98學年度第1學期 微積分甲二組期中考解答

1. (10%) Let $f(x) = e^x \cdot \ln(2 + \sin x)$.

   (a) Find $f'(x)$.

   (b) Find $f'(0)$.

   Sol:

   (a) $f'(x) = e^x \ln(2 + sinx) + e^x \dfrac{cosx}{2 + sinx}$

   (b) $f'(0) = 1 \cdot \ln 2 + 1 \cdot \dfrac{1}{2} = \ln 2 + \dfrac{1}{2}$

2. (10%) (a) Find $\lim\limits_{x \to 5} \dfrac{2^x - 32}{x - 5}$.

   (b) Find $\lim\limits_{x \to -\infty} \sqrt{x^2 + 4x + 5} - \sqrt{x^2 + x + 1}$.

   Sol:

   (a)

$$\lim_{x \to 5} \frac{2^x - 32}{x - 5} = \frac{d}{dx} 2^x |_{x=5}$$

   or you can use the L'Hospital's rule after identifying the limit is of the indeterminate form $\frac{0}{0}$, Anyhow, the answer must have to do with the differentiation of 2^x at $x = 5$, which is

$$\frac{d}{dx} 2^x |_{x=5} = 2^x \ln 2 |_{x=5} = 32 \ln 2$$

   (b) We first do some simplification:

$$\sqrt{x^2 + 4x + 5} - \sqrt{x^2 + x + 1} = \frac{(x^2 + 4x + 5) - (x^2 + x + 1)}{\sqrt{x^2 + 4x + 5} + \sqrt{x^2 + x + 1}} = \frac{3x + 4}{\sqrt{x^2 + 4x + 5} + \sqrt{x^2 + x + 1}}$$

   Hence we have

$$\begin{aligned} \lim_{x \to -\infty} \sqrt{x^2 + 4x + 5} - \sqrt{x^2 + x + 1} &= \lim_{x \to -\infty} \frac{3 + \frac{4}{x}}{\frac{\sqrt{x^2+4x+5}}{x} + \frac{\sqrt{x^2+x+1}}{x}} \\ &= \lim_{x \to -\infty} \frac{3 + \frac{4}{x}}{-\sqrt{\frac{x^2+4x+5}{x^2}} - \sqrt{\frac{x^2+x+1}{x^2}}} \\ &= \frac{3}{-1 - 1} = \frac{3}{-2} \end{aligned}$$

3. (10%) Find $\lim_{x\to\infty} \dfrac{\sin(x^{-2}+e^{-x})-\sin(x^{-2})}{e^{-x}}$.

Sol:

Method 1.

Let $f(x)=\sin x$. By Mean Value Theorem,

$f(\sin(x^{-2}+e^{-x})-f(x^{-2})=f'(c)e^{-x}$, where $c\in(x^{-2},x^{-2}+e^{-x})$.

So $c\to 0$ as $x\to\infty$. Therefore,

$$\begin{aligned}\lim_{x\to\infty}\frac{\sin(x^{-2}+e^{-x})-\sin(x^{-2})}{e^{-x}} &= \lim_{c\to 0}\frac{f'(c)e^{-x}}{e^{-x}}=\lim_{c\to 0}f'(c)\\ &= \lim_{c\to 0}\cos(c^{-2})=\cos 0=1.\end{aligned}$$

Method 2.

$$\begin{aligned}\lim_{x\to\infty}\frac{\sin(x^{-2}+e^{-x})-\sin(x^{-2})}{e^{-x}} &= \lim_{x\to\infty}\frac{\sin x^{-2}\cos e^{-x}+\cos x^{-2}\sin e^{-x}-\sin x^{-2}}{e^{-x}}\\ &= \lim_{x\to\infty}\left(\frac{\sin x^{-2}(\cos e^{-x}-1)}{e^{-x}}+\frac{\sin e^{-x}\cos x^{-2}}{e^{-x}}\right)\\ &= (\lim_{x\to\infty}\sin x^{-2})(\lim_{x\to\infty}\frac{\cos e^{-x}-1}{e^{-x}})+(\lim_{x\to\infty}\cos x^{-2})(\lim_{x\to\infty}\frac{\sin e^{-x}}{e^{-x}})\\ &= 0\cdot 0+1\cdot 1=1.\end{aligned}$$

4. (10%) For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}+a+\frac{b}{x}\right)=0.$$

Sol:

By l'Hospital rule,we have

$$\lim_{x\to 0}\frac{1-\cos x+ax^2+bx}{x^2}=\lim_{x\to 0}\frac{\sin x+2ax+b}{2x}$$

if the RHS exists. Thus we may assume the RHS exits to try to find a and b

Because the LHS exists,we have

$$\lim_{x\to 0}\sin x+2ax+b=0\Rightarrow b=0$$

Again by l'Hospital rule, we have

$$\lim_{x\to 0}\frac{\sin x+2ax+b}{2x}=\lim_{x\to 0}\frac{\cos x+2a}{2}=a+\frac{1}{2}$$

The above statement implies $a = \frac{-1}{2}$

5. (10%) A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate 3 cm/s. How fast is the distance from the particle to the origin changing at this moment?

Sol:

Method 1.

$('=\frac{d}{dt})$

$s = \sqrt{(x^2+y^2)},\ s' = \frac{2xx'+2yy'}{2\sqrt{x^2+y^2}},$

$y = \sqrt{x},\ y' = \frac{x'}{2\sqrt{x}},$

$s' = \frac{2\cdot 4 + 2\cdot 2\cdot \frac{1}{2\sqrt{4}}}{2\sqrt{16+4}}\cdot 3 = \frac{27}{4\sqrt{5}}\ cm/s$

Method 2.

$s = \sqrt{(x^2+y^2)} = \sqrt{(x^2+(\sqrt{x})^2)} = \sqrt{(x^2+x)}$

$s' = \frac{2x+1}{2\sqrt{x^2+x}}\frac{dx}{dt}$

$s' = \frac{2\cdot 4+1}{2\sqrt{4^2+4}}\cdot 3 = \frac{27}{4\sqrt{5}}\ cm/s$

6. (15%) Find y' and y'' of the curve $x^2+y^2 = (2x^2+2y^2-x)^2$ at the point $(0, \frac{1}{2})$.

Sol:

implicit differentiation.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(2x^2+2y^2-x)^2$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = (4x + 4y\frac{dy}{dx} - 1)\cdot x \cdot (2x^2+2y^2-x) \tag{1}$$

Plug in $(x, y) = (0, \frac{1}{2})$

$$\frac{dy}{dx}\Big|_{(0,\frac{1}{2})} = (2\frac{dy}{dx}\Big|_{(0,\frac{1}{2})} - 1)\cdot 2\cdot(2\cdot\frac{1}{4}) \Rightarrow \frac{dy}{dx}\Big|_{(0,\frac{1}{2})} = 1$$

Next, differentiate (1) w.r.t to x:

$$2+2(\frac{dy}{dx})^2+2y\frac{d^2y}{dx^2} = (4+4(\frac{dy}{dx})^2+4y\frac{d^2y}{dx^2})\cdot 2\cdot(2x^2+2y^2-x)+(4x+4y\frac{dy}{dx}-1)\cdot 2\cdot(4x+4y\frac{dy}{dx}-1)$$

$$\text{Plug in} \begin{cases} (x,y) = (0,\frac{1}{2}) \\ \frac{dy}{dx}\Big|_{(0,\frac{1}{2})} = 1 \end{cases}$$

$$\Rightarrow 2+2+1\cdot \frac{d^2y}{dx^2}\Big|_{(0,\frac{1}{2})} = (4+4+2\,\frac{d^2y}{dx^2}\Big|_{(0,\frac{1}{2})})\cdot 2\cdot(2\cdot\frac{1}{4}) + (2\cdot 1-1)\cdot 2\cdot(2\cdot 1-1)$$

$$4+\frac{d^2y}{dx^2}\Big|_{(0,\frac{1}{2})} = (8+2\,\frac{d^2y}{dx^2}\Big|_{(0,\frac{1}{2})}) + 2 \Rightarrow \frac{d^2y}{dx^2}\Big|_{(0,\frac{1}{2})} = -6$$

7. (15%) Consider the function $f(x) = \dfrac{x^2+1}{\sqrt{x^2-4}}$, for $x < -2$ and $x > 2$.

   (a) Find the intervals of increase and decrease. (b) Find the maximum and the minimum.

   (c) Find the asymptotes.

Sol:

(a), (b) $f'(x) = \dfrac{\sqrt{x^2-4}\cdot(2x) - (x^2+1)\cdot\frac{2x}{2\sqrt{x^2-4}}}{x^2-4} = \dfrac{x(x-3)(x+3)}{(x^2-4)^{\frac{3}{2}}}$

Increasing interval: $[-3,-2)$ and $[3,\infty)$.

Descending interval: $(-\infty,-3]$ and $(-2,3]$.

So $f(x)$ only has minima at $x=-3,3$. $\Longrightarrow$ Minimum: $f(-3) = f(3) = 2\sqrt{5}$.

(c) There are four asymptotes to the function. As x approach ± 2, f(x) go to infinite. Hence $x = \pm 2$ are asymptotes. Moreover, to see if there exists slant asymptotes,i.e $mx+b$, we need to observe

$$m = \lim_{x\to\infty} \frac{x^2+1}{x\sqrt{x^2-4}} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{\sqrt{1-\frac{4}{x^2}}} = 1$$

$$\begin{aligned} b &= \lim_{x\to\infty} f(x) - x \\ &= \lim_{x\to\infty} \frac{x^2+1-x\sqrt{x^2-4}}{\sqrt{x^2-4}} \\ &= \lim_{x\to\infty} \frac{(x^2+1)-(x\sqrt{x^2-4})}{\sqrt{x^2-4}} \frac{(x^2+1)+(x\sqrt{x^2-4})}{(x^2+1)+(x\sqrt{x^2-4})} \\ &= \lim_{x\to\infty} \frac{6x^2+1}{\sqrt{x^2-4}(x^2+1+x\sqrt{x^2-4})} = 0 \end{aligned}$$

By definition of asymptote, $y = 1 \cdot x + 0$ is an asymptote. Since f(x)=f(-x), y=-x is also an asymptote too.

8. (10%) Evaluate $\lim_{n\to\infty} \frac{1}{n}\left[\frac{\ln\left(\frac{n+1}{n}\right)}{\frac{n+1}{n}} + \frac{\ln\left(\frac{n+2}{n}\right)}{\frac{n+2}{n}} + \cdots + \frac{\ln\left(\frac{n+n}{n}\right)}{\frac{n+n}{n}}\right]$.

   Sol:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\ln(1+\frac{k}{n})}{1+\frac{k}{n}} = \int_{x=1}^{2} \frac{\ln x}{x} dx$$
$$= \int_{u=0}^{\ln 2} u du$$

   where $u = \ln x$, $= \frac{1}{2}(\ln 2)^2$

9. (10%) Find the derivative of $h(x) = \int_0^{\sin x} \sqrt{1+r^3}\, dr$.

   Sol:

$$h(x) = \int_0^{\sin x} \sqrt{1+r^3} dr$$
$$\text{Let } G(x) = \int_0^{x} \sqrt{1+r^3} dr$$
$$\frac{d}{dx}G(x) = \sqrt{1+x^3}$$
$$h(x) = G(\sin x)$$
$$\frac{d}{dx}h(x) = \frac{d}{d\sin x}G(\sin x)\frac{d\sin x}{dx} = \sqrt{1+\sin^3 x}\cos x$$