mathbf{H}\cdot d\mathbf{S}$.

(c) Compute $\displaystyle\iint_{S_1\cup S_2} \mathbf{H}\cdot d\mathbf{S}$. (Hint. Use the Divergence Theorem.)



Sol:

(a) Note that the surface S_3 is a piece of the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = \rho_0^2\}$. The spherical coordinate works. So let $x = \rho_0\sin\phi\cos\theta$, $y = \rho_0\sin\phi\sin\theta$, and $z = \rho_0\cos\phi$. Notice that $\theta \in [0, \pi/3]$ and $\phi \in [0, \pi]$. (2 pts)

The parametric representation is given by

$$r(\theta, \phi) = (\rho_0\sin\phi\cos\theta, \rho_0\sin\phi\sin\theta, \rho_0\cos\phi), \quad \theta \in [0, \pi/3],\ \phi \in [0, \pi]. \text{ (2 pts)}$$

(b) Since S_3 is a piece of the sphere, the outward normal of S_3 is given by

$$\mathbf{n}(x,y,z) = \frac{(x,y,z)}{\rho_0}, \qquad (x,y,z) \in S_3.$$

And $\mathbf{H}(x,y,z)\cdot\mathbf{n}(x,y,z) = \dfrac{y^2}{\rho_0}$. So the integral is given by

$$\iint_{S_3} \mathbf{H}\cdot d\mathbf{S} = \iint_{S_3} \mathbf{H}\cdot\mathbf{n}dS = \iint_{R} \frac{y^2}{\rho_0}\rho_0^2\sin\phi d\theta d\phi = \int_0^{\pi/3}\int_0^{\pi} \rho_0^3 \sin^3\phi \sin^2\theta d\phi d\theta \quad \text{(4 pts)}$$

Here $R = [0,\pi/3]\times[0,\pi]$. Note that

$$\int_0^{\pi/3}\int_0^{\pi} \rho_0^3\sin^3\phi\sin^2\theta d\phi d\theta = \rho_0^3\int_0^{\pi/3}\sin^2\theta d\theta \int_0^{\pi}\sin^3\phi d\phi$$

For the first integral, consider $\sin^2\theta = (1-\cos 2\theta)/2$.

$$\int_0^{\pi/3}\sin^2\theta d\theta = \int_0^{\pi/3}\frac{1-\cos 2\theta}{2}d\theta = \left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi/3} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

For the later one, we have

$$\int_0^{\pi}\sin^3\phi d\phi = -\int_0^{\pi}(1-\cos^2\phi)d\cos\phi = \left(\frac{1}{3}\cos^3\phi - \cos\phi\right)\Bigg|_0^{\pi} = \frac{4}{3}$$

So

$$\iint_{S_3}\mathbf{H}\cdot d\mathbf{S} = \frac{4}{3}\rho_0^3\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) \quad \text{(3 pts)}$$

(c) $\nabla\cdot\mathbf{H} = z+1$. So by divergence theorem,

$$\iint_{S_1\cup S_2}\mathbf{H}\cdot d\mathbf{S} + \iint_{S_3}\mathbf{H}\cdot d\mathbf{S} = \iiint_E \nabla\cdot\mathbf{H}dV = \iiint_E (z+1)dV, \quad \text{(4 pts)}$$

where E is the solid bounded by $S_1\cup S_2\cup S_3$. Firstly, note that E is symmetric with respect to xy-plane. Therefore,

$$\iiint_E zdV = 0$$

So

$$\iiint_E (z+1)dV = \iiint_E dV = \frac{4}{3}\pi\rho_0^3\frac{\pi/3}{2\pi} = \frac{2}{9}\pi\rho_0^3 \quad \text{(3 pts)}$$

since the volume of E is equal to the volume of the sphere times $\dfrac{\pi/3}{2\pi} = 1/6$. Now

$$\iint_{S_1\cup S_2}\mathbf{H}\cdot d\mathbf{S} = \iiint_E\nabla\cdot\mathbf{H}dV - \iint_{S_3}\mathbf{H}\cdot d\mathbf{S} = \frac{2}{9}\pi\rho_0^3 - \frac{4}{3}\rho_0^3\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) = \frac{\sqrt{3}}{6}\rho_0^3$$